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Uniqueness Results and Asymptotic Behaviour of Nonlinear Schrödinger–Kirchhoff Equations

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Abstract: In this paper, we first study the uniqueness and symmetry of solution of nonlinear Schrödinger–Kirchhoff equations with constant coefficients. Then, we show the uniqueness of the solution of nonlinear Schrödinger–Kirchhoff equations with the polynomial potential. In the end, we investigate the asymptotic behaviour of the positive least energy solutions to nonlinear Schrödinger–Kirchhoff equations with vanishing potentials. The vanishing potential means that the zero set of the potential is non-empty. The uniqueness results of Schrödinger equations and the scaling technique are used in our proof. The elliptic estimates and energy analysis are applied in the proof of the asymptotic behaviour of the above Schrödinger–Kirchhoff-type equations.

Keywords: Schrödinger–Kirchhoff equations; uniqueness; symmetry; asymptotic behaviour

MSC: 35J60; 35J20; 35B38

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1. Introduction

In this paper, we first show the uniqueness and symmetry of solution of the following nonlinear Schrödinger–Kirchhoff equation:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + cv = dv^p \quad \text{in } \mathbb{R}^3, \\ v>0, \quad v\in H^1(\mathbb{R}^3), \end{cases} \tag{1}$$

where $1<p<5$. The coefficients a, b, c and d in the equation are positive constants. Then, we prove a uniqueness result of the following nonlinear Schrödinger–Kirchhoff equation with potential $|x|^m$:

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + |x|^m v = v^p \quad \text{in } \mathbb{R}^3, \\ v>0, \quad v\in H^1(\mathbb{R}^3), \end{cases} \tag{2}$$

where $m>0$ and $3<p<5$.

In the second part of this paper we deal with the asymptotic behaviour of least energy solutions of the following Schrödinger–Kirchhoff equations:

$$\begin{cases} -\left(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla v|^2\right)\Delta v + V(x)v = v^p \quad in\ \mathbb{R}^3, \\ v>0, \quad v\in H^1(\mathbb{R}^3), \end{cases} \tag{3}$$

where $3<p<5$ and $\varepsilon>0$ is small. Furthermore, the equation has a vanishing potential $V(x)$ in the following sense:

(V1) $V(x):\mathbb{R}^3\to\mathbb{R}$ is nonnegative and continuous, $V_\infty := \liminf_{|x|\to\infty} V(x) > 0$.

(V2) The potential V can vanish, i.e., the set $\mathcal{Z} := \{x\in\mathbb{R}^3 | V(x)=0\}$ is non-empty. Moreover, $0\in\mathcal{Z}$.

The above equations are related to the stationary analogues of the following equation proposed by Kirchhoff [1]:

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2\right)\Delta u = g(x,t). \tag{4}$$

Equation (4), with a nonlocal term $\int_{\Omega} |\nabla u|^2$ in it, extends the classical D'Alembert's wave equation. Concerning (4), early studies were Bernstein [2], Pohozaev [3] and Lions [4]. These years, an enormous amount of research on the elliptic Kirchhoff equations has been done. Perera and Zhang [5], using the Yang index, proved the existence of nontrivial solutions of Kirchhoff equations. In [6], using the method of invariant sets of descent flow, sign changing solutions were obtained by Zhang and Perera. The uniqueness result was proven in [7] by Anello. Since we can not give a comprehensive list of references here, we merely refer to [8–15].

Recently, many authors studied the following Schrödinger–Kirchhoff equations with a small parameter $\varepsilon > 0$:

$$\begin{cases} -(\varepsilon^2 a + \varepsilon b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^3, \\ u > 0, \quad u \in H^1(\mathbb{R}^3). \end{cases} \tag{5}$$

In [16], He and Zou proved the existence and concentration of least energy solutions of (5) with subcritical nonlinearity. In [17], Wang et al. treated (5) with critical nonlinearity. In [18], Figueiredo et al. considered the Schrödinger–Kirchhoff Equation (5) with the almost optimal Berestycki–Lions nonlinearity. In [19], Sun and Zhang obtained the existence and concentration results of least energy solutions with competing potentials. More results are in [20–24], etc. From these papers, we can see that either problem (1) or (2) is often related to the limiting equations of (5). Thus, the uniqueness results of (1) or (2) are important when one studies the asymptotic behaviour of (5) as $\varepsilon \to 0^+$.

In [19], we studied the uniqueness of the ground state solution of (1) for the case $3 < p < 5$. In [25], the authors proved the uniqueness of solutions of (1) when $c = d = 1$ and $1 < p < 5$. In this paper, we take a different approach from [25] to obtain the uniqueness results for (1), which also allows us to obtain the symmetry result for the solution. Furthermore, the uniqueness for Schrödinger–Kirchhoff Equation (2) with potential $|x|^m$ is also considered in this paper.

We first prove the following uniqueness result:

Theorem 1. *The solution of Equation (1) is unique (up to translation) and radially symmetric.*

Theorem 2. *Let a, b, $m > 0$, $3 < p < 5$, then Equation (2) admits a unique solution.*

In [26], Sun and Zhang treated the nonlinear Schrödinger–Kirchhoff equations with a critical frequency. They obtained the existence results of least energy solutions for (3). But the paper [26] only concerns the asymptotic behaviour of least energy solutions for the finite case. In this paper, we deal with the asymptotic behaviour for problem (3) for the flat case and the infinite case:

(V3) The flat case:

$$\mathrm{int}(\mathcal{Z}) \textit{ is not empty}, \mathcal{Z} = \overline{\mathrm{int}(\mathcal{Z})}, \mathrm{int}(\mathcal{Z}) \textit{ is a connected domain},$$

where $\mathrm{int}(\mathcal{Z})$ is the set of interior points of $\mathcal{Z}$; $\mathcal{Z}$ is defined in (V2).

(V4) The infinite case: we assume that for $|x| \le 1$,

$$V(x) = \exp(-\frac{1}{|x|}).$$

Consider the following problem:

$$\begin{cases} -a\Delta u = u^p & in\ \mathrm{int}(\mathcal{Z}), \\ u > 0 & in\ \mathrm{int}(\mathcal{Z}), \\ u = 0 & on\ \partial\mathrm{int}(\mathcal{Z}). \end{cases} \tag{6}$$

Then problem (6) has a least energy solution U with the least energy I_U:

$$I_U = \frac{1}{2}\int_{\mathrm{int}(\mathcal{Z})} a|\nabla U|^2 - \frac{1}{p+1}\int_{\mathrm{int}(\mathcal{Z})} U^{p+1}.$$

In [26], we have proved the existence of least energy solutions v_ε of nonlinear Schrödinger–Kirchhoff Equation (3). Now, concerning the asymptotic behaviour of the least energy solutions v_ε for the flat case, we have the following result:

Theorem 3. *Assume that (V1), (V2) and (V3) hold. Let Γ_ε denote the energy functional associated to (3), then*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2(p+1)/(p-1)}\Gamma_\varepsilon(v_\varepsilon) = I_U. \tag{7}$$

Furthermore, up to a subsequence, the function $\varepsilon^{-2/(p-1)}v_\varepsilon$ converges pointwise to some least energy solution U of (6) on $int(\mathcal{Z})$ and to 0 on $\mathbb{R}^3 \setminus int(\mathcal{Z})$ as $\varepsilon \to 0$. For each $\delta > 0$, $\varepsilon^{-2/(p-1)}v_\varepsilon$ converges uniformly on $\{x \in \mathbb{R}^3 | dist(x, \partial int(\mathcal{Z})) \geq \delta\}$.

In the end, we deal with the asymptotic behaviour for problem (3) for the infinite case. Consider the following problem:

$$\begin{cases} -a\Delta u = u^p & in\ B_1, \\ u > 0 & in\ B_1, \\ u = 0 & on\ \partial B_1, \end{cases} \tag{8}$$

where $B_1 := \{x \in \mathbb{R}^3 | |x| < 1\}$. We have the following result:

Theorem 4. *Assume that (V1), (V2) and (V4) hold. Let v_ε be the least energy solutions of nonlinear Schrödinger–Kirchhoff Equation (3) proven in [26], and let Γ_ε denote the energy functional associated to (3), then*

$$\lim_{\varepsilon\to 0}(\varepsilon g(\varepsilon))^{-2(p+1)/(p-1)} g(\varepsilon)^{-3}\Gamma_\varepsilon(v_\varepsilon) = I(B_1), \tag{9}$$

where $g(\varepsilon) := -log\varepsilon^2$ and $I(B_1)$ is the ground energy of (8). Moreover, for each $\delta > 0$ and up to a subsequence, the function $(\varepsilon g(\varepsilon))^{-2/(p-1)}v_\varepsilon(\frac{x}{g(\varepsilon)})$ converges uniformly to $\bar{W}$ on $\{x \in \mathbb{R}^3 | dist(x, \partial B_1) \geq \delta\}$ as $\varepsilon \to 0$, where W is a least energy solution of (8) and

$$\bar{W}(x) = \begin{cases} W(x) & for\ x \in B_1, \\ 0 & for\ x \notin B_1. \end{cases}$$

We organize this paper as follows. In Sections 2 and 3, the uniqueness results in Theorems 1 and 2 are proved. In Section 4, we study the asymptotic behaviour of least energy solutions of nonlinear Schrödinger–Kirchhoff Equations (3) for the flat case. In Section 5, we study the asymptotic behaviour of least energy solutions of nonlinear Schrödinger–Kirchhoff Equations (3) for the infinite case.

2. Uniqueness Result for Equations with Constant Coefficients

In this section, we will use a scaling technique to obtain the uniqueness result in Theorem 1.

Proof of Theorem 1. Assume that v is a solution of (1), let $v(x) = \lambda u(\mu x)$, with $\mu^2 = c,\ \lambda = (c/d)^{\frac{1}{p-1}}$, then $u(x)$ satisfies

$$-(a + \frac{b\lambda^2}{\mu}\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u + u = u^p \ \text{ in } \mathbb{R}^3. \tag{10}$$

Therefore, to prove the uniqueness of the solution of Equation (1), it is equivalent to prove the uniqueness for (10), and without a loss of generality, it suffices to consider the case $c = d = 1$ in (1):

$$\begin{cases} -(a + b\int_{\mathbb{R}^3}|\nabla v|^2)\Delta v + v = v^p & \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3), \end{cases} \tag{11}$$

where $1 < p < 5$, a, b are positive.

First, we can know that (11) has a positive solution v_1 from [27]. Now, by elliptic estimates (see Theorem 4.1 in [28], for example), $v_1(x) \to 0$ as $|x| \to \infty$. Then, by translation, we know that v_1 satisfies

$$v_1 > 0,\ v_1(\infty) = 0,\ v_1(0) = \max\ v_1(x). \tag{12}$$

Next, we prove that v_1 is unique. Otherwise, if v_2 is another solution which satisfies (11); let

$$K_1 := a + b\int_{\mathbb{R}^3}|\nabla v_1|^2,\ K_2 := a + b\int_{\mathbb{R}^3}|\nabla v_2|^2.$$

Then, $v_i (i = 1, 2)$ satisfies the following problem:

$$-\Delta v + \frac{1}{K_i}v = \frac{1}{K_i}v^p \ \text{ in } \mathbb{R}^3.$$

Let $\bar{u}_i(x) := u_i(\sqrt{K_i}x)$, then $\bar{u}_i(x)$ is a solution of:

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^3, \\ u > 0,\ u(\infty) = 0,\ u(0) = \max\ u(x). \end{cases} \tag{13}$$

From [29], the problem (13) has a unique solution. Thus, $\bar{u}_1(x) \equiv \bar{u}_2(x)$, i.e., $v_1(\sqrt{K_1}x) = v_2(\sqrt{K_2}x)$. Therefore,

$$v_2(x) = v_1(\sqrt{\frac{K_1}{K_2}}x). \tag{14}$$

Then

$$K_2 \ = \ a + b\int_{\mathbb{R}^3}|\nabla v_2|^2 = a + b\sqrt{\frac{K_2}{K_1}}\int_{\mathbb{R}^3}|\nabla v_1|^2.$$

It implies that $K_2 = a + \sqrt{\frac{K_2}{K_1}}(K_1 - a)$, i.e.,

$$\frac{(K_2 - a)^2}{K_2} = \frac{(K_1 - a)^2}{K_1}. \tag{15}$$

Let us define that $f(x) := \frac{(x-a)^2}{x}$, $x > 0$, then $f'(x) = \frac{x^2 - a^2}{x^2}$. Thus, for $x > a$, $f(x)$ is strictly increasing function. As $K_1, K_2 > a$, from (15), we have that $K_1 = K_2$. Then, by (14), we can imply that $u_1 = u_2$.

Furthermore, by [30], the solution $\bar{u}_1(x)$ of (13) is radially symmetric, and $\bar{u}_1(x) = v_1(\sqrt{K_1}x)$ implies that v_1 is also radially symmetric. □

3. Uniqueness Result for Equations with Potential $|x|^m$

In this section, we consider problem (2) and prove the result in Theorem 2.

Proof of Theorem 2. First, the existence of solutions of (2) can be seen in [31] for example. We denote a positive solution of (2) by u_1. By elliptic estimates, $u_1(x) \to 0$ as $|x| \to \infty$. Now, we prove that u_1 is the unique solution of (2). Otherwise, let u_2 is another solution of (2). Let

$$K_1 := a + b\int_{\mathbb{R}^3} |\nabla u_1|^2,\ K_2 := a + b\int_{\mathbb{R}^3} |\nabla u_2|^2.$$

Then $u_i (i = 1, 2)$ satisfies

$$-\Delta u + \frac{|x|^m}{K_i} u = \frac{1}{K_i} u^p.$$

Let $w_i(x) := \beta_i u_i(\alpha_i x)$, where $\alpha_i = K_i^{\frac{1}{m+2}}$, $\beta_i = K_i^{\frac{-m}{(m+2)(p-1)}}$. Then, $w_i(x)$ is the solution of

$$\begin{cases} -\Delta w + |x|^m w = w^p, & x \in \ \mathbb{R}^3, \\ w > 0,\ w(\infty) = 0. \end{cases} \tag{16}$$

Now, by [32], we know that the solution of (16) is unique. It yields that $w_1(x) \equiv w_2(x)$, i.e.,

$$\beta_1 u_1(\alpha_1 x) = \beta_2 u_2(\alpha_2 x).$$

Therefore,

$$u_2(x) = \frac{\beta_1}{\beta_2} u_1(\frac{\alpha_1}{\alpha_2} x). \tag{17}$$

Then,

$$\begin{aligned} K_2 &= a + b\int_{\mathbb{R}^3} |\nabla u_2|^2 \\ &= a + b\frac{\beta_1^2}{\beta_2^2}\frac{\alpha_2}{\alpha_1}\int_{\mathbb{R}^3} |\nabla u_1|^2 \\ &= a + (\frac{K_2}{K_1})^{\frac{p-1+2m}{(m+2)(p-1)}}(K_1 - a). \end{aligned}$$

From above, we can determine that

$$\frac{(K_2 - a)^{\frac{(m+2)(p-1)}{p-1+2m}}}{K_2} = \frac{(K_1 - a)^{\frac{(m+2)(p-1)}{p-1+2m}}}{K_1}. \tag{18}$$

For simplicity, we define $k := \frac{(m+2)(p-1)}{p-1+2m}$, and let

$$f(x) := \frac{(x-a)^k}{x},\ x > 0.$$

Then

$$f'(x) = \frac{(x-a)^{k-1}((k-1)x + a)}{x^2}. \tag{19}$$

Since

$$k - 1 = \frac{(p-3)m + p - 1}{p - 1 + 2m} > 0,$$

we get that $f(x)$ is strictly increasing for $x > a$. Now, by $K_1, K_2 > a$, we can know $K_1 = K_2$ from (18). Then, by the definition of α_i, β_i and from (17), we know that $u_1 = u_2$. This completes our proof that u_1 is the unique solution of (2). □

4. Asymptotic Behaviour of Ground State Solutions for the Flat Case

Let v_ε be the least energy solution of (3), which is proved in [26]. Now, let

$$w_\varepsilon(x) := \varepsilon^{-\frac{2}{p-1}} v_\varepsilon(x),$$

then, w_ε is a least energy solution of the problem

$$-(a+\varepsilon^\alpha b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)\Delta w_\varepsilon + \frac{1}{\varepsilon^2}V(x)w_\varepsilon = w_\varepsilon^p, \tag{20}$$

where $\alpha = \frac{5-p}{p-1} > 0$ by the assumption $3 < p < 5$.

Assume that I_ε is the energy functional associated to problem (20); then, by direct computations,

$$\begin{aligned} I_\varepsilon(w_\varepsilon) &= \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon^2}V(x)w_\varepsilon^2 + \varepsilon^\alpha\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} w_\varepsilon^{p+1} \\ &= \varepsilon^{-2(p+1)/(p-1)}\Gamma_\varepsilon(v_\varepsilon), \end{aligned}$$

where Γ_ε is the energy functional associated to (3). We have

Lemma 1.

$$\limsup_{\varepsilon\to 0} I_\varepsilon(w_\varepsilon) \le I_U,$$

where I_U is the least energy of Equation (6).

Proof. Given $R > 0$, let $\varphi_R \in C_0^\infty(\mathbb{R}^3)$ be such that $\varphi_R \equiv 1$ on $B_R(0) = \{x \in \mathbb{R}^3 | |x| \le R\}$, $\varphi_R \equiv 0$ in $\mathbb{R}^3 \backslash B_{R+1}(0)$, $0 \le \varphi_R \le 1$, $|\nabla\varphi_R| \le c$, where c is a positive constant. Define $v_R := \varphi_R w_0$, where w_0 is the least energy solution of (6) (we regard $w_0 \equiv 0$ on $\mathbb{R}^3 \setminus \mathrm{int}(\mathcal{Z})$). Then, we can get a unique $\theta > 0$ such that $\theta v_R \in \bar{\mathcal{N}}_\varepsilon$, where $\bar{\mathcal{N}}_\varepsilon$ is the Nehari manifold with respect to (20), i.e.,

$$\theta^{p-1}\int_{\mathbb{R}^3} v_R^{p+1} = \int_{\mathbb{R}^3}(a|\nabla v_R|^2 + \frac{1}{\varepsilon^2}V(x)v_R^2) + \theta^2\varepsilon^\alpha b(\int_{\mathbb{R}^3}|\nabla v_R|^2)^2,$$

which implies that

$$\begin{aligned} \theta^{p-1} &= \frac{\int_{\mathbb{R}^3} a|\nabla v_R|^2}{\int_{\mathbb{R}^3} v_R^{p+1}} + \frac{\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3} V(x)v_R^2}{\int_{\mathbb{R}^3} v_R^{p+1}} + \theta^2\frac{\varepsilon^\alpha b(\int_{\mathbb{R}^3}|\nabla v_R|^2)^2}{\int_{\mathbb{R}^3} v_R^{p+1}} \\ &=: I_1 + I_2 + \theta^2 I_3. \end{aligned} \tag{21}$$

Since w_0 is a least energy solution of (6), then $I_1 \to 1$ as $R \to \infty$. As $w_0 \equiv 0$ on $\mathbb{R}^3 \setminus \mathrm{int}(\mathcal{Z})$ and $V(x) \equiv 0$ on $\mathrm{int}(\mathcal{Z})$, it yields that

$$\int_{\mathbb{R}^3} V(x)v_R^2 = 0$$

and so, $I_2 = 0$. By $\alpha > 0$, $I_3 \to 0$ for fixed $R > 0$ and $\varepsilon \to 0^+$. Now, define the function

$$h(m,n,\theta) = \theta^{p-1} - \theta^2 m - n.$$

We have that $h(0,1,1) = 0$ and $\frac{\partial h}{\partial\theta}(0,1,1) \neq 0$. Then, the implicit function theorem tells us that there exists a function $\theta = \theta(m,n)$, which satisfies that $h(m,n,\theta) = 0$ near $(0,1,1)$, and $\theta(m,n)$ is continuous near $(0,1)$. Thus, $\theta(m,n) \to \theta(0,1) = 1$, as $m \to 0$ and $n \to 1$. Therefore by (21), for fixed and sufficiently large $R > 0$, θ is just close to 1 as $\varepsilon \to 0^+$.

Now

$$
\begin{aligned}
& I_\varepsilon(w_\varepsilon) = \inf_{v\in\bar{\mathcal{N}}_\varepsilon} I_\varepsilon(v) \le I_\varepsilon(\theta v_R) \\
= \quad & \theta^2[\tfrac{1}{2}\int_{\mathbb{R}^3}(a|\nabla v_R|^2 + \tfrac{1}{\varepsilon^2}V(x)v_R^2) + \theta^2\varepsilon^\alpha\tfrac{b}{4}(\int_{\mathbb{R}^3}|\nabla v_R|^2)^2 - \tfrac{\theta^{p-1}}{p+1}\int_{\mathbb{R}^3} v_R^{p+1}] \\
= \quad & \theta^2[\tfrac{1}{2}\int_{\mathbb{R}^3} a|\nabla v_R|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^3} v_R^{p+1} + \theta^2\varepsilon^\alpha\tfrac{b}{4}(\int_{\mathbb{R}^3}|\nabla v_R|^2)^2 + \tfrac{1-\theta^{p-1}}{p+1}\int_{\mathbb{R}^3} v_R^{p+1}].
\end{aligned} \tag{22}
$$

As $\frac{1}{2}\int_{\mathbb{R}^3} a|\nabla v_R|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} v_R^{p+1} \to I_U$ when $R \to \infty$, we can get that the last quantity in (22) is just close to I_U if R is sufficiently large. Then letting $\varepsilon \to 0^+$, we have proven that $\limsup\limits_{\varepsilon\to 0} I_\varepsilon(w_\varepsilon) \le I_U$. □

Now, by Lemma 1 and

$$
I_\varepsilon(w_\varepsilon) = \frac{1}{4}\int_{\mathbb{R}^3} a|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon^2}V(x)w_\varepsilon^2 + (\frac{1}{4} - \frac{1}{p+1})\int_{\mathbb{R}^3} w_\varepsilon^{p+1},
$$

by our assumption $3 < p < 5$, we have that $\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2$ and $\int_{\mathbb{R}^3} w_\varepsilon^{p+1}$ are bounded for small $\varepsilon > 0$. Thus, by combining (iii) of Theorem 1.1 in [26], we know that $\|w_\varepsilon\|_{H^1(\mathbb{R}^3)}$ is bounded for small $\varepsilon > 0$. Now, we have some $w \in H^1(\mathbb{R}^3)$ such that up to a subsequence, w_ε converges weakly in $H^1(\mathbb{R}^3)$ and pointwise to w. Moreover, from (iii) of Theorem 1.1 in [26], we see that $w = 0$ on $\mathbb{R}^3 \setminus \mathrm{int}(\mathcal{Z})$ and that $w_\varepsilon \to w$ in $L^{p+1}(\mathbb{R}^3)$. Now, we test $\phi \in C_0^\infty(\mathrm{int}(\mathcal{Z}))$ on Equation (20),

$$
0 = \int_{\mathbb{R}^3} a\nabla w_\varepsilon\nabla\phi - w_\varepsilon^p\phi + \varepsilon^\alpha b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2\int_{\mathbb{R}^3}\nabla w_\varepsilon\nabla\phi.
$$

Thus, letting $\varepsilon \to 0$, we have

$$
\begin{cases}
-a\Delta w = w^p & in\ \mathrm{int}(\mathcal{Z}), \\
w \ge 0 & in\ \mathrm{int}(\mathcal{Z}), \\
w = 0 & on\ \partial\mathrm{int}(\mathcal{Z}).
\end{cases} \tag{23}
$$

Now, by Lemma 1, we can get that $\frac{1}{2}\int_{\mathrm{int}(\mathcal{Z})} a|\nabla w|^2 - \frac{1}{p+1}\int_{\mathrm{int}(\mathcal{Z})} w^{p+1} = I_U$, and hence, $w = U$ on $\mathrm{int}(\mathcal{Z})$, where U is a least energy solution of (23). Moreover, by elliptic estimates, we can show that for any compact subset $A \subset \mathrm{int}(\mathcal{Z})$, the convergence is uniform. Thus, for each $\delta > 0$, $w_\varepsilon \to w$ uniformly on $\{x \in \mathbb{R}^3 | \mathrm{dist}(x, \partial\mathrm{int}(\mathcal{Z})) \ge \delta\}$. By now, we have proven Theorem 3.

5. Asymptotic Behaviour of Ground State Solutions for the Infinite Case

In this section, we consider the asymptotic behaviour of the least energy solutions of nonlinear Schrödinger–Kirchhoff Equation (3) for the infinite case. Here suppose that (V1), (V2) and (V4) hold. Let v_ε be the least energy solutions to problem (3) proven in [26], and let Γ_ε denote the energy functional associated to (3), i.e.,

$$
\Gamma_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^3}(\varepsilon^2 a|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2) + \frac{b}{4}\varepsilon(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(x)|v_\varepsilon|^{p+1}.
$$

Define $g(\varepsilon) := -\log\varepsilon^2$ and $w_\varepsilon(x) := (\varepsilon g(\varepsilon))^{-2/(p-1)} v_\varepsilon(\frac{x}{g(\varepsilon)})$. Then, w_ε satisfies that

$$
-(a + \varepsilon^{\frac{p+5}{p-1}} g^{\frac{7p-1}{p-1}}(\varepsilon) b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)\Delta w_\varepsilon + (\varepsilon g(\varepsilon))^{-2} V(\frac{x}{g(\varepsilon)}) w_\varepsilon = w_\varepsilon^p.
$$

From the definition of $g(\varepsilon)$ and direct computations, one can obtain

$$
\varepsilon^{\frac{p+5}{p-1}} g^{\frac{7p-1}{p-1}}(\varepsilon) \to 0, \tag{24}
$$

as $\varepsilon \to 0$. Thus, for $|x| \leq g(\varepsilon)$, by (V4), we have

$$-(a+\varepsilon^{\frac{p+5}{p-1}}g^{\frac{7p-1}{p-1}}(\varepsilon)b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)\Delta w_\varepsilon+(\varepsilon g(\varepsilon))^{-2}\exp(-\frac{|g(\varepsilon)|}{|x|})w_\varepsilon=w_\varepsilon^p.$$

By the definition of $g(\varepsilon)$, it is

$$-(a+\varepsilon^{\frac{p+5}{p-1}}g^{\frac{7p-1}{p-1}}(\varepsilon)b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)\Delta w_\varepsilon+(\varepsilon g(\varepsilon))^{-2}\varepsilon^{\frac{2}{|x|}}w_\varepsilon=w_\varepsilon^p, \tag{25}$$

for $|x| \leq g(\varepsilon)$. Thus, for each compact set $B_r \subset B_1$, we have

$$\lim_{\varepsilon\to 0}\max_{x\in B_r}(\varepsilon g(\varepsilon))^{-2}\varepsilon^{\frac{2}{|x|}}=0. \tag{26}$$

Moreover, for any $d > 1$,

$$\lim_{\varepsilon\to 0}\min\{(\varepsilon g(\varepsilon))^{-2}\varepsilon^{\frac{2}{|x|}}|d\leq |x|\leq g(\varepsilon)\}=\infty. \tag{27}$$

Now, we consider the problem:

$$\begin{cases} -a\Delta w=w^p & in\ B_1, \\ w>0 & in\ B_1, \\ w=0 & on\ \partial B_1. \end{cases} \tag{28}$$

From the estimations (24) and (26), we can determine that

$$\limsup_{\varepsilon\to 0}\tilde{\Gamma}_\varepsilon(w_\varepsilon)\leq \frac{1}{2}\int_{B_1}a|\nabla W|^2-\frac{1}{p+1}\int_{B_1}W^{p+1}=:I(B_1),$$

where W is a ground state solution of (28) and

$$\begin{aligned}\tilde{\Gamma}_\varepsilon(w_\varepsilon) &= \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla w_\varepsilon|^2+(\varepsilon g(\varepsilon))^{-2}V(\frac{x}{g(\varepsilon)})w_\varepsilon^2)+\varepsilon^{\frac{p+5}{p-1}}g^{\frac{7p-1}{p-1}}(\varepsilon)\frac{b}{4}(\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2)^2- \\ &\frac{1}{p+1}\int_{\mathbb{R}^3}w_\varepsilon^{p+1}=(\varepsilon g(\varepsilon))^{-2(p+1)/(p-1)}g(\varepsilon)^{-3}\Gamma_\varepsilon(v_\varepsilon).\end{aligned}$$

Then, by elliptic estimates and (27), we can know that

$$\lim_{\varepsilon\to 0}||w_\varepsilon||_{L^\infty(\{x\in\mathbb{R}^3|d<|x|\leq g(\varepsilon)\})}=0.$$

Moreover, from Theorem 1.1 in [26], we can deduce that

$$\lim_{\varepsilon\to 0}||w_\varepsilon||_{L^\infty(\{x\in\mathbb{R}^3||x|\geq g(\varepsilon)\})}=0.$$

Therefore, by similar arguments used in the flat case and the finite case, we can determine that

$$\lim_{\varepsilon\to 0}(\varepsilon g(\varepsilon))^{-2(p+1)/(p-1)}g(\varepsilon)^{-3}\Gamma_\varepsilon(v_\varepsilon)=I(B_1).$$

Furthermore, for each $\delta > 0$, the function $(\varepsilon g(\varepsilon))^{-2/(p-1)}v_\varepsilon(\frac{x}{g(\varepsilon)})$ converges (up to a subsequence) uniformly to $\bar{W}$ on $\{x \in \mathbb{R}^3|\mathrm{dist}(x, \partial B_1) \geq \delta\}$ as $\varepsilon \to 0$, where W is a ground state solution of (28) and

$$\bar{W}(x)=\begin{cases} W(x) & \text{for } x\in B_1, \\ 0 & \text{for } x\notin B_1. \end{cases}$$

By now, we have proven Theorem 4.

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