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Well-Posedness and Energy Decay Rates for a Timoshenko-Type System with Internal Time-Varying Delay in the Displacement

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Abstract: In this work, we consider a Timoshenko-type system in a bounded one-dimensional domain under Dirichlet conditions with time-varying delay and internal friction damping acting in the displacement. First, we show that the system is well-posed using semi-group theory. Then, under appropriate assumptions on the weights of the delay, the stability of system is obtained via a suitable Lyapunov functional.

Keywords: Timoshenko-type systems; time-varying delays; well-posedness; energy decay; Lyapunov functions; partial differential equations

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1. Introduction

In 1921, Timoshenko [1] introduced the problem of a thick beam given by the following system of coupled hyperbolic equations:

$$\begin{aligned} &\rho u_{tt}(x,t) - (Ku_x - \varphi)_x(x,t) = 0, \quad (x,t) \in (0,L)\times]0,+\infty[,\\ &I_\rho \varphi_{tt}(x,t) - (EL\varphi_x)_x - K(u_x - \varphi)(x,t) = 0, \quad (x,t) \in (0,L)\times]0,+\infty[, \end{aligned} \tag{1}$$

where u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficient ρ is the density, I_ρ is the polar moment of inertia of a cross section, E is Young's modulus of elasticity, I is the moment of inertia of a cross section, and k is the shear modulus.

In the late 19th century, researchers became interested in studying the deformations in elastic structures such as beams, plates, and shells when rotational inertia and shear deformation form the main hypotheses. With the beginning of the 21st century, authors' interest in studying the system in (1) increased and results related to existence and asymptotic behavior were achieved. The stability of the Timoshenko system with different types of damping has been studied—we refer the reader to [2–6] and their references.

Problem (1) has been studied by Kim and Renardy [2] under the following two boundary conditions:

$$\begin{aligned} &K\varphi(L,t) - Ku_x(L,t) = \alpha u_t(L,t), \ \ \forall t > 0,\\ &EI\varphi_x(L,t) + \beta\varphi_t(L,t) = 0, \ \ \forall t > 0, \end{aligned}$$

as they proved the exponential decay of the natural energy of (1) by multiplier techniques.

Soufyane and Wehbe in [7] showed that Problem (1) with unique locally distributed feedback is uniformly stable if and only if the wave speeds are equal; otherwise, it is asymptotically stable. Shi and Feng [8] studied a nonuniform Timoshenko beam and showed that the beam's vibration decays exponentially under some locally distributed controls. This was carried out using the frequency multiplier method.

In this article, we study a more general Timoshenko problem than the problems that have been studied, with the delay term appearing as the control term in the first equation.

The introduction of the term delay $\mu u_t(t-\tau(t))$ makes the problem different from that addressed in the literature.

Many works have shown that the presence of a delay in a partial differential equation problem is a source of instability unless additional control terms or conditions are used; see, for example, references [9–12].

Several researchers treated the Timoshenko system with internal constant delay acting in one equation or in two equations; as we mention here, one of the first results was obtained by Said-Houari and Laskri [13]. They studied the following Timoshenko system:

$$\begin{aligned}
&\rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0,\\
&\rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) - K(\varphi_x+\psi)(x,t) + a_0\psi_t(x,t) + a\psi_t(x,t-\tau) = 0,
\end{aligned} \tag{2}$$

In $(0,1)\times\mathbb{R}_+$, the authors of [13] proved the stability of (2) in the case of the equal-speed propagation under the condition ($a < a_0$). Moreover, Said-Houari and Rahali [14] studied System (2) in the presence of a viscoelastic damping of the form $\int_0^t g(s)\psi_{xx}(t-s)ds$ acting on the second equation. They proved that the energy total of this problem decays exponentially in the case of equal wave speeds, and $0 < a = \mu_1 \le a_0 = \mu_2$.

In 2013, Muhammad Kafini et al. [15] considered the following Timoshenko system of thermoelasticity of type III with constant delay:

$$\begin{array}{ll}
(\rho_1\varphi_{tt} - K(\varphi_x+\psi)_x)(x,t) = 0, & \textit{in }]0,1[\times]0,+\infty[,\\
(\rho_2\psi_{tt} - b\psi_{xx} - K(\varphi_x+\psi) + \beta\theta_{tx})(x,t) = 0, & \textit{in }]0,1[\times]0,+\infty[,\\
(\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} + \mu_1\theta_{txx})(x,t) + \mu_2\theta_{txx}(x,t-\tau) = 0, & \textit{in }]0,1[\times]0,+\infty[,\\
\theta(x,0) = \theta_0, \theta_t(x,0) = \theta_1, \psi(x,0) = \psi_0, \psi_t(x,0) = \psi_1, & \textit{in }]0,1[,\\
\varphi(x,0) = \varphi_0, \varphi_t(x,0) = \varphi_1, & \textit{in }]0,1[,\\
\theta_{tx}(x,t-\tau) = f_0(x,t-\tau), & \textit{in }]0,1[\times]-\tau,0[,\\
\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, & \textit{on }]0,+\infty[,\\
\theta_x(0,t) = \theta_x(1,t) = 0, & \textit{on }]-\tau,+\infty[.
\end{array}$$

The authors in this article showed that the energy decreases exponentially in the case of equal wave speeds and polynomially otherwise (under suitable conditions on the coefficients and the initial data). A one-dimensional linear thermoelastic system of Timoshenko type with delay is considered in [16]. Well-posedness and stability of the system are established by using the well known Lyapunov functional. The results in our article are obtained using the Lyapunov functional, as in [16], but with another choice for the functions, i.e., constructing the Lyapunov functional. This choice is imposed by the nature of our system, which is totally different from the one previously studied.

Almeida Junior et al. [17] studied the asymptotic behavior of solutions for two dissipative Bresse–Timoshenko systems without a "second spectrum" and with a delay term in the internal feedback, one on the vertical displacement and the other on angular rotation, which are given by

$$\begin{array}{ll}
\rho_1 y_{tt} - K(y_x+\psi)_x + \mu_1 y_t + \mu_2 y_t(x,t-\tau) = 0, & \textit{in }]0,L[\times]0,+\infty[,\\
-\rho_2\psi_{ttx} - b\psi_{xx} - K(y_x+\psi) = 0, & \textit{in }]0,L[\times]0,+\infty[,
\end{array} \tag{3}$$

and

$$\begin{array}{ll}
\rho_1 y_{tt} - K(y_x+\psi)_x = 0, & \textit{in }]0,L[\times]0,+\infty[,\\
-\rho_2\psi_{ttx} - b\psi_{xx} - K(y_x+\psi) + \mu_1\psi_t + \mu_2\psi_t(x,t-\tau) = 0, & \textit{in }]0,L[\times]0,+\infty[.
\end{array} \tag{4}$$

The result of System (2) was extended to the case of time-varying delay by Kirane et al. [14].

Systems (3) and (4) were studied by Feng et al. [18] with time-dependent delay terms. The authors used the appropriate Lyapunov function to demonstrate the exponential decay results.

We mention here that the nonlinear Timoshenko system subject to variable delay and internal feedback was considered by Xin-Guang Yang et al. [19] as follows:

$$\begin{array}{ll} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = h(x), & in\ ]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) - K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) & \\ \qquad\qquad +\mu_2\psi_t(x,t-\tau(t)) + f(\psi(x,t)) = g(x), & in\ ]0,1[\times]0,+\infty[, \end{array}$$

with the Dirichlet boundary condition:

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad \forall t > 0.$$

After proving that the problem is well-posed, the authors demonstrated the existence of the finite-dimensional global and exponential attractors by using the concept of quasi-stability used by Lasiecka and Chueshov in [20,21].The motivation to introduce a time-dependent delay is that, in previous papers, fixed delays have mostly been considered, except for a few works—see [22–26], which can be considered as the most widely cited papers that deal with these types of problems. However, to show the influence of a time-dependent delay, we should make a comparison to previous results. With time-varying weight and time-varying delay, the authors in [22] studied the global well-posedness and exponential stability for a Rao–Nakra sandwich beam equation (see [25,26]). The aim of [23] was to consider the Timoshenko system in thermoelasticity of second sound with a time-varying delay, where the questions of well-posedness and stability were investigated; one can also see the results in [24].

For systems with two internal time delays, we mention the work of Said Houari and Sofiane [27]:

$$\begin{array}{ll} \rho_1 y_{tt} - K(y_x+\psi)_x + a_1 y_t(x,t-\tau) = 0, & in\ ]0,L[\times]0,+\infty[, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(y_x+\psi) + a_2\psi_t(x,t-\tau) = 0, & in\ ]0,L[\times]0,+\infty[, \end{array}$$

with the following boundary controls:

$$k(y_x+\psi)(L,t) = -\alpha y_t(L,t), \quad b\psi_x(L,t) = -\mu\psi_t(L,t).$$

The stability of this Timoshenko system was proven under some smallness conditions on L and the weights of the delays.

In [28], Aissa Guesmia and Abdelaziz Soufyane considered a Timoshenko-type system with delay terms:

$$\begin{array}{l} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x+\psi)_x(x,t) + \lambda_1\varphi_t(x,t) + \mu_1\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x+\psi)(x,t) + \lambda_2\psi_t(x,t) + \mu_2\psi_t(x,t-\tau_2) = 0, \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \;\; \psi_t(x,0) = \psi_1(x), \\ \varphi_t(x,-\rho\tau_1) = f_1(x,-\rho\tau_1), \;\; \psi_t(x,-\rho\tau_2) = f_2(x,-\rho\tau_2), \end{array} \tag{5}$$

under the Dirichlet–Dirichlet boundary conditions:

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \tag{6}$$

or the Dirichlet–Neumann boundary conditions:

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0, \tag{7}$$

for $x \in]0,L[$, $t > 0$, $\rho \in]0,1[$, $\mu_j \in \mathbb{R}$, $L,\rho_j,k_j,\tau_j > 0$, $\lambda_j \geq 0, (j=1,2)$, $(\varphi,\psi) :]0,L[\times]0,+\infty[\longrightarrow \mathbb{R}^2$ is the state of (5) with (6) or (7), φ_0, φ_1, ψ_0, $\psi_1 :]0,L[\longrightarrow \mathbb{R}$, and $f_j :]0,L[\times]-\tau_j,0[\longrightarrow \mathbb{R}$, $(j=1,2)$. The authors of this article have demonstrated the well-posedness and asymptotic behavior of (5) with (6) or (7) in the case of equal-velocity wave propagation as well as in the opposite case. Precisely, they proved the exponential stability in the case of equal-speed wave propagation and the polynomial stability in the opposite case. It is known, at least in this field of research, that, if we add more damping terms to evolutionary systems, this weakens the scientific value of the problem, particularly from a mathematics point of view, which is not our case. This makes the problem weak, and the stabilization process can be facilitated despite the presence of some positive points, which are mainly represented by the interactions between the different parameters of the damping terms. This case is in [29], where a system similar to (5) is considered with three damping terms (discrete delay, complementary frictional damping, and infinite memory).

The continuation of this work is organized as follows: In Section 2, we introduce the problem and we consider the hypotheses for the coefficients present in (8). In Section 3, we present some preliminaries, and our main results are presented in Section 4, using the semi-group theory of linear operators found in [30,31] to prove the well-posedness result. Then, the exponential decay of the energy of our problem is obtained in Section 5.

2. Position of Problem and Hypothesis

A new mathematical model of a Timoshenko-type system is constructed, taking into account internal friction damping, in which the effects of time-dependent delay are considered. This generalization is analyzed in the process of thermomechanical loading.

Now, we propose to study the exponential stability of the following Timoshenko-type system subject to a time-dependent delay term acting on the following equation:

$$\begin{array}{ll} [\rho_1 u_{1tt} - k(u_{1x} + pu_2)_x - k_0(u_{1x} + pu_2)_{tx} + (\alpha u_1 - \gamma\beta u_2)_{xxxx}](x,t) & \\ \qquad + \mu_1 u_{1t}(x,t) + \mu_2 u_{1t}(x, t - \tau(t)) = 0 & \text{in }]0,l[\times]0,+\infty[, \\ [\rho_2 u_{2tt} - bu_{2xx} + pk(u_{1x} + pu_2) + pk_0(u_{1x} + pu_2)_t](x,t) & \\ \qquad + \beta(u_2 - \gamma u_1)_{xxxx}(x,t) = 0 & \text{in }]0,l[\times]0,+\infty[, \end{array} \tag{8}$$

where $u_1(x,t)$ and $u_2(x,t)$ are the unknowns, which represent the transverse displacement of the plate and the rotation angle of a filament of the plate, respectively, l is the curvature of the beam, $\mu_1 u_{1t}$ represent frictional damping, $\tau(t)$ represents time-varying delay to the system, ρ_i; $(i = 1;2)$, μ_1, k, α, β, and γ are strictly positive constants, and μ_2 is a real number.

From now on, we consider for System (8) the following initial conditions:

$$\begin{array}{ll} u_i(x,0) = u_i^0(x), \;\; u_{it}(x,0) = u_i^1(x), \;\; (i = 1;2), & \text{in }]0,l[\\ u_{1t}(x, t - \tau(0)) = f_0(x, t - \tau(0)), & \text{in }]0,l[\times]0,\tau(0)[, \end{array} \tag{9}$$

and the Dirichlet boundary conditions:

$$u_i(x,t) = 0, \;\; (i = 1;2), \;\; x \in \{0,l\}, \quad \forall t \geq 0. \tag{10}$$

First, we consider the following hypotheses.

Hypothesis 1. *The delay function $\tau(t)$ is a $C^1(\mathbb{R}_+)$ continuous function which satisfies*

$$\tau \in W^{2,\infty}([0,T]), \;\; \forall T > 0, \tag{11}$$

and there exist positive constants τ_0, τ_1, and $d > 0$, such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \;\; \forall t > 0, \tag{12}$$

and

$$0 < \tau'(t) \leq d < 1, \;\; \forall t > 0. \tag{13}$$

Hypothesis 2. *There exists a constant $d > 0$, such that*

$$|\mu_2| \leq \sqrt{1-d}\mu_1, \;\; \forall t > 0. \tag{14}$$

Remark 1. *If we look at the function $f(t) = t - \tau(t)$, Condition (12) implies that f is a strictly increasing function. This means that the delayed information arrives in chronological order.*

3. Preliminaries and Main Results

Due to Datko et al. [12] and also [32], we consider the following changes of variables:

$$z(x,\rho,t) = u_{1t}(x, t - \rho\tau(t)), \quad (x,\rho,t) \in]0,l[\times]0,1[\times]0,+\infty[. \tag{15}$$

We can easily check that z satisfies the following relationship:

$$\tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0, \ (x,\rho,t) \in]0,l[\times]0,1[\times]0,+\infty[. \tag{16}$$

Using these new variables, System (8) is converted to the following equivalent form:

$$\begin{array}{ll} [\rho_1 u_{1tt} - k(u_{1x}+pu_2)_x - k_0(u_{1x}+pu_2)_{tx} + (\alpha u_1 - \gamma\beta u_2)_{xxxx}](x,t) & \\ \quad +\mu_1 u_{1t}(x,t) + \mu_2 z(x,1,t) = 0 & \text{in }]0,l[\times]0,+\infty[, \\ [\rho_2 u_{2tt} - bu_{2xx} + pk(u_{1x}+pu_2) + pk_0(u_{1x}+pu_2)_t](x,t) & \\ \quad +\beta(u_2 - \gamma u_1)_{xxxx}(x,t) = 0 & \text{in }]0,l[\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\rho\tau'(t))z_\rho(x,\rho,t) = 0, & \text{in }]0,l[\times]0,1[\times]0,+\infty[. \end{array} \tag{17}$$

System (17) is equipped with the following initial and boundary conditions:

$$\begin{array}{ll} u_i(x,0) = u_i^0(x), \ u_{it}(x,0) = u_i^1(x), \ (i=1;2), & \text{in }]0,l[\\ u_{1t}(x,t-\tau(0)) = f_0(x,t-\tau(0)), & \text{in }]0,l[\times]0,\tau(0)[, \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), & \text{in }]0,l[\times]0,1[, \\ z(x,0,t) = u_{1t}(x,t), & \text{in }]0,l[\times]0,+\infty[, \\ u_i(x,t) = 0, \ (i=1;2), \ x \in \{0,l\}, \ \forall t \geq 0. & \end{array} \tag{18}$$

From now on, we use the following symbols:

$$u_i := u_i(x,t), \ (i=1;2) \quad \textit{and} \quad z(\rho) := z(x,\rho,t).$$

To announce our stability results, we define the energy function associated with (8) by

$$\begin{array}{rl} E(t) = & \frac{1}{2}\Big[k\|u_{1x}+pu_2\|^2 + (\alpha - \beta\gamma^2)\|u_{1xx}\|^2 + \beta\|\gamma u_{1xx} - u_{2xx}\|^2 \\ & +b\|u_{2x}\|^2 + \sum_{i=1}^2 \rho_i\|u_{it}\|^2 + \mu_1\xi\tau(t)\int_0^1 \|z^2(\rho)\|^2 d\rho\Big]. \end{array} \tag{19}$$

The main goal of our problem is to prove the following result.

Theorem 1. *Assume that Hypothesis 1 and Hypothesis 2 hold.*

Then, for any $\mathcal{U}_0 = (u_1^0, u_1^1, u_2^0, u_2^1, f_0)^T \in \mathcal{H}$, there exist positive constants ζ and ω, independent of t, such that the solution $\mathcal{U} = (u_1, u_{1t}, u_2, u_{2t}, z)^T$ of Problems (8) and (9) satisfies

$$E(t) \leq \zeta e^{-\omega t} \quad \forall t \in \mathbb{R}_+. \tag{20}$$

In the next section, we are concerned with the existence, uniqueness, and smoothness of the solution of (17) and (18) based on the classical Lumiere–Phillips theory, which is found in [31,33].

4. Well-Posedness

We also use h' to denote the derivative when the function h has only one variable. The notation ∂y denotes the derivative with respect to y and w_y denotes the derivative of w with respect to y.

We introduce the following notations. We note $\|.\|_X$ as the usual norm defined on the Banach space X and $\langle.\rangle$ and $\|.\|$ as the inner product and the norm defined on $L^2(0,l)$, respectively.

First, we transform Systems (17) and (18) to the first-order differential system in (28) below. For this, we adopt the technique in [9–34]. Then, we prove that the operator $\mathcal{A}$, given in (25), generates a contraction semi-group on the Hilbert space $\mathcal{H}$ given in (21).

Now, we introduce $\phi_i = u_{it}$, $(i = 1;2)$ and consider the following energy space:

$$\mathcal{H} = \left(H_0^1(0,l) \times L^2(0,l)\right)^2 \times L^2((0,l) \times (0,1)). \tag{21}$$

The space $\mathcal{H}$ is equipped with the inner product, which is defined as follows:

$$\begin{aligned}\left\langle U, \widetilde{U}\right\rangle_{\mathcal{H}} = \; & \int_0^l (\alpha - \beta\gamma^2)\partial_{xx}u_1\partial_{xx}\widetilde{u_1} + \sum_{i=1}^2 \rho_i\phi_i\widetilde{\phi_i}dx \\ & + \int_0^l \beta(\gamma\partial_{xx}u_1 - \partial_{xx}u_2)(\gamma\partial_{xx}\widetilde{u_1} - \partial_{xx}\widetilde{u_2})dx \\ & + \int_0^l k(\partial_x u_1 + pu_2)(\partial_x\widetilde{u_1} + p\widetilde{u_2}) + b\partial_x u_2\partial_x\widetilde{u_2}dx \\ & + \int_0^l \mu_1\xi\tau(t)\int_0^1 z(\rho)\widetilde{z}(\rho)d\rho dx,\end{aligned} \tag{22}$$

for any $U = (u_1, \phi_1, u_2, \phi_2, z)$, $\widetilde{U} = (\widetilde{u_1}, \widetilde{\phi_1}, \widetilde{u_2}, \widetilde{\phi_2}, \widetilde{z})$ in $\mathcal{H}$.

Moreover, by Hypothesis 1 and Hypothesis 2, we also assume that there is a positive constant ξ that, for any $t > 0$, satisfies

$$\frac{|\mu_2|}{(1-\tau'(t))\mu_1} < \xi < 2 - \frac{|\mu_2|}{\sqrt{1-d}\mu_1}. \tag{23}$$

And from there, we deduce the norm associated with this space:

$$\begin{aligned}\|U\|_{\mathcal{H}} = \; & k\|u_{1x} + pu_2\|^2 + (\alpha - \beta\gamma^2)\|u_{1xx}\|^2 + \sum_{i=1}^2 \rho_i\|\phi_i\|^2 \\ & + b\|u_{2x}\|^2 + \beta\|\gamma u_{1xx} - u_{2xx}\|^2 + \mu_1\xi\tau(t)\int_0^1 \|z(\rho)\|^2 d\rho.\end{aligned} \tag{24}$$

Now, we define the differential operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ by the following matrix:

$$\begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{1}{\rho_1}(-\alpha\partial_x^4 + k\partial_x^2) & \frac{k_0}{\rho_1}\partial_x^2 - \frac{\mu_1}{\rho_1}I & \frac{\beta\gamma}{\rho_1}\partial_x^4 + \frac{pk}{\rho_1}\partial_x & \frac{pk_0}{\rho_1}\partial_x & -\frac{\mu_2}{\rho_1}I_{|_{\rho=1}} \\ 0 & 0 & 0 & I & 0 \\ \frac{\beta\gamma}{\rho_2}\partial_x^4 - \frac{pk}{\rho_2}\partial_x & -\frac{pk_0}{\rho_2}\partial_x & -\frac{\beta}{\rho_2}\partial_x^4 + \frac{b}{\rho_2}\partial_x^2 - \frac{p^2k}{\rho_2}I & -\frac{p^2k_0}{\rho_2}I & 0 \\ 0 & 0 & 0 & 0 & \delta_1(t)\partial_\rho \end{pmatrix} \tag{25}$$

where $\delta_1(t) = \frac{\rho\tau'(t)-1}{\tau(t)}$, with the domain

$$D(\mathcal{A}) = \{U \in H;\ \partial_\rho z \in L^2(0,l;L^2(0,l)) \ \textit{and}\ z(0) = u_{1t}\}, \tag{26}$$

where

$$H = \left((H^2(0,l) \cap H_0^1(0,l)) \times H^1(0,l)\right)^2 \times L^2(0,l;L^2(0,l)). \tag{27}$$

Under the above definitions, for any $U = (u_1, \phi_1, u_2, \phi_2, z)^T$ and $U_0 = \left(u_1^0, u_1^1, u_2^0, u_2^1, f_0(., -\rho\tau(0))\right)^T$ in $\mathcal{H}$, System (17) can be written as the following Cauchy problem in $\mathcal{H}$:

$$\begin{aligned}&\mathcal{A}U(t) = U'(t), \quad \textit{in}\ ]0,+\infty[, \\ &U(0) = U_0.\end{aligned} \tag{28}$$

Observe that $D(\mathcal{A}(t))$ is independent of the time t. This means that

$$D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \forall t > 0.$$

By the classical semi-group theory, we obtain our well-posedness result in the following theorem.

Theorem 2. *Assume that Hypothesis 1 and Hypothesis 2 are satisfied and* (23) *holds; then, for any $U_0 \in \mathcal{H}$ Problem* (8)*, has a unique solution*

$$U \in C([0,+\infty[,\mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$*, then*

$$U \in C([0,+\infty[, D(\mathcal{A})) \cap C^1([0,+\infty[, \mathcal{H}).$$

Proof. In order to prove Theorem 2, we show that the operator $\mathcal{A}$ generates a C_0 semi-group in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative.

For $U = (u_1, \phi_1, u_2, \phi_2, z)^T \in D(\mathcal{A})$, we have

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle \quad = \quad & (\alpha - \beta\gamma^2)\langle \phi_{1xx}, u_{1xx}\rangle - k\langle u_{1x} + pu_2, \phi_{1x}\rangle - k_0\langle \phi_{1x} + p\phi_2, \phi_{1x}\rangle \\
& -\alpha\langle u_{1xx}, \phi_{1xx}\rangle + \beta\gamma\langle u_{2xx}, \phi_{1xx}\rangle - \mu_1\|\phi_1\|^2 - \mu_2\langle z(1), \phi_i\rangle \\
& -pk\langle u_{1x} + pu_2, \phi_2\rangle - pk_0\langle \phi_{1x} + p\phi_2, \phi_2\rangle - \beta\langle u_{2xx}, \phi_{2xx}\rangle \\
& +\beta\gamma\langle u_{1xx}, \phi_{2xx}\rangle + \beta\langle \gamma\phi_{1xx} - \phi_{2xx}, \gamma u_{1xx} - u_{2xx}\rangle \\
& +k\langle \phi_{1x} + p\phi_2, u_{1x} + pu_2\rangle - \mu_1\xi\int_0^1\langle (1-\rho\tau'(t))\partial_\rho z(\rho), z(\rho)\rangle d\rho \\
= \quad & -k_0\|\phi_{1x} + p\phi_2\|^2 - \mu_1\|\phi_1\|^2 - \mu_2\langle z(1), \phi_1\rangle \\
& -\xi\mu_1\int_0^1\langle (1-\rho\tau'(t))\partial_\rho z(\rho), z(\rho)\rangle d\rho \\
\leq \quad & -k_0\|\phi_{1x} + p\phi_2\|^2 - \mu_1\|\phi_1\|^2 + \tfrac{|\mu_2|}{2\varepsilon_1}\|\phi_1\|^2 + \tfrac{\varepsilon_1|\mu_2|}{2}\|z(1)\|^2 \\
& -\tfrac{\xi\mu_1}{2}\left[(1-\tau'(t))\|z(1)\|^2 - \|\phi_1\|^2 + \tau'(t)\int_0^1\|z(\rho)\|^2 d\rho\right] \\
= \quad & -k_0\|\phi_{1x} + p\phi_2\|^2 - \left(\mu_1 - \tfrac{|\mu_2|}{2\varepsilon_1} - \tfrac{\mu_1\xi}{2}\right)\|\phi_1\|^2 \\
& -\left(\tfrac{\mu_1\xi(1-\tau'(t))}{2} - \tfrac{\varepsilon_1|\mu_2|}{2}\right)\|z(1)\|^2 - \tfrac{\xi\mu_1\tau'(t)}{2}\int_0^1\|z(\rho)\|^2 d\rho.
\end{aligned} \tag{29}$$

We can choose $\varepsilon_1 = \sqrt{1-d}$ and, from (13), (14), and (23), for all $t > 0$, we obtain

$$\lambda_1 = \mu_1 - \frac{|\mu_2|}{2\varepsilon_1} - \frac{\mu_1\xi}{2} \geq 0,$$

and

$$\lambda_2 = \frac{\mu_1\xi(1-\tau'(t))}{2} - \frac{\varepsilon_1|\mu_2|}{2} \geq 0.$$

Hence, from (29) we deduce that the operator $\mathcal{A}$ is dissipative.

Next, we prove that $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$.

For this, we seek a solution $U = (u_1, \phi_1, u_2, \phi_2, z)^T \in D(\mathcal{A})$ of the equation $(\lambda I - \mathcal{A})U = F$, where $F = (f_1, f_2, f_3, f_4, f_5)^T$, that is,

$$\begin{cases}
\lambda u_1 - \phi_1 = f_1 \\
\lambda\phi_1 - \dfrac{1}{\rho_1}[k(u_{1xx} + pu_{2x}) + k_0(\phi_{1xx} + p\phi_{2x}) + (\beta\gamma u_2 - \alpha u_1)_{xxxx} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\mu_1\phi_1 - \mu_2 z(1)] = f_2 \\
\lambda u_2 - \phi_2 = f_3 \\
\lambda\phi_2 - \dfrac{1}{\rho_2}[-pk(u_{1x} + pu_2) - pk_0(\phi_{1x} + p\phi_2) + bu_{2xx} + \beta(\gamma u_1 - u_2)_{xxxx}] = f_4 \\
\lambda z + \frac{1-\rho\tau'(t)}{\tau(t)} z_\rho = f_5.
\end{cases} \tag{30}$$

Then, from the first and third equations in (30), we have

$$\begin{cases}
\phi_1 = \lambda u_1 - f_1 \\
\phi_2 = \lambda u_2 - f_3.
\end{cases} \tag{31}$$

The last equation in (30) is equivalent to

$$g(t,\rho)z(\rho) + z_\rho(\rho) = \tfrac{1}{\lambda}g(t,\rho)f_5, \tag{32}$$

where

$$g(t,\rho) = \frac{\lambda\tau(t)}{1-\rho\tau'(t)}.$$

Then, by solving the ordinary differential Equation (32) and noting that $z(0) = \phi_1 = \lambda u_1 - f_1$, we obtain

$$z(\rho) = \lambda e^{G(t,\rho)} u_1 - \left(f_1 - \frac{1}{\lambda} \int_0^\rho g(t,y) f_5(x,y) e^{-G(t,y)} dy \right) e^{G(t,\rho)}, \tag{33}$$

where

$$\begin{cases} G(t,\rho) = -\int_0^\rho g(t,\sigma) d\sigma, & \text{if } \tau'(t) \neq 0, \\ G(t,\rho) = -\rho\lambda\tau(t), & \text{if } \tau'(t) = 0. \end{cases} \tag{34}$$

Substituting (31) and (33) into the second and fourth equations in (30), we have

$$\begin{cases} \lambda\vartheta_1(t) u_1 - (k + k_0\lambda)(u_{1x} + pu_2)_x + (\alpha u_1 - \beta\gamma u_2)_{xxxx} \\ \qquad = \vartheta_1(t) f_1 + \rho_1 f_2 - k_0(\partial_x f_1 + p f_3)_x + \mu_2 f_7 \\ \lambda^2 \rho_2 u_2 + p(k + k_0\lambda)(u_{1x} + pu_2) - \beta(\gamma u_1 - u_2)_{xxxx} - bu_{2xx} \\ \qquad = \rho_2 \lambda f_3 + pk_0(\partial_x f_1 + p f_3) + \rho_2 f_4, \end{cases} \tag{35}$$

where

$$\vartheta_1(t) = \lambda\rho_1 + \mu_1 + \mu_2 e^{G(t,1)},$$

and

$$\begin{cases} f_7 = \left(\frac{1}{\lambda} \int_0^1 g(t,y) f_5(x,y) e^{-G(t,y)} dy \right) e^{G(t,1)}, & \text{if } \tau'(t) \neq 0, \\ f_7 = \left(\tau(t) \int_0^1 f_5(x,y) e^{\lambda\tau(t)} dy \right) e^{-\lambda\tau(t)}, & \text{if } \tau'(t) = 0, \end{cases}$$

with

$$\begin{cases} G(t,1) = ln(1 - \tau'(t))^{\frac{\lambda\tau(t)}{\tau'(t)}}, & \text{if } \tau'(t) \neq 0, \\ G(t,1) = -\lambda\tau(t), & \text{if } \tau'(t) = 0, \end{cases}$$

We use these in order to solve the following equations:

$$\begin{array}{r} \lambda\vartheta_i(t) u_i + (-1)^i p^{i-1}(k + k_0\lambda)\partial_x^{2-i}(\partial_x u_1 + pu_2) + ((2-i)\alpha + (i-1)\beta)\partial_x^4 u_i - \gamma\beta\partial_x^4 u_{3-i} \\ = \vartheta_i(t) f_{2i-1} + \rho_i f_{2i} + (-1)^i k_0 p^{i-1} \partial_x^{2-i}(\partial_x f_1 + p f_3) + (2-i)\mu_i f_i, \quad i = 1; 2. \end{array} \tag{36}$$

where $\vartheta_i(t) = \rho_i\lambda + (2-i)(\mu_1 + \mu_2 e^{G(t,1)}), \ \ i = 1; 2.$

We use a standard procedure for these, multiplying (36) by φ_1 if $i = 1$ and by φ_2 if $i = 2$, where $\varphi_i \in H_0^1(]0, l[)$. By summing the resulting equations and then integrating by parts with respect to x, we obtain the following variational formulation:

$$a((u_1, u_2), (\varphi_1, \varphi_2)) = \mathcal{L}(\varphi_1, \varphi_2), \tag{37}$$

where the bi-linear form $a : \left(H_0^1(]0, l[)\right)^2 \times \left(H_0^1(]0, l[)\right)^2 \longrightarrow \mathbb{R}$ and the linear form $\mathcal{L} : \left(H_0^1(]0, l[)\right)^2 \longrightarrow \mathbb{R}$ are given by

$$\begin{aligned} a((u_1, u_2), (\varphi_1, \varphi_2)) = \ & \lambda\vartheta_1\langle u_1, \varphi_1\rangle + (k + k_0\lambda)\langle u_{1x} + pu_2, \varphi_{1x} + p\varphi_2\rangle \\ & + \alpha\langle u_{1xx}, \varphi_{1xx}\rangle - \beta\gamma\langle u_{2xx}, \varphi_{1xx}\rangle - \beta\gamma\langle u_{1xx}, \varphi_{2xx}\rangle \\ & + \beta\langle u_{2xx} + \varphi_{2xx}\rangle + \lambda^2\rho_2\langle u_2, \varphi_2\rangle + b\langle u_{2x}, \varphi_{2x}\rangle, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\varphi_1, \varphi_2) = \ & \vartheta_1\langle f_1, \varphi_1\rangle + k_0\langle f_{1x} + p f_3, \varphi_{1x} + p\varphi_2\rangle + \mu_2\langle f_7, \varphi_1\rangle \\ & + \rho_1\langle f_2, \varphi_1\rangle + \rho_2\lambda\langle f_3, \varphi_2\rangle + \rho_2\langle f_4, \varphi_2\rangle. \end{aligned}$$

It is easy to check that a is coercive; by choosing the test functions $\varphi_1 = u_1$ and $\varphi_2 = u_2$, we obtain

$$\begin{aligned} a((u_1, u_2), (u_1, u_2)) \ = \ & (k + k_0\lambda)\langle u_{1x} + pu_2, u_{1x} + pu_2\rangle + \alpha\langle u_{1xx}, u_{1xx}\rangle \\ & - 2\beta\gamma\langle u_{2xx}, u_{1xx}\rangle + \beta\langle u_{2xx}, u_{2xx}\rangle + \lambda\vartheta_1\langle u_1, u_1\rangle \\ & + \lambda^2\rho_2\langle u_2, u_2\rangle + b\langle u_{2x}, u_{2x}\rangle \\ = \ & (k + k_0\lambda)\|u_{1x} + pu_2\|^2 + \left(\alpha - \beta\gamma^2\right)\|u_{1xx}\|^2 \\ & + \beta\|\gamma u_{1xx} - u_{2xx}\|^2 + \lambda\vartheta_1\|u_1\|^2 + \lambda^2\rho_2\|u_2\|^2 + b\|u_{2x}\|^2. \end{aligned}$$

Moreover, it is not difficult to show that the operators a and $\mathcal{L}$ are continuous linear. Thus, by Lax–Milgram theorem, we have proven that the problem (37) admits a unique solution $(u_1, u_2) \in \left(H_0^1(0,l)\right)^2$ for all $(\varphi_1, \varphi_2) \in \left(H_0^1(0,l)\right)^2$. This means that the operator $\lambda I - \mathcal{A}$ is surjective for any fixed $t > 0$ and $\lambda > 0$. Thus, by applying the Lumer–Phillips theorem (see [31]) to Problem (28), we have proven that operator $\mathcal{A}$ generates a strongly continuous semigroup of $S(t)$ on $\mathcal{H}$. $\square$

5. Exponential Stability

In this section, we are interested in studying asymptotic behavior. We show that the solution to Problems (8)–(10) is exponentially stable. To achieve this goal, we construct a functional $L(t)$ that is equivalent to the energy $E(t)$, such that $\partial_t L$ has a negative multiple of E. For this, we consider the following lemmas.

Our objective in the first result indicates that the energy is a non-increasing function and is uniformly bounded above by $E(0)$.

Lemma 1. *Assume that Hypothesis 1 and Hypothesis 2 hold. Then, for any regular solution of Problems (17) and (18) and for any $t \geq 0$, the derivative of energy $E(t)$ satisfies the following estimate:*

$$\frac{d}{dt}E(t) \leq -k_0\|u_{1xt} + pu_{2t}\|^2 - \lambda_1\|u_{1t}\|^2 - \lambda_2\|z(1)\|^2. \tag{38}$$

Proof. First, by multiplying the first and second equations of (17) by u_{1t} and u_{2t}, respectively, and then integrating by parts over $[0,l]$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[k\|u_{1x} + pu_2\|^2 + (\alpha - \beta\gamma^2)\|u_{1xx}\|^2 + b\|u_{2x}\|^2 + \beta\|(\gamma u_1 - u_2)_{xx}\|^2 \\ + \textstyle\sum_{i=1}^2 \rho_i\|u_{it}\|^2\Big] = -k_0\|u_{1xt} + pu_{2t}\|^2 - \mu_1\|u_{1t}\|^2 - \mu_2\int_0^l z(1)u_{1t}dx. \end{aligned} \tag{39}$$

Then, multiplying the third equation in (17) by $\xi\mu_1 z(\rho)$ and then integrating by parts over $[0,l] \times [0,1]$, we obtain

$$\begin{aligned} \frac{\mu_1\xi}{2}\frac{d}{dt}\left(\tau(t)\int_0^l\int_0^1 z^2(\rho)d\rho dx\right) &= \frac{\mu_1\xi}{2}\tau'(t)\int_0^l\int_0^1 z^2(\rho)d\rho dx \\ &\quad - \frac{\mu_1\xi}{2}\int_0^1 \frac{d}{d\rho}(1-\rho\tau'(t))\int_0^l\int_0^1 z^2(\rho)d\rho dx \\ &\quad - \frac{\mu_1\xi\tau'(t)}{2}\int_0^l\int_0^1 z^2(\rho)d\rho dx \\ &= -\frac{\mu_1\xi(1-\tau'(t))}{2}\int_0^l z^2(1)dx + \frac{\mu_1\xi}{2}\int_0^l u_{1t}^2 dx. \end{aligned} \tag{40}$$

Then, adding up (39) and (40), we have

$$\begin{aligned} \frac{d}{dt}E(t) = \; & -k_0\|u_{1xt} + pu_{2t}\|^2 - \left(\mu_1 - \frac{\mu_1\xi}{2}\right)\|u_{1t}\|^2 \\ & - \frac{\mu_1\xi(1-\tau'(t))}{2}\int_0^l\|z\|^2(1)dx - \mu_2\int_0^l z(1)u_{1t}dx. \end{aligned} \tag{41}$$

Using Young's inequality, the last term in the above equality can be estimated as follows:

$$-\mu_2\int_0^l z(1)u_{1t}dx \leq \frac{|\mu_2|}{2\varepsilon_2}\|u_{1t}\|^2 + \frac{\varepsilon_2|\mu_2|}{2}\|z(1)\|^2. \tag{42}$$

Plugging the above results into (41) and taking into account (23) and **(H2)**, we obtain (38), and $E(t) \leq E(0)$ for all $t \geq 0$.

This completes the proof of Lemma 1. $\square$

Now, we have Lemma 2.

Lemma 2. *Let $(u_1, u_{1t}, u_2, u_{2t}, z)$ be the solution of (17) and (18); then, the functional $\mathcal{G}$, defined by*

$$\mathcal{G}(t) = \frac{k_0}{2}\int_0^l (u_{1x} + pu_2)^2 dx + \frac{\mu_1}{2}\int_0^l u_1^2 dx + \sum_{i=1}^2\int_0^l \rho_i u_i u_{it} dx, \tag{43}$$

satisfies the following estimate:

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(t) \leq \; & -k\|u_{1x} + pu_2\|^2 - \beta\|\gamma u_{1xx} - u_{2xx}\|^2 + \textstyle\sum_{i=1}^2 \rho_i\|u_{it}\|^2 \\ & -b\|u_{2x}\|^2 - (\alpha_0 - \varepsilon_1 C_p)\|u_{1xx}\|^2 + \frac{\mu_2^2}{4\varepsilon_1}\|z(1)\|^2, \end{aligned} \tag{44}$$

where $\alpha_0 = \alpha - \beta\gamma^2 > 0$.

Proof. Multiplying the first equation in (17) by u_1 and then integrating over $[0, l]$ using integration by parts and zero boundary condition for u_1 and u_2, we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^l \rho_1 u_{1t}u_1 + \frac{\mu_1}{2}u_1^2 dx = \rho_1\|u_{1t}\|^2 - \alpha\|u_{1xx}\|^2 + \beta\gamma\int_0^l u_{2xx}u_{1xx}dx - \mu_2\int_0^l z(1)u_1 dx \\ - \int_0^l k_0(u_{1tx} + pu_{2t})u_{1x} + k(u_{1x} + pu_2)u_{1x}dx. \end{aligned} \tag{45}$$

Similarly, multiplying the second equation in (17) by u_2, we have

$$\begin{aligned} \frac{d}{dt}\int_0^l \rho_2 u_{2t}u_2 dx = {} & \rho_2\|u_{2t}\|^2 - \beta\|u_{2xx}\|^2 + \beta\gamma\int_0^l u_{1xx}u_{2x}dx \\ & -b\|u_{2x}\|^2 - p\int_0^l k_0(u_{1tx} + pu_{2t})u_2 + k(u_{1x} + pu_2)u_2 dx. \end{aligned} \tag{46}$$

And adding up (45) and (46), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(t) \; & = -k\|u_{1x} + pu_2\|^2 - \alpha\|u_{1xx}\|^2 - \beta\|u_{2xx}\|^2 - b\|u_{2x}\|^2 \\ & \; +2\beta\gamma\int_0^l u_{2xx}u_{1xx}dx - \mu_2\int_0^l z(1)u_1 dx + \sum_{i=1}^2 \rho_i\|u_{it}\|^2 \\ & = -k\|u_{1x} + pu_2\|^2 - (\alpha - \beta\gamma^2)\|u_{1xx}\|^2 - \beta\|\gamma u_{1xx} - u_{2xx}\|^2 \\ & \; -b\|u_{2x}\|^2 - \mu_2\int_0^l z(1)u_1 dx + \sum_{i=1}^2 \rho_i\|u_{it}\|^2. \end{aligned} \tag{47}$$

Finally, for the last integral, applying Young's and Poincaré's inequalities, we have

$$\begin{aligned} \mu_2\int_0^l z(1)u_1 dx & \leq \varepsilon_1\|u_1\|^2 + \frac{\mu_2^2}{4\varepsilon_1}\|z(1)\|^2 \\ & \leq \varepsilon_1 C_p\|u_{1xx}\|^2 + \frac{\mu_2^2}{4\varepsilon_1}\|z(1)\|^2. \end{aligned} \tag{48}$$

where Cp is the Poincaré's constant. Substituting (48) into (47), we obtain (44). □

Next, let us introduce the functional

$$\mathcal{I}(t) = \sum_{i=1}^{2}\gamma^{i-1}\rho_i\int_0^l u_{it}(\gamma u_1 - u_2)dx, \quad \forall t \geq 0. \tag{49}$$

Lemma 3. *Let* $(u_1, u_{1t}, u_2, u_{2t}, z)$ *be a solution of* (17) *and* (18)*; then, the functional* $\mathcal{H}$ *satisfies*

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) \leq \; & -(\gamma\rho_2 - \varepsilon_0')\|u_{2t}\|^2 + \frac{1}{4\varepsilon_1'}\|u_{1xx}\|^2 + \frac{k_0^2}{4}\left(\frac{1}{\varepsilon_7'} + \frac{(p\gamma)^2}{\varepsilon_8'}\right)\|u_{1xt} + pu_{2t}\|^2 \\ & + \left(\alpha_0^2\varepsilon_1' + (b\gamma)^2\varepsilon_4' + C_p C(\varepsilon)\right)\|\gamma u_{1xx} - u_{2xx}\|^2 \\ & + \left(\gamma\rho_1 + \frac{\gamma^2(\gamma\rho_2 - \rho_1)^2}{4\varepsilon_0'} + \frac{\mu_1^2}{4\varepsilon_2'}\right)\|u_{1t}\|^2 + \frac{1}{4}\left(\frac{1}{\varepsilon_5'} + \frac{1}{\varepsilon_6'}\right)\|u_{1x} + pu_2\|^2 \\ & + \frac{\mu_2^2}{4\varepsilon_3'}\|z(1)\|^2 + \frac{C_p}{4\varepsilon_4'}\|u_{2x}\|^2, \end{aligned} \tag{50}$$

where $C(\varepsilon) = \varepsilon_2' + \varepsilon_3' + k^2\left(\varepsilon_5' + (p\gamma)^2\varepsilon_6'\right) + \varepsilon_7' + \varepsilon_8'$.

Proof. Multiplying the first and second equation in (17) by $\gamma u_1 - u_2$ and $\gamma(\gamma u_1 - u_2)$, respectively, then adding the two results, integrating over $(0, l)$ with respect to x, and using integration by parts and the boundary conditions (18), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) = \; & -\int_0^l [k(u_{1x} + pu_2) + k_0(u_{1xt} + pu_{2t})][\gamma u_{1x} - u_{2x} + \gamma p(\gamma u_1 - u_2)]dx \\ & -\alpha_0\int_0^l (\gamma u_{1xx} - u_{2xx})u_{1xx}dx + \gamma\sum_{i=1}^2(-1)^{i-1}\rho_i\int_0^l u_{i\,t}^2 dx \\ & +\gamma(\gamma\rho_2 - \rho_1)\int_0^l u_{2t}u_{1t}dx - \mu_1\int_0^l u_{1t}(\gamma u_1 - u_2)dx \\ & -\mu_2\int_0^l z(1)(\gamma u_1 - u_2)dx + b\gamma\int_0^l u_2(\gamma u_{1xx} - u_{2xx})dx. \end{aligned} \tag{51}$$

By Young's and Poincaré's inequalities and (18), we arrive at (50). □

Now, we define the functional $\mathcal{K}$ by

$$\mathcal{K}(t) = \sum_{i=1}^{2}\int_0^l \gamma^{i-1}\rho_i u_{it}u_1 + \frac{\mu_1}{2}(2 - i)u_i^2 dx, \quad \forall t \geq 0. \tag{52}$$

Lemma 4. *Let $(u_1, u_{1t}, u_2, u_{2t}, z)$ be a solution of (17) and (18); then, the functional $\mathcal{K}$ satisfies*

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) \leq \quad & -\left(\alpha_0 - \varepsilon_4''(b\gamma)^2 - C'(\varepsilon)C_p\right)\|u_{1xx}\|^2 + \left(\rho_1 + \frac{(\gamma\rho_2)^2}{4\varepsilon_3''}\right)\|u_{1t}\|^2 \\ & +\varepsilon_3''\|u_{2t}\|^2 + \frac{k_0^2}{4}\left(\frac{(p\gamma)^2}{\varepsilon_6''} + \frac{1}{\varepsilon_1''}\right)\|u_{1xt} + pu_{2t}\|^2 \\ & +\frac{1}{4}\left(\frac{1}{\varepsilon_0''} + \frac{1}{\varepsilon_5''}\right)\|u_{1x} + pu_2\|^2 + \frac{C_p}{4\varepsilon_4''}\|u_{2x}\|^2 + \frac{\mu_2^2}{4\varepsilon_2''}\|z(1)\|^2, \end{aligned} \tag{53}$$

where $C'(\varepsilon) = \varepsilon_1'' + \varepsilon_2'' + k^2\left(\varepsilon_0'' + (p\gamma)^2\varepsilon_5''\right) + \varepsilon_6''$.

Proof. Multiplying the first and second equation in (17) by u_1 and γu_1, respectively, then adding the two results, integrating over $(0, l)$ with respect to x, and using integration by parts and the boundary conditions (18), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) = \quad & \rho_1\int_0^l u_{1t}^2 dx - \int_0^l [k(u_{1x} + pu_2) + k_0(u_{1xt} + pu_{2t})]u_{1x}dx \\ & -\alpha_0\int_0^l u_{1xx}^2 dx - \mu_2\int_0^l z(1)u_1 dx + \gamma\rho_2\int_0^l u_{2t}u_{1t}dx \\ & -p\gamma\int_0^l [k(u_{1x} + pu_2) + k_0(u_{1xt} + pu_{2t})]u_1 dx + b\gamma\int_0^l u_2 u_{1xx}dx, \end{aligned} \tag{54}$$

and, by using Young's and Poincaré's inequalities, we conclude the proof of this lemma. □

As in [35], in this last lemma, we introduce the functional

$$J(t) = \xi\tau(t)\int_0^l\int_0^1 e^{-2\rho\tau(t)}z^2(\rho)d\rho dx. \tag{55}$$

Lemma 5. *Let $(u_1, u_{1t}, u_2, u_{2t}, z)$ be a solution of (17) and (18); then, the functional $J(t)$ satisfies the following estimate:*

$$\frac{d}{dt}J(t) \leq -\xi(1 - \tau'(t))e^{-2\tau(t)}\|z(1)\|^2 + \xi\|u_{1t}\|^2 - 2\xi\tau(t)e^{-2\tau(t)}\int_0^1\|z(\rho)\|^2 d\rho. \tag{56}$$

Proof. Deriving the functional $J(t)$ and using the last two equations of (17), we obtain

$$\begin{aligned} \frac{d}{dt}J(t) \quad & = \xi\tau'(t)\int_0^1 e^{-2\rho\tau(t)}\|z(\rho)\|^2 d\rho - 2\xi\tau'(t)\tau(t)\int_0^1 \rho e^{-2\rho\tau(t)}\|z(\rho)\|^2 d\rho \\ & -\xi\int_0^1 (1 - \rho\tau'(t))e^{-2\rho\tau(t)}\frac{d}{d\rho}\|z(\rho)\|^2 d\rho \\ & = \xi\tau'(t)\int_0^1 e^{-2\rho\tau(t)}\|z(\rho)\|^2 d\rho - 2\xi\tau'(t)\tau(t)\int_0^1 \rho e^{-2\rho\tau(t)}\|z(\rho)\|^2 d\rho \\ & -\xi\left[(1 - \rho\tau'(t))e^{-2\rho\tau(t)}\|z(\rho)\|^2\right]_{\rho=0}^{\rho=1} \\ & +\xi\int_0^1\|z(\rho)\|^2\partial_\rho\left((1 - \rho\tau'(t))e^{-2\rho\tau(t)}\right)d\rho \\ & \leq -\xi\left[(1 - \rho\tau'(t))e^{-2\rho\tau(t)}\|z(\rho)\|^2\right]_{\rho=0}^{\rho=1} - 2\xi\tau(t)e^{-2\tau(t)}\int_0^1\|z(\rho)\|^2 d\rho. \end{aligned} \tag{57}$$

The proof is, therefore, finished. □

Now, we are in a position to prove our main result.

Proof of Theorem 1. Let us define the Lyapunov functional:

$$L(t) = NE(t) + \mathcal{G}(t) + N_1\mathcal{I}(t) + N_2\mathcal{K}(t) + J(t),$$

where N and N_i, $i = \overline{1,2}$, are positive constants that will be chosen later.

First, we check that the function L satisfies the following relationship:

$$\omega_1 E(t) \leq L(t) \leq \omega_2 E(t), \quad \forall t \geq 0, \tag{58}$$

where all values of ω_i, $i = \overline{1,2}$, are positive constants.

From (19), (43), (49), (52), and (55), we have

$$
\begin{aligned}
|L(t)-NE(t)| \leq & \left[\frac{k_0}{2}\int_0^l (u_{1x}+pu_2)^2dx+\frac{\mu_1}{2}\int_0^l u_1^2dx+\sum_{i=1}^2\int_0^l \rho_i|u_i||u_{it}|dx\right] \\
&+N_1\left[\rho_1\int_0^l|u_{1t}||(\gamma u_1-u_2)|dx+\gamma\rho_2\int_0^l|u_{2t}||(\gamma u_1-u_2)|dx\right] \\
&+N_2\left[\rho_1\int_0^l|u_{1t}||u_1|dx+\gamma\rho_2\int_0^l|u_{2t}||u_1|dx+\frac{\mu_1}{2}\int_0^l u_1^2dx\right] \\
&+\xi\tau(t)\int_0^l\int_0^1 e^{-2\rho\tau(t)}z^2(\rho)d\rho dx.
\end{aligned} \tag{59}
$$

Applying Young's and Poincaré's inequalities and from the facts that $\tau(t)\leq\tau_1,\ \forall t\geq 0$, and $e^{-2\tau(t)\sigma}\leq 1,\ \ \forall(t,\sigma)\in]0,+\infty[\times[0,1]$, we have

$$
\begin{aligned}
|L(t)-NE(t)| \leq & c_3'\|\gamma u_{1xx}-u_{2xx}\|^2+c_4'\sum_{i=1}^2\|u_{ixx}\|^2+\sum_{i=1}^2 c_i\|u_{it}\|^2 \\
&+c_5\int_0^1\|z(\rho)\|^2d\rho+c_6\|u_{1x}+pu_2\|^2.
\end{aligned} \tag{60}
$$

Note that

$$
\int_0^l u_{2xx}^2dx\leq 2\int_0^l(\gamma u_{1xx}-u_{2xx})^2dx+2\gamma^2\int_0^l u_{1xx}^2dx. \tag{61}
$$

Inserting (61) into (60), we have

$$
\begin{aligned}
|L(t)-NE(t)| \leq & c_3\|\gamma u_{1xx}-u_{2xx}\|^2+c_4\|u_{1xx}\|^2+\sum_{i=1}^2 c_i\|u_{it}\|^2 \\
&+c_5\int_0^1\|z(\rho)\|^2d\rho+c_6\|u_{1x}+pu_2\|^2 \\
\leq & CE(t),
\end{aligned} \tag{62}
$$

where all values of $c_i,\ \ i=\overline{1,6}$, are positive real numbers and

$$
C=2\max\left\{\frac{c_1}{\rho_1},\frac{c_2}{\rho_2},\frac{c_3}{\beta},\frac{c_4}{\alpha_0},\frac{c_5}{\mu_1\xi\tau(t)},\frac{c_6}{k}\right\}.
$$

Thus, we can choose N large enough, such that $\omega_1=N-C>0$ and $\omega_2=N+C$. This shows that Relation (58) is true.

By combining (38), (44), (50), (53), and (56), we obtain the following estimates:

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq & -\left[(\gamma\rho_2-\varepsilon_0')N_1-\rho_2-\varepsilon_3''N_2\right]\|u_{2t}\|^2 \\
&-\left[\left(\alpha_0-\varepsilon_1C_p\right)+\left(\alpha_0-\varepsilon_4''(b\gamma)^2-C'(\varepsilon)C_p\right)N_2-\frac{1}{4\varepsilon_1'}N_1\right]\|u_{1xx}\|^2 \\
&-\left[\beta-\left(\alpha_0^2\varepsilon_1'+(b\gamma)^2\varepsilon_4'+C(\varepsilon)C_p\right)N_1\right]\|\gamma u_{1xx}-u_{2xx}\|^2 \\
&-\left[k-\frac{1}{4}\left(\frac{1}{\varepsilon_5'}+\frac{1}{\varepsilon_6'}\right)N_1-\frac{1}{4}\left(\frac{1}{\varepsilon_0''}+\frac{1}{\varepsilon_5''}\right)N_2\right]\|u_{1x}+pu_2\|^2 \\
&-\left[k_0N-\frac{k_0^2}{4}\left(\frac{1}{\varepsilon_7'}+\frac{(p\gamma)^2}{\varepsilon_8'}\right)N_1-\frac{k_0^2}{4}\left(\frac{1}{\varepsilon_1''}+\frac{(p\gamma)^2}{\varepsilon_6''}\right)N_2\right]\|u_{1xt}+pu_{2t}\|^2 \\
&-\left[\lambda_1N-\rho_1-\left(\gamma\rho_1+\frac{\mu_1^2}{4\varepsilon_2'}+\frac{\gamma^2(\gamma\rho_2-\rho_1)^2}{4\varepsilon_0'}\right)N_1-\xi\right. \\
&\left.-\left(\rho_1+\frac{(\gamma\rho_2)^2}{4\varepsilon_3''}\right)N_2\right]\|u_{1t}\|^2 \\
&-\left[b-\frac{C_p}{4\varepsilon_4'}N_1-\frac{C_p}{4\varepsilon_4''}N_2\right]\|u_{2x}\|^2 \\
&-\left[\lambda_2N+\xi(1-\tau'(t))e^{-2\tau(t)}-\frac{\mu_2^1}{4\varepsilon_1}-\frac{\mu_2^2}{4\varepsilon_3'}N_1-\frac{\mu_2^2}{4\varepsilon_2''}N_2\right]\|z(1)\|^2 \\
&-2\xi\tau(t)e^{-2\tau(t)}\int_0^1\|z(\rho)\|^2d\rho.
\end{aligned}
$$

First, we take

$$
\varepsilon_3''=\frac{\rho_2}{N_2},\ N_1=\frac{5}{2\gamma},
$$

then, if we pick

$$
\varepsilon_1<\frac{\alpha_0}{C_p},
$$

we obtain

$$\begin{aligned}
\frac{d}{dt}L(t) \quad & \leq -\left[\tfrac{1}{2}\rho_2 - \tfrac{5}{2\gamma}\varepsilon_0'\right]\|u_{2t}\|^2 \\
& -\left[\alpha_0(1+N_2) - \varepsilon_1 C_p - \tfrac{5}{8\gamma\varepsilon_1'} - \varepsilon_4''(b\gamma)^2 N_2 - \left(k^2\varepsilon_0'' + (pk\gamma)^2\varepsilon_5'' + \delta_0\right)C_p N_2\right]\|u_{1xx}\|^2 \\
& -\left[\beta - \tfrac{5b^2\gamma}{2}\varepsilon_4' - \tfrac{5\alpha_0^2}{2\gamma}\varepsilon_1' - \left(k^2\varepsilon_5' + (pk\gamma)^2\varepsilon_6' + \delta_1\right)\tfrac{5C_p}{2\gamma}\right]\|\gamma u_{1xx} - u_{2xx}\|^2 \\
& -\left[k - \tfrac{5}{8\gamma}\left(\tfrac{1}{\varepsilon_5'} + \tfrac{1}{\varepsilon_6'}\right) - \left(\tfrac{1}{4\varepsilon_0''} + \tfrac{1}{4\varepsilon_5''}\right)N_2\right]\|u_{1x} + pu_2\|^2 \\
& -\left[b - \tfrac{5C_p}{8\gamma\varepsilon_4'} - \tfrac{C_p}{4\varepsilon_4''}N_2\right]\|u_{2x}\|^2 \\
& -\left[k_0 N - \tfrac{5k_0^2}{8\gamma}\left(\tfrac{1}{\varepsilon_7'} + \tfrac{(p\gamma)^2}{\varepsilon_8'}\right) - \tfrac{k_0^2}{4}\left(\tfrac{1}{\varepsilon_1''} + \tfrac{(p\gamma)^2}{\varepsilon_6''}\right)N_2\right]\|u_{1xt} + pu_{2t}\|^2 \\
& -\left[\lambda_1 N - \rho_1 - \tfrac{5}{2\gamma}\left(\gamma\rho_1 + \tfrac{\mu_1^2}{4\varepsilon_2'} + \tfrac{\gamma^2(\rho_1-\gamma\rho_2)^2}{4\varepsilon_0'}\right) - \xi - \left(\rho_1 + \tfrac{(\gamma\rho_2)^2}{4\varepsilon_3''}\right)N_2\right]\|u_{1t}\|^2 \\
& -\left[\lambda_2 N - \tfrac{\mu_1^2}{4\varepsilon_1} - \tfrac{5\mu_2^2}{8\gamma\varepsilon_3'} - \tfrac{\mu_2^2}{4\varepsilon_2''}N_2 + \xi(1-\tau'(t))e^{-2\tau(t)}\right]\|z(1)\|^2 \\
& -2\xi\tau(t)e^{-2\tau(t)}\int_0^1 \|z(\rho)\|^2 d\rho,
\end{aligned}$$

where $\delta_0 = \max\{\varepsilon_1'', \varepsilon_2'', \varepsilon_6''\}$ and $\delta_1' = \max\{\varepsilon_2', \varepsilon_3', \varepsilon_7', \varepsilon_8'\}$.

Next, by setting

$$\begin{aligned}
&\varepsilon_5' = \varepsilon_6' = \tfrac{5(1+\lambda)}{2\gamma\lambda k^3 d}, \ \ \varepsilon_0'' = \varepsilon_5'' = \tfrac{(1+\lambda)N_2}{\lambda k^3 d}, \ \textit{where } d = \left(1+(p\gamma)^2\right)C_p \textit{ and } \lambda > 0, \\
&\varepsilon_4' = \tfrac{5C_p(1+\lambda)}{8\gamma\lambda b^2}, \ \ \varepsilon_4'' = \tfrac{(1+\lambda)C_p N_2}{\lambda(b\gamma)^2}, \quad \textit{and } \varepsilon_1' = \tfrac{5(1+\lambda)}{8\gamma\lambda\alpha_0(1+N_2)},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{d}{dt}L(t) \quad & \leq -\left[\tfrac{1}{2}\rho_2 - \tfrac{5}{2\gamma}\varepsilon_0'\right]\|u_{2t}\|^2 \\
& -\left[-\tfrac{1+\lambda}{\lambda}C_p N_2^2 + \left(\tfrac{\alpha_0}{1+\lambda} - \tfrac{1+\lambda}{\lambda}\right)N_2 + \tfrac{\alpha_0}{1+\lambda} - 3\delta_0 C_p N_2\right]\|u_{1xx}\|^2 \\
& -\left[\beta - \tfrac{25}{4}\tfrac{1+\lambda}{\lambda}(\kappa_1+\kappa_2) - \tfrac{20C_p}{2\gamma}\delta_1\right]\|\gamma u_{1xx} - u_{2xx}\|^2 \\
& -\left(k - \tfrac{d\lambda k^3}{1+\lambda}\right)\|u_{1x} + pu_2\|^2 \\
& -\left[\tfrac{b}{4(1+\lambda)}\left(4(1+\lambda) - \lambda\left(4+\gamma^2\right)b\right)\right]\|u_{2x}\|^2 \\
& -\left[k_0 N - \tfrac{5k_0^2}{8\gamma}\left(\tfrac{1}{\varepsilon_7'} + \tfrac{(p\gamma)^2}{\varepsilon_8'}\right) - \tfrac{k_0^2}{4}\left(\tfrac{1}{\varepsilon_1''} + \tfrac{(p\gamma)^2}{\varepsilon_6''}\right)N_2\right]\|u_{1xt} + pu_{2t}\|^2 \\
& -\left[\lambda_1 N - \rho_1 - \tfrac{5}{2\gamma}\left(\gamma\rho_1 + \tfrac{\mu_1^2}{4\varepsilon_2'} + \tfrac{\gamma^2(\rho_1-\gamma\rho_2)^2}{4\varepsilon_0'}\right) - \xi - \left(\rho_1 + \tfrac{(\gamma\rho_2)^2}{4\varepsilon_3''}\right)N_2\right]\|u_{1t}\|^2 \\
& -\left[\lambda_2 N - \tfrac{\mu_1^2}{4\varepsilon_1} - \tfrac{5\mu_2^2}{8\gamma\varepsilon_3'} - \tfrac{\mu_2^2}{4\varepsilon_2''}N_2 + \xi(1-\tau'(t))e^{-2\tau(t)}\right]\|z(1)\|^2 \\
& -2\xi\tau(t)e^{-2\tau(t)}\int_0^1 \|z(\rho)\|^2 d\rho,
\end{aligned} \tag{63}$$

where $\lambda_0 = \frac{(1+\lambda)}{\lambda}$, $\alpha_1 = \frac{\alpha_0}{1+\lambda} - \lambda_0$, $\kappa_1 = \frac{C_p}{4} + \frac{1}{\gamma^2 k}$, and $\kappa_2 = \frac{\alpha_0}{4\gamma^2(1+N_2)}$.

Obviously, for $0 < N_2 < \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\lambda_0 C_p(\alpha_1+\lambda_0)}}{2\lambda_0 C_p}$, $\alpha_1 + \lambda_0 + \alpha_1 N_2 - \lambda_0 C_p N_2^2 > 0$.

Then, we take small enough values of κ_1 and κ_1, such that

$$\beta - \frac{25}{4}\frac{1+\lambda}{\lambda}(\kappa_1+\kappa_2) > 0.$$

After that, we pick ε_0' small enough that

$$\eta_1 = \frac{1}{2}\rho_2 - \frac{5}{2\gamma}\varepsilon_0' > 0,$$

and pick ε_1, δ_0, and δ_1' small enough that

$$\eta_2 = -\lambda_0 C_p N_2^2 + \alpha_1 N_2 + \alpha_1 + \lambda_0 - 3\delta_0 C_p N_2 > 0,$$

$$\eta_3 = \beta - \frac{25}{4}\lambda_0(\kappa_1 + \kappa_2) - \frac{20C_p}{2\gamma}\delta_1 > 0.$$

Finally, we choose N large enough so that

$$k_0 N - \frac{5k_0^2}{8\gamma}\left(\frac{1}{\varepsilon_7'} + \frac{(p\gamma)^2}{\varepsilon_8'}\right) - \frac{k_0^2}{4}\left(\frac{1}{\varepsilon_1''} + \frac{(p\gamma)^2}{\varepsilon_6''}\right) N_2 > 0,$$

$$\eta_4 = \lambda_1 N - \rho_1 - \frac{5}{2\gamma}\left(\gamma\rho_1 + \frac{\mu_1^2}{4\varepsilon_2'} + \frac{\gamma^2(\rho_1 - \gamma\rho_2)^2}{4\varepsilon_0'}\right) - \xi - \left(\rho_1 + \frac{(\gamma\rho_2)^2}{4\varepsilon_3''}\right) N_2 > 0,$$

$$\eta_5 = \lambda_2 N - \frac{\mu_1^2}{4\varepsilon_1} - \frac{5\mu_2^2}{8\gamma\varepsilon_3'} - \frac{\mu_2^2}{4\varepsilon_2''} N_2 + \xi(1 - \tau'(t)) e^{-2\tau(t)} > 0.$$

Therefore, from (19), we can conclude that there exists a positive constant $K_0 > 0$, such that (63) becomes

$$\frac{d}{dt} L(t) \le -K_0 E(t), \quad \forall t > 0, \tag{64}$$

By (64) and $\mathcal{L} \sim E$, we deduce that

$$\frac{d}{dt} L(t) \le -k_1 L(t), \quad \forall t \ge 0.$$

By integrating this differential inequality, we obtain

$$L(t) \le L(0) e^{-k_1 t}, \quad \forall t > 0.$$

Consequently, using (58), we find (20) with $\lambda = \frac{L(0)}{N-C}$ and $\omega = \frac{k_0}{N+C}$. This completes the proof of Theorem 1. □

6. Conclusions

It is well known that most researchers discussed the study of the Tymoshenko system with a delay in one of its equations or with two fixed delays. That is why we decided to propose this type of one-dimensional system for Tymoshenko under Dirichlet–Dirichlet conditions, which differs from others in that it contains internal frictional damping, a time-dependent delay acting on the vertical displacement in symmetrical point of view.

In this work, we showed the existence of a unique solution by using the semigroup theory. By introducing an appropriate Lyapunov functional, the exponential stability of the system is obtained if the weights of the time delays are small.

We can conclude that the application of this type of problem is very rich. It is found in all areas of modern physics and in many branches of applied science. Our novelty is located in the following points:

1. We considered a new non-classical model on the Timoshenko-type system with a time-varying internal delay in the displacement;
2. The existence, uniqueness, and smoothness of the solution are shown based on the classical Lumiere–Phillips theory;
3. We have clearly outlined and minimized the impact of the weight of the time-varying delay compared to the weight of the frictional term;
4. Our results can be seen as an extension of many recent related works.

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References

1. Timoshenko, S. On the correction for shear of the differential equation for transverse vibrations of prismaticbars. *Phil. Mag.* **1921**, *41*, 744–746. [CrossRef]
2. Kim, J.U.; Renardy, Y. Boundary control of the Timoshenko beam. *SIAM J. Control Optim.* **1987**, *25*, 1417–1429. [CrossRef]
3. Messaoudi, S.A.; Mustafa, M.I. On the internal and boundary stabilization of Timoshenko beams. *Nonlinear Differ. Equations Appl.* **2008**, *15*, 655–671. [CrossRef]
4. Messaoudi, S.A.; Said-Houari, B. Uniform decay in a Timoshenko system with past history. *J. Math. Anal. Appl.* **2009**, *360*, 458–475. [CrossRef]
5. Messaoudi, S.A.; Soufyame, A. Boundary stabilization of a nonlinear system of Timoshenko type. *Nonlinear Anal.* **2007**, *67*, 2107–2112. [CrossRef]
6. Tatar, N.E. Stabilization of a viscoelastic Timoshenko beam. *Appl. Anal.* **2013**, *92*, 27–43. [CrossRef]
7. Soufyane, A.; Wehbe, A. Uniform stabilization for the Timoshenko beam by a locally distributed damping. *Elecron. J. Differ. Equ.* **2003**, *29*, 1–14.
8. Shi, D.-H.; Feng, D.-X. Exponential decay of Timoshenko beam with locally distributedfeedback. *IMA J. Math. Cont. Inf.* **2001**, *18*, 395–403. [CrossRef]
9. Nicaise, S.; Pignotti, C. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.* **2006**, *5*, 1561–1585. [CrossRef]
10. Racke, R. Instability of coupled systems with delay. *Commun. Pure Appl. Anal.* **2012**, *11*, 1753–1773. [CrossRef]
11. Datko, R. Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. *SIAM J. Control Optim.* **1988**, *26*, 697–713. [CrossRef]
12. Datko, R.; Lagnese, J.; Polis, M.P. An example on the effect of time delays in boundary feedback stabilization of wave equations. *SIAM J. Control Optim.* **1986**, *24*, 152–156. [CrossRef]
13. Said-Houari, B.; Laskri, Y. A stability result of a Timoshenko system with a delay term in the internal feedback. *Appl. Math. Comput.* **2010**, *217*, 2857–2869. [CrossRef]
14. Said-Houari, B.; Rahali, R. A stability result for a Timoshenko system with past history and a delay term in the internal feedback. *Dyn. Syst. Appl.* **2011**, *20*, 327–354.
15. Kafini, M.; Messaoudi, S.A.; Mustafa, M.I. Energy decay rates for a Timoshenko-type system of thermoelasticityof type III with constant delay. *Appl. Anal.* **2013**, *93*, 1201–1216. [CrossRef]
16. Kafini, M.; Messaoudi, S.A.; Mustafa, M.I.; Apalara, T. Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay. *Z. Angew. Math. Phys.* **2015**, *66*, 1499–1517. [CrossRef]
17. Junior, D.S.A.; Elishakoff, I.; Ramos, A.J.A.; Miranda, L.G.R. The hypothesis of equal wave speeds for stabilization of Timoshenko beam is not necessary anymore: The time delay cases. *IMA J. Appl. Math.* **2019**, *84*, 1–34.
18. Feng, B.; Junior, D.S.A.; dos Santos, M.J.; Rosàrio Miranda, L.G. A new scenario for stability of nonlinear Bresse-Timoshenko type systems with time dependent delay. *Z. Angew. Math. Mech.* **2019**, *100*, e201900160. [CrossRef]
19. Yang, X.-G.; Zhang, J.; Lu, Y. Dynamics of the Nonlinear Timoshenko System with Variable Delay. *Appl. Math. Optim.* **2018**, *83*, 297–326. [CrossRef]
20. Chueshov, I.D.; Lasiecka, I. *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*; Memoirs of the American Mathematical Society; American Mathematical Society: Providence, RI, USA, 2008; Volume 195.
21. Chueshov, I.D.; Lasiecka, I. *Von Karman Evolution Equations. Well-Posedness and Long Time Dynamics*; Springer Monographs in Mathematics; Springer: New York, NY, USA, 2010.
22. Feng, B.; Raposo, C.; Nonato, C.; Soufyane, A. Analysis of exponential stabilization for Rao-Nakra sandwich beam with time-varying weight and time-varying delay: Multiplier method versus observability. *Math. Control Relat. Fields* **2023**, *13*, 631–663. [CrossRef]
23. Khalili, Z.; Ouchenane, D. Exponential stability for a Timoshenko thermoelastic system with second sound and a time-varying delay term in the internal feedback. *Asymptot. Anal.* **2023**, *132*, 131–152. [CrossRef]
24. Mukiawa, S.; Enyi, C.; Messaoudi, S. Stability of thermoelastic Timoshenko beam with suspenders and time-varying feedback. *Adv. Contin. Discret. Model.* **2023**, *2023*, 7. [CrossRef]
25. Nonato, C.; Raposo, C.; Feng, B. Exponential stability for a thermoelastic laminated beam with nonlinear weights and time-varying delay. *Asymptot. Anal.* **2022**, *126*, 157–185.
26. Raposo, C.; Ayala, Y.; Nonato, C. Laminated beams with time-varying delay. *Osaka J. Math.* **2021**, *58*, 929–945.
27. Said-Houari, B.; Soufyane, A. Stability result of the Timoshenko system with delay and boundary feedback. *IMA J. Math. Control Inf.* **2012**, *29*, 383–389. [CrossRef]
28. Guesmia, A.; Soufyane, A. On the stability of Timoshenko-type systems with internal frictional dampings and discrete time delays. *Appl. Anal.* **2016**, *96*, 2075–2101. [CrossRef]
29. Guesmia, A. Well-posedness and energy decay for Timoshenko systems with discrete time delay under frictional damping and/or infinite memory in the displacement. *Afr. Mat.* **2017**, *28*, 1253–1284. [CrossRef]
30. Kato, T. Linear and quasi-linear equations of evolution of hyperbolic type. In *Hyperbolicity*; Prato, G.D., Geymonat, G., Eds.; Springer: Berlin/Heidelberg, Germany, 2011; Volume 72.
31. Pazy, A. *Semi-Groups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.

32. Nicaise, S.; Valein, J.; Fridman, E. Stability of the heat and of the wave equations with boundary time-varying delays. *Disc. Cont. Dyn. Syst.* **2009**, *2*, 559–581.
33. Liu, Z.; Zheng, S. *Semigroups Associated with Dissipative Systems*; CRC Press: Boca Raton, FL, USA, 1999; Volume 398.
34. Nicaise, S.; Pignotti, C. Exponential stability of the wave equation with boundary time-varying delay. *Disc. Cont. Dyn. Syst.* **2011**, *3*, 693–722. [CrossRef]
35. Kirane, M.; Said-Houari, B.; Anwar, M.N. Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks. *Commun. Pure Appl. Anal.* **2011**, *10*, 667–686. [CrossRef]