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Global Existence to Cauchy Problem for 1D Magnetohydrodynamics Equations

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Abstract: Magnetohydrodynamics are widely used in medicine and biotechnology, such as drug targeting, molecular biology, cell isolation and purification. In this paper, we prove the existence of a global strong solution to the one-dimensional compressible magnetohydrodynamics system with temperature-dependent heat conductivity in unbounded domains and a large initial value by the Lagrangian symmetry transformation, when the viscosity μ is constant and the heat conductivity κ, which depends on the temperature, satisfies $\kappa = \bar{\kappa}\theta^b (b > 1)$.

Keywords: compressible MHD equations; temperature-dependent heat conductivity; global strong solution; Cauchy problem

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1. Introduction

Magnetohydrodynamics (MHDs) is the coupling of fluid dynamics to electromagnetism. MHD finds its way into a very wide range of physical objects, medicine and biotechnology. For example, the physical applications include levitation melting, the casting and stirring of liquid metals, and aluminum reduction. The applications in medicine and biotechnology include drug targeting, molecular biology, and cell isolation and purification; see [1,2]. The compressible planar MHD flows, which are uniform in the transverse directions, read as

$$\begin{cases} \rho_t + (\rho u)_y = 0, \\ (\rho u)_t + (\rho u^2 + P + \frac{1}{2}|\mathbf{b}|^2)_y = (\lambda u_y)_y, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w} - \mathbf{b})_y = (\mu \mathbf{w}_y)_y, \\ (\mathbf{b})_t + (u\mathbf{b} - \mathbf{w})_y = (\nu \mathbf{b}_y)_y, \\ (\rho e)_t + \left[u(\rho e + P + \frac{1}{2}|\mathbf{b}|^2) - \mathbf{w}\cdot\mathbf{b}\right]_y = (\kappa e_y)_y + (\lambda u u_y + \mu \mathbf{w}\cdot\mathbf{w}_y + \nu \mathbf{b}\cdot\mathbf{b}_y)_y. \end{cases} \tag{1}$$

where $t > 0$ and $y \in \mathbb{R}$ are the time variable and the spatial variable, respectively, where the unknowns $\rho \geq 0$, $u \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^2$ and e denote the density, the longitudinal velocity, the transverse velocity, the transverse magnetic field, and the internal energy of the flow, respectively. The parameters $\lambda(\rho,\theta)$ and $\mu(\rho,\theta)$ denote the viscosity coefficients, $\nu = \nu(\rho,\theta)$ is the magnetic diffusion coefficient of the magnetic field, and $\kappa = \kappa(\rho,\theta)$ is the heat conductivity. All the parameters are generally related to the density and temperature of the flow.

For technical convenience, we transform the problem (1) into Lagrangian variables. To this end, we introduce the Lagrangian symmetry variable

$$x = \int_{y(t)}^{y} \rho(t,z)dz,$$

where $y(t)$ is the particle path satisfying $y'(t) = u(t, y(t))$. The Lagrangian version of the system (1) can be written as

$$\begin{cases} v_t = u_x, \\ u_t + (P + \frac{1}{2}|\mathbf{b}|^2)_x = (\frac{\lambda u_x}{v})_x, \\ \mathbf{w}_t - \mathbf{b}_x = (\frac{\mu \mathbf{w}_x}{v})_x, \\ (v\mathbf{b})_t - \mathbf{w}_x = (\frac{\nu \mathbf{b}_x}{v})_x, \\ e_t + Pu_x = (\frac{\kappa \theta_x}{v})_x + \frac{\lambda}{v} u_x^2 + \frac{\mu}{v}|\mathbf{w}_x|^2 + \frac{\nu}{v}|\mathbf{b}_x|^2. \end{cases} \tag{2}$$

In this paper, we consider a perfect gas for magnetohydrodynamic flow, that is,

$$P = R\frac{\theta}{v}, \quad e = c_v \theta, \tag{3}$$

where R is a positive constant and c_v is the heat capacity of the gas at a constant volume. The system (2) is supplemented with the following initial condition:

$$(u,\ v,\ \mathbf{w},\ \mathbf{b},\ \theta)|_{t=0} = (u_0,\ v_0,\ \mathbf{w}_0,\ \mathbf{b}_0,\ \theta_0), \quad x \in \mathbb{R}, \tag{4}$$

and the far-field condition

$$\lim_{|x| \to \infty} (v(x,t), u(x,t),\ \mathbf{w}(x,t),\ \mathbf{b}(x,t),\ \theta(x,t)) = (1, 0, 0, 0, 1), \quad t > 0. \tag{5}$$

Many works in the literature have studied system (2). The existence and uniqueness of local smooth solutions were proved in [3]. When μ and κ are constants, for perfect gases with small smooth initial data, the existence of a global solution was proved in [4], Chen and Wang [5] established the existence and uniqueness of global solutions to the free-boundary problem and initial boundary problem; for other existential results, see [6–8]. The large-time behavior of the solution was studied in [9–11], where the exponential stability of the solution was established.

When κ depends on v and θ, μ is a constant, under the technical condition that $\kappa(v, \theta)$ satisfies

$$C^{-1}(1 + \theta^q) \leq \kappa(v, \theta) = \kappa(\theta) \leq C(1 + \theta^q)$$

for some $q \geq 2$. Chen and Wang in [12] obtained the existence, uniqueness and the Lipschitz-continuous dependence of global strong solutions to the initial boundary problem with large initial data, satisfying

$$0 < \inf v_0 \leq v_0(x) \leq \sup v_0 < \infty,\ \rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0 \in H^1, \inf \theta_0(x) \geq 0.$$

A similar result was obtained in [10] for real gas. Fan, Huang and Li [6] obtained a global existence theorem with vacuum and large data. Hu and Ju proved the global existence of a solution for the case $\kappa = \bar{\kappa}\theta^q, (q > 0)$ in [7].

In this paper, we consider the Cauchy problem for (2)–(4) with the far-field condition (5). Assume that the heat conductivity coefficient κ satisfies

$$\mu = \bar{\mu},\ \kappa(\theta) = \bar{\kappa}\theta^b, \tag{6}$$

for simplicity, let $\lambda = \mu = \nu = R = c_v = \bar{\kappa} = 1$. The global existence of a solution to Equations (2)–(4) with the far-field condition (5) is obtained for the Cauchy problem with $b > 1$. For the unbounded domain, which loses compactness, $v(x,t), \theta(x,t)$ are not vanishing as $|x|$ tends to infinity. This brings a lot of difficulty.

It needs to be emphasized that the well-posedness problem of (2)–(4) with (5) is a big open problem with a large initial when μ, κ are both dependent on v and θ.

Notation 1. *(1) For $p \geq 1, L^p = L^p(\mathbb{R})$ denotes the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, $W^{k,p} = W^{k,p}(\mathbb{R})$ denotes the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k = W^{k,2}(\mathbb{R})$. $Q_T = [0,T] \times \mathbb{R}$.*

(2) Throughout this paper, the letter C denotes a positive constant, which may be different from line to line.

The following is the main results of this paper.

Theorem 1. *Assume that μ and κ satisfy (6) for some positive constants $\bar{\mu}$ and $\bar{\kappa}$. If the initial data $(v_0,\ u_0,\ \mathbf{b}_0,\ \boldsymbol{w}_0,\ \theta_0)(x)$ are compatible with (2)–(4) together with the far-field condition (5), satisfying*

$$(v_0 - 1,\ u_0,\ \mathbf{b}_0,\ \boldsymbol{w}_0,\ \theta_0 - 1)(x) \in H^1 \times H^2 \times H^2 \times H^2 \times H^2, \tag{7}$$

and there are constants $\underline{v}$, $\underline{\theta}$, such that

$$0 < \underline{v} \leq v_0(x),\ \ 0 < \underline{\theta} \leq \theta_0(x). \tag{8}$$

Then for any $T > 0$, there exists a unique global strong solution $(v,\ u,\ \mathbf{b},\ \boldsymbol{w},\ \theta)$ satisfying

$$\begin{cases} \|(v-1,\ u,\ \mathbf{b},\ \boldsymbol{w},\ \theta-1)(\cdot,t)\|^2_{H^1(\mathbb{R})} + \int_0^t \|(v-1,\ u,\ \mathbf{b},\ \boldsymbol{w},\ \theta-1)(\cdot,s)\|^2_{H^1(\mathbb{R})} ds \leq C, \\ \|(u,\ \mathbf{b},\ \boldsymbol{w},\ \theta-1)(\cdot,t)\|^2_{H^2(\mathbb{R})} \\ \quad + \int_0^t \|\ u_{xt},\ u_{xx}\ \theta_{xt},\ \theta_{xx},\ \mathbf{b}_{xx},\ \mathbf{b}_{xt},\ \boldsymbol{w}_{xx},\ \boldsymbol{w}_{xt})(\cdot,s)\|^2_{L^2(\mathbb{R})} ds \leq C, \end{cases} \tag{9}$$

here and below, we denote C as some constant that depends on T.

If in addition to the initial data, $(v_0,\ u_0,\ \mathbf{b}_0,\ \boldsymbol{w}_0, \theta_0)(x)$ satisfies

$$v_0(x) \in C^{1+\alpha},\ \ u_0(x) \in C^{2+\alpha},\ \ \mathbf{b}_0 \in C^{2+\alpha},\ \ \boldsymbol{w}_0 \in C^{2+\alpha},\ \ \theta_0 \in C^{2+\alpha}. \tag{10}$$

for some $\alpha \in (0,1)$ has a unique global classical solution $(v,\ u,\ \mathbf{b},\ \boldsymbol{w},\ \theta)(x,t)$ such that, for any fixed $T > 0$,

$$\begin{aligned} &v \in C^{1+\alpha,\frac{\alpha}{2}}(\mathbb{R} \times [0,T]),\ u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]),\ \mathbf{b} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]), \\ &\boldsymbol{w} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]),\ \theta \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]), \end{aligned} \tag{11}$$

and for each $(x,t) \in \mathbb{R} \times [0,T)$, $(v,\ u,\ \mathbf{b},\ \boldsymbol{w},\ \theta)(x,t)$ satisfies (9).

Based on the local existence, the existence of a global solution will be established by extending the local solution in time with the help of the global a priori estimates stated in (9). It is clear that (9) is sufficient to extend the local strong solution to global one by a standard continuity argument. Assume that there exists a finite maximal time T^* for the unique strong solution $(v,\ u,\ \mathbf{b},\ \mathbf{w},\ \theta)(x,t)$ of (2). Then estimates in (9) assure that $(v,\ u,\ \mathbf{b},\ \mathbf{w},\ \theta)(x,T^*)$ satisfies the conditions in local solution for the initial data. Applying the estimates in (9) for (2) with initial time T^*, which extends the existence of a strong solution to the time interval $[T^*, T^* + T_1)$ for some $T_1 > 0$. This contradicts the assumption that T^* is the maximal existence time.

We now outline the main ideas and difficulties in our problem, comparing to previous results. The existence of strong solutions for the initial-boundary problem and free-boundary problem can be easily obtained due to pioneering works, e.g., Kazhikhov [13], Chen and Wang [5,12], and Fan [6]. In our case, we will follow the basic framework laid out in [14] with extra attention to the new difficulties. The key step to prove the global solution is to obtain the bounds of $v(x,t)$ and $\theta(x,t)$ from below and above. Jiang proved the uniform positive lower and upper bounds on $v(x,t)$ in [15] by a decent localized version of the expression for $v(x,t)$ when κ is a constant to one-dimensional compressible Navier–Stokes equations. Li and Liang deduced the uniform positive lower and upper bounds on the temperature $\theta(x,t)$ in [16] by a smart test function method and space separation technique. This approach cannot be applied to the case when $\kappa = \bar{\kappa}(1+\theta^b)$ to (2), since with the strong nonlinearity, it is difficult to obtain the bounds of $\theta(x,t)$ and $v(x,t)$ from below and above. Another main reason is that the full pressure $p + \mathbf{b}^2/2$ in MHD does not have the simple special structure as the pressure p in the Navier–Stokes equations. To overcome such a difficulty, motivated by [7,17–19], we obtain the high-order estimates on $\theta(x,t)$ and $v(x,t)$. It should be pointed out that the crucial techniques of proofs in [19] cannot be adapted directly here since the domain is unbounded and it loses compactness. In this paper, we obtain the lower bound of the temperature when $t \in [0,T]$ is induced by the comparison principle.

2. Some Priori Estimates

In this section, we will perform a sequence of estimates, which are stated in the following as lemmas. In particular, we prove the volume $v(x,t)$ is bounded from below and above. This is a key step in the proof of global existence. Assume that $(v,\ u,\ \theta,\ \mathbf{b},\ \mathbf{w})(x,t)$ is the unique local strong solution of (2)–(4) with the far-field condition (5) defined on $\mathbb{R} \times [0,T]$ for some $T > 0$.

Lemma 1. *There is a positive constant e_0 and C independent of T, such that*

$$\begin{aligned}&\sup_{0\le t<\infty}\int_{\mathbb{R}}\left(\frac{1}{2}(u^2+\boldsymbol{w}^2+v\boldsymbol{b}^2)+(v-\ln v-1)+(\theta-\ln\theta-1)\right)dx\\&+\int_0^\infty\int_{\mathbb{R}}\left(\frac{u_x^2}{v\theta}+\frac{\theta^b\theta_x^2}{v\theta^2}+\frac{\boldsymbol{w}_x^2}{v\theta}+\frac{\boldsymbol{b}_x^2}{v\theta}\right)dxdt\le e_0,\end{aligned}\tag{12}$$

where the constant e_0 depending only on $\|(v_0-1,u_0,\theta_0-1)\|_{H^1(\mathbb{R})}, \underline{v}, \underline{\theta}$.

Proof. Multiplying $(2)_1$ by $(1-v^{-1})$, $(2)_2$ by u, $(2)_3$ by $\mathbf{w}$, $(2)_4$ by $\mathbf{b}$, and $(2)_5$ by $(1-\theta^{-1})$, and adding them altogether, we obtain

$$\begin{aligned}&\left(\frac{1}{2}(u^2+\mathbf{w}^2+v\mathbf{b}^2)+(v-\ln v-1)+(\theta-\ln\theta-1)\right)_t+\frac{u_x^2}{v}+\frac{\theta^b\theta_x^2}{v\theta^2}\\&+\frac{\mathbf{w}_x^2}{v\theta}+\frac{\mathbf{b}_x^2}{v\theta}=(\frac{uu_x}{v})_x-(\frac{u\theta}{v})_x+u_x+\left((1-\theta^{-1})\frac{\theta^b\theta_x^2}{v}\right)_x,\end{aligned}\tag{13}$$

using Taylor's theorem, (6), and Sobolev's embedding theorem ($H^1 \hookrightarrow L^\infty$), we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{1}{2}(u_0^2+\mathbf{w}_0^2+v_0\mathbf{b}_0^2)+(v_0-\ln v_0-1)+(\theta_0-\ln\theta_0-1)\right)dx\\&\le C(1+\|(v_0-1,u_0,\mathbf{w}_0,\mathbf{b}_0,\theta_0-1)\|^2_{H^1(\mathbb{R})}).\end{aligned}$$

Integrating (13) over $\mathbb{R}\times[0,\infty)$ and using the far-field condition (5), we obtain (12). This finishes the proof of Lemma 1. □

Lemma 2. *There exists a positive constant C such that*

$$C^{-1}\le v(x,t)\le C+C\int_0^t\max_{x\in[k,k+1]}\theta(x,t)+\frac{v\boldsymbol{b}^2}{2}dt,\tag{14}$$

Proof. For any $x\in\mathbb{R}$, denoting $k=[x]$, we have by Lemma 1

$$\int_k^{k+1}[(v-\ln v-1)+(\theta-\ln\theta-1)]dx\le e_0,\tag{15}$$

which together with Jensen's inequality yields

$$\alpha_1 \le \int_k^{k+1} v(x,t)dx \le \alpha_2,\ \alpha_1 \le \int_k^{k+1} \theta(x,t)dx \le \alpha_2, \tag{16}$$

where $0 < \alpha_1 < 1 < \alpha_2$ are two roots of

$$x - \ln x - 1 = e_0.$$

Moreover, it follows from (15) that for any $t > 0$, there exists some $b_k(t) \in [k, k+1]$ such that

$$(v - \ln v - 1) + (\theta - \ln\theta - 1)(b_k(t), t) \le e_0,$$

which implies

$$\alpha_1 \le v(b_k(t), t) \le \alpha_2,\ \alpha_1 \le \theta(b_k(t), t) \le \alpha_2. \tag{17}$$

Letting $\sigma \triangleq \frac{u_x}{v} - \frac{\theta}{v} - \frac{\mathbf{b}^2}{2} = (\ln v)_t - \frac{\theta}{v} - \frac{\mathbf{b}^2}{2}$, we write $(2)_2$ as

$$u_t = \sigma_x. \tag{18}$$

Integrating (18) over $[k, x] \times [0, t]$ leads to

$$\int_k^x (u - u_0)dy = (\ln v - \ln v_0) - \int_0^t \left(\frac{\theta}{v} + \frac{\mathbf{b}^2}{2}\right)ds - \int_0^t \sigma(k,s)ds,$$

which gives

$$v(x,t) = B_k(x,t)Y_k \exp\int_0^t \left(\frac{\theta}{v} + \frac{\mathbf{b}^2}{2}\right)ds, \tag{19}$$

where

$$B_k \triangleq v_0(x) \exp\left\{\int_k^x (u(y,t) - u_0(y))dy\right\},$$

and

$$Y_k(t) \triangleq \exp\left\{\int_0^t \sigma(k,s)ds\right\}.$$

Thus, multiplying (19) by $\frac{1}{\theta + \frac{v\mathbf{b}^2}{2}}$, one has

$$v(x,t) = B_k(x,t)Y_k(t)\left(1 + \int_0^t \frac{\theta(x,\tau) + \frac{v\mathbf{b}^2}{2}}{B_k(x,\tau)Y_k(\tau)}d\tau\right). \tag{20}$$

Since

$$\left|\int_k^x (u(y,t) - u_0(y))dy\right| \le \left(\int_k^{k+1} u^2 dy\right)^{\frac{1}{2}} + \left(\int_k^{k+1} u_0^2 dy\right)^{\frac{1}{2}} \le C,$$

here, we have

$$C^{-1} \le B_k(x,t) \le C. \tag{21}$$

Moreover, integrating (20) with respect to x over $[k, k+1]$ gives

$$\frac{1}{Y_k(t)}\int_k^{k+1} v(x,t)dx = \int_k^{k+1} B_k(x,t)\left(1 + \int_0^t \frac{\theta(x,\tau) + \frac{v\mathbf{b}^2}{2}}{B_k(x,\tau)Y_k(\tau)}d\tau\right)dx,$$

which yields

$$C^{-1} \le \frac{1}{Y_k(t)} \le C + C\int_0^t \frac{1}{Y_k(\tau)}d\tau, \tag{22}$$

which we have used (16), (21), and the following simple fact:

$$\int_k^{k+1} \frac{\theta(x,\tau)B_k(x,\tau)}{B_k(x,\tau)}dx \le C\int_k^{k+1}\theta(x,\tau)dx \le C. \tag{23}$$

Applying Grönwall's inequality to (22) gives

$$C^{-1} \le \frac{1}{Y_k(t)} \le C, \tag{24}$$

which together with (20), (21) and (24) implies that for $(x,t) \in [k,k+1]\times[0,T]$.

$$C^{-1} \le v(x,t) \le C + C\int_0^t \max_{x\in[k,k+1]}\theta(x,t) + \frac{v\mathbf{b}^2}{2}dt. \tag{25}$$

This finishes the proof of Lemma 2. □

Now we give the estimation on θ from below by the comparison principle.

Lemma 3. *There exists a positive constant C such that for all $(x,t)\in\mathbb{R}\times[0,T]$,*

$$C \le \theta(x,t).$$

Proof. For $t\in[0,T]$, letting $\Theta = \frac{1}{\theta}$, and rewriting the Equation $(2)_5$ as

$$-\frac{\Theta_t}{\Theta^2} = -\frac{u_x}{v\Theta} - \left(\frac{\kappa(\Theta)\Theta_x}{v\Theta^2}\right)_x + \frac{u_x^2}{v} + \frac{\mathbf{w}_x^2}{v} + \frac{\mathbf{b}_x^2}{v}, \tag{26}$$

so

$$\begin{aligned}
\Theta_t &= \frac{\Theta u_x}{v} + \Theta^2\left(\frac{\kappa(\Theta)\Theta_x}{v\Theta^2}\right)_x - \frac{\Theta^2 u_x^2}{v} - \frac{\Theta^2\mathbf{w}_x^2}{v} - \frac{\Theta^2\mathbf{b}_x^2}{v}\\
&= \left(\frac{\kappa(\Theta)\Theta_x}{v}\right)_x - \frac{2\kappa(\Theta)\Theta_x^2}{v\Theta} - \frac{\Theta^2 u_x^2}{v} + \frac{\Theta u_x}{v} - \frac{\Theta^2\mathbf{w}_x^2}{v} - \frac{\Theta^2\mathbf{b}_x^2}{v}\\
&= \left(\frac{\kappa(\Theta)\Theta_x}{v}\right)_x - \frac{2\kappa(\Theta)\Theta_x^2}{v\Theta} - \frac{\Theta^2}{v}(u_x^2 - \frac{u_x}{\Theta}) - \frac{\Theta^2\mathbf{w}_x^2}{v} - \frac{\Theta^2\mathbf{b}_x^2}{v}\\
&= \left(\frac{\kappa(\Theta)\Theta_x}{v}\right)_x - \frac{2\kappa(\Theta)\Theta_x^2}{v\Theta} - \frac{\Theta^2}{v}(u_x - \frac{1}{2\Theta})^2 + \frac{1}{4v} - \frac{\Theta^2\mathbf{w}_x^2}{v} - \frac{\Theta^2\mathbf{b}_x^2}{v},
\end{aligned} \tag{27}$$

which implies

$$\Theta_t \le \left(\frac{\kappa(\Theta)\Theta_x}{v}\right)_x + \frac{1}{4v} \le \left(\frac{\kappa(\Theta)\Theta_x}{v}\right)_x + C. \tag{28}$$

Define the operator $L := -\frac{\partial}{\partial_t} + \left(\frac{\kappa(\cdot)}{v}(\cdot)_x\right)_x$ and then

$$\begin{cases} L\tilde{\Theta} < 0 \quad \text{on} \quad [0,\infty)\times\mathbb{R},\\ \tilde{\Theta}\mid_{t=0} \ge 0 \quad \text{on} \quad \mathbb{R},\\ \tilde{\Theta}\mid_{x\to\infty} \ge 0 \quad \text{on} \quad [0,\infty), \end{cases} \tag{29}$$

where $\tilde{\Theta}(x,t) = Ct + \sup_{\mathbb{R}}\Theta_0 - \Theta(x,t)$, and by the comparison theorem, we obtain

$$\min_{(x,t)\in\bar{Q}_T} \tilde{\Theta}(x,t) \ge 0,$$

which implies

$$\theta(x,t) \geq C. \tag{30}$$

This completes the proof of Lemma 3. □

By using the Lemmas 2 and 3, we obtain the upper bound of $v(x,t)$.

Lemma 4. *There exists a positive constant C, it holds*

$$v(x,t) \leq C. \tag{31}$$

for all $(x,t) \in \times[0,T]$.

Proof. Thanks to Lemma 3, we have

$$\begin{aligned}
&\left|\theta^{\frac{b+1}{2}}(x,t) - \theta^{\frac{b+1}{2}}(b_k(t),t)\right| \\
&= \left|\int_{b_k(t)}^{x} \left(\theta^{\frac{b+1}{2}}\right)_x dx\right| \\
&= \left|\int_{b_k(t)}^{x} \frac{\theta^{b/2}\theta_x}{\theta^{1/2}} dx\right| \\
&\leq \frac{b+1}{2}\left(\int_k^{k+1} \frac{\theta^b\theta_x^2}{\theta^2 v} dx\right)^{\frac{1}{2}} \left(\int_k^{k+1} \theta v dx\right)^{\frac{1}{2}} \\
&\leq C\left(\int_k^{k+1} \frac{\theta^b\theta_x^2}{\theta^2 v} dx\right)^{\frac{1}{2}} \max_{x\in[k,k+1]} v^{1/2}(x,t),
\end{aligned}$$

which together with (17) yields that for any $t > 0$,

$$\max_{x\in[k,k+1]} \theta(x,t) \leq C\big(1 + \int_k^{k+1} \frac{\theta^b\theta_x^2}{\theta^2 v} dx \max_{x\in[k,k+1]} v(x,t)\big).$$

Substituting it into (25), one has

$$\begin{aligned}
v(x,t) \quad &\leq C + C\int_0^t \max_{x\in[k,k+1]} \theta(x,t) + \frac{v\mathbf{b}^2}{2} dt \\
&\leq C + C\int_0^t \left(\int_k^{k+1} \frac{\theta^b\theta_x^2}{\theta^2 v} dx + \frac{\mathbf{b}^2}{2}\right) \max_{x\in[k,k+1]} v(x,t) dt.
\end{aligned} \tag{32}$$

Then we just need to prove

$$\int_0^t \mathbf{b}^2 ds \leq C,$$

and by using the Grönwall inequality, the upper bound of $v(x,t)$ from above can be obtained. From (12) and the boundedness of $\theta(x,t) \geq C$, we have the following estimate for $b \geq 1$:

$$\begin{aligned}
\int_0^t \mathbf{b}^2 ds &\leq C + C\int_0^t \int_k^{k+1} |\mathbf{b}||\mathbf{b}_x| dxds \\
&\leq C + C\int_0^t \int_k^{k+1} v|\mathbf{b}|^2 dx \max_{[k,k+1]} \theta ds + C\int_0^t \int_k^{k+1} \frac{|\mathbf{b}_x|^2}{v\theta} dxds \\
&\leq C + C\int_0^t \max_{[k,k+1]} \theta ds + C\int_0^t \int_k^{k+1} \frac{|\mathbf{b}_x|^2}{v\theta} dxds \\
&\leq C + C\int_0^t \max_{[k,k+1]} \theta^b ds, \qquad (33)
\end{aligned}$$

on the other hand, we have

$$\begin{aligned}
&\int_0^t \max_{[k,k+1]} \theta^b ds \\
&\leq C\int_0^t \int_k^{k+1} \theta^b dxds + C\int_0^t \int_k^{k+1} (\theta^b)_x dxds \\
&\leq C\int_0^t \max_{[k,k+1]} \theta^{b-1} ds + C\int_0^t \int_k^{k+1} \frac{\theta^b \theta_x^2}{\theta^2 v} dxds \\
&\quad + \epsilon_1 \int_0^t \int_k^{k+1} \theta^b v dxds \\
&\leq C + \epsilon \int_0^t \max_{[k,k+1]} \theta^b ds + \epsilon_1 \int_0^t \max_{[k,k+1]} \theta^b \int_k^{k+1} v dxds \\
&\leq C + \epsilon_2 \int_0^t \max_{[k,k+1]} \theta^b ds, \qquad (34)
\end{aligned}$$

hence, we have

$$\int_0^t \max_{[k,k+1]} \theta^b ds \leq C.$$

Plugging it into (33), we have

$$\int_0^t \mathbf{b}^2 ds \leq C.$$

Then, combining with (32) and using the Grönwall inequality, it yields

$$v(x,t) \leq C. \qquad (35)$$

This is combined with (25) and the fact that C is dependent on T. The proof of Lemma is finished. □

3. Proof of Theorem 1

In this section, we apply the results obtained in Section 2 to prove Theorem 1. The key to studying the global existence of the solution is to obtain the high-order estimations as well as the upper bound of $\theta(x,t)$.

Lemma 5. *There exists a positive constant C such that*

$$\int_0^T \int_{\mathbb{R}} (u_x^2 + \theta^{-1}\theta_x^2) dxdt \leq C, \qquad (36)$$

for $b \geq 1$ and all $(x,t) \in \mathbb{R} \times [0,T]$.

Proof. For a small constant δ, we have

$$\begin{aligned}
\int_0^T \sup_{x\in\mathbb{R}} \theta^{b+1} dt &\leq C\int_0^T \sup_{x\in\mathbb{R}} (\theta-2)_+^{b+1} dt + C \\
&\leq C\int_0^T \int_{\mathbb{R}} (\theta-2)_+^{b} |\theta_x| dx dt + C \\
&\leq C\int_0^T \int_{(\theta>2)(t)} \theta^{b} |\theta_x| dx dt + C \\
&\leq C\int_0^T \left(\int_{(\theta>2)(t)} \theta^{b+2} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \theta^{b-2}\theta_x^2 dx\right)^{\frac{1}{2}} dt + C \\
&\leq C\int_0^T \left(\sup_{x\in\mathbb{R}} \theta^{b+1}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \theta^{b-2}\theta_x^2 dx\right)^{\frac{1}{2}} dt + C \\
&\leq \frac{1}{2}\int_0^T \sup_{x\in\mathbb{R}} \theta^{b+1} dt + \int_0^T \int_{\mathbb{R}} \theta^{b-2}\theta_x^2 dx dt + C,
\end{aligned} \tag{37}$$

here, we use the fact

$$\sup_{0\leq t\leq T} \int_{(\theta>2)(t)} \theta dx \leq C \sup_{0\leq t\leq T} \int_{\mathbb{R}} (\theta - \log\theta - 1) dx \leq C. \tag{38}$$

Combining (12), (14), (31) and (37) yields

$$\int_0^T \sup_{x\in\mathbb{R}} \theta^{b+1} dt \leq C, \tag{39}$$

which implies

$$\int_0^T \sup_{x\in\mathbb{R}} \theta dt \leq C\int_0^T \sup_{x\in\mathbb{R}} (1+\theta^{b+1}) dt \leq C, \tag{40}$$

Next, integrating the momentum equation $(2)_2$ multiplied by u with respect to x over $\mathbb{R}$, after integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\int_{\mathbb{R}} u^2 dx\right)_t + \int_{\mathbb{R}} \frac{u_x^2}{v} dx \\
\leq\ & \int_{\mathbb{R}} \frac{\theta}{v}|u_x| dx + \frac{1}{2}\int_{\mathbb{R}} |\mathbf{b}|^2 |u_x| dx \\
=\ & \int_{\mathbb{R}} \frac{\theta-1}{v}|u_x| dx + \int_{\mathbb{R}} \frac{|u_x|}{v} dx + \frac{1}{2}\int_{\mathbb{R}} |\mathbf{b}|^2 |u_x| dx \\
\leq\ & C\int_{\mathbb{R}} (\theta-1)^2 dx + C\int_{\mathbb{R}} (v-1)^2 dx + \frac{1}{2}\int_{\mathbb{R}} \frac{u_x^2}{v} dx + C\int_{\mathbb{R}} \mathbf{b}^4 dx \\
\leq\ & C\int_{(\theta>2)(t)} \theta^2 dx + C + C\int_{\mathbb{R}} (v-1)^2 dx + \frac{1}{2}\int_{\mathbb{R}} \frac{u_x^2}{v} dx + C\int_{\mathbb{R}} \mathbf{b}^4 dx \\
\leq\ & C\sup_{x\in\mathbb{R}} \theta + C + \frac{1}{2}\int_{\mathbb{R}} \frac{u_x^2}{v} dx,
\end{aligned}$$

where in the last inequality we have used (12), (14), (38). Combining this with (40) gives

$$\int_0^T \int_{\mathbb{R}} u_x^2 dx dt \leq C. \tag{41}$$

Finally, if $b \geq 1$, we have

$$\int_0^T \int_{\mathbb{R}} \theta^{-1}\theta_x^2 dx dt \leq C\int_0^T \int_{\mathbb{R}} \theta^{b-2}\theta_x^2 dx dt \leq C. \tag{42}$$

This completes the proof. □

The following lemma gives estimates on the L^2 norm of v_x.

Lemma 6. *For any $t \in [0,T]$, there exists a constant C independent of time, it holds that*

$$\sup_{0\le t<T}\int_{\mathbb{R}} v_x^2 dx + \int_0^T\int_{\mathbb{R}} \theta v_x^2 dxdt \le C. \tag{43}$$

Proof. First, integrating $(2)_2$ multiplied by $\frac{v_x}{v}$ over $\mathbb{R}$, we obtain after using $(2)_1$ that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\frac{v_x^2}{v^2}dx = \int_{\mathbb{R}}\left(\frac{\theta}{v}\right)_x\frac{v_x}{v}dx + \int_{\mathbb{R}}\mathbf{b}\mathbf{b}_x\frac{v_x}{v}dx + \int_{\mathbb{R}}u_t\frac{v_x}{v}dx\\
&= \int_{\mathbb{R}}\frac{\theta_x v_x}{v^2}dx - \int_{\mathbb{R}}\frac{\theta v_x^2}{v^3}dx + \int_{\mathbb{R}}\mathbf{b}\mathbf{b}_x\frac{v_x}{v}dx + \frac{d}{dt}\int_{\mathbb{R}}\frac{uv_x}{v}dx + \int_{\mathbb{R}}\frac{u_x^2}{v}dx\\
&\le C\int_{\mathbb{R}}\frac{\theta_x^2}{v\theta}dx + C\sup_{x\in\mathbb{R}}\mathbf{b}^2\int_{\mathbb{R}}\frac{v_x^2}{v^2}dx + \mathbf{b}^2\int_{\mathbb{R}}\frac{\mathbf{b}_x^2}{v\theta}dx - \frac{1}{2}\int_{\Omega}\frac{\theta v_x^2}{v^3}dx\\
&+\frac{d}{dt}\int_{\mathbb{R}}\frac{uv_x}{v}dx + \int_{\mathbb{R}}\frac{u_x^2}{v}dx,
\end{aligned}\tag{44}$$

which together with (12), (36) and the bounds of $v(x,t)$, which yields

$$\int_{\mathbb{R}}\frac{v_x^2}{v^2}dx + \int_{Q_T}\frac{\theta v_x^2}{v^3}dxdt \le C + \int_{Q_T}\frac{\theta_x^2}{v\theta}dxdt \le C.$$

This finishes the proof. □

Lemma 7. *For any $0 \le t \le T$, we have the following estimate*

$$\sup_{0\le t\le T}\int_{\mathbb{R}}(u_x^2 + \boldsymbol{b}_x^2 + \boldsymbol{w}_x^2)dx + \int_{Q_T}(u_t^2 + \boldsymbol{b}_t^2 + \boldsymbol{w}_t^2 + u_{xx}^2 + \boldsymbol{b}_{xx}^2 + \boldsymbol{w}_{xx}^2)dxdt \le C. \tag{45}$$

Proof. We rewrite the momentum equation in the following form:

$$u_t - \frac{u_{xx}}{v} = -\frac{u_x v_x}{v^2} - \frac{\theta_x}{v} + \frac{\theta v_x}{v^2} + \mathbf{b}\mathbf{b}_x. \tag{46}$$

Multiplying both sides of (46) by u_{xx}, $(2)_3$ and $(2)_4$ by $\mathbf{w}_{xx}$ and $\mathbf{b}_{xx}$, respectively, and integrating x over $\mathbb{R}$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2)dx + \int_{\mathbb{R}}(u_{xx}^2 + \mathbf{b}_{xx}^2 + \mathbf{w}_{xx}^2)dx\\
&\le \left|\int_{\mathbb{R}}\frac{u_x v_x}{v^2}u_{xx}dx\right| + \left|\int_{\mathbb{R}}\frac{\theta_x}{v}u_{xx}dx\right| + \left|\int_{\mathbb{R}}\frac{\theta v_x}{v^2}u_{xx}dx\right| + \left|\int_{\mathbb{R}}\mathbf{b}\mathbf{b}_x u_{xx}dx\right|\\
&+\left|\int_{\mathbb{R}}\frac{v_x\mathbf{w}_x\mathbf{w}_{xx}}{v^2}dx\right| + \left|\int_{\mathbb{R}}\mathbf{b}_x\mathbf{w}_{xx}dx\right| + \left|\int_{\mathbb{R}}\frac{u_x\mathbf{b} - \mathbf{w}_x}{v}\mathbf{b}_{xx}dx\right| + \left|\int_{\Omega}\frac{v_x\mathbf{b}_x\mathbf{b}_{xx}}{v^3}dx\right|\\
&\le \frac{1}{4}\int_{\mathbb{R}}\frac{u_{xx}^2 + \mathbf{b}_{xx}^2 + \mathbf{w}_{xx}^2}{v}dx + C\int_{\mathbb{R}}(u_x^2 v_x^2 + v_x^2\theta^2 + \theta_x^2 + \mathbf{b}^2\mathbf{b}_x^2 + v_x^2\mathbf{w}_x^2\\
&+\mathbf{b}_x^2 + \mathbf{w}_x^2 + u_x^2\mathbf{b}^2 + v_x^2\mathbf{b}_x^2)dx,
\end{aligned}\tag{47}$$

here, we have the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}}(u_x^2v_x^2+v_x^2\theta^2+\theta_x^2)dx \\
\leq\ & C\left(\sup_{x\in\mathbb{R}}u_x^2+\sup_{x\in\mathbb{R}}\theta^2\right)\int_{\mathbb{R}}v_x^2dx+\int_{\mathbb{R}}\theta_x^2dx \\
\leq\ & C\sup_{x\in\mathbb{R}}u_x^2+C\sup_{x\in\mathbb{R}}(\theta-2)_+^2+C+\int_{\mathbb{R}}\theta_x^2dx \\
\leq\ & \epsilon\int_{\mathbb{R}}u_{xx}^2dx+C(\epsilon)\int_{\mathbb{R}}u_x^2dx+C+C\int_{\mathbb{R}}\theta_x^2dx,
\end{aligned}\tag{48}$$

and in the last inequality on the above estimate, we used

$$\begin{aligned}
\sup_{x\in\mathbb{R}}u_x^2 \leq\ & \int_{\mathbb{R}}|(u_x^2)_x|dx \\
\leq\ & C\left(\int_{x\in\mathbb{R}}u_{xx}^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}u_x^2dx\right)^{\frac{1}{2}} \\
\leq\ & \epsilon\int_{x\in\mathbb{R}}u_{xx}^2dx+C(\epsilon)\int_{x\in\mathbb{R}}u_x^2dx,
\end{aligned}\tag{49}$$

and

$$\begin{aligned}
\sup_{x\in\mathbb{R}}(\theta-2)_+^2 =\ & \sup_{x\in\mathbb{R}}\left(\int_x^\infty\partial_y(\theta-2)_+(y,t)dy\right)^2 \\
\leq\ & \left(\int_{(\theta>2)(t)}|\theta_y|dy\right)^2\leq C\int_{\mathbb{R}}\theta_x^2dx,
\end{aligned}$$

and inequality (49) also holds both for $\mathbf{b}_x$ and $\mathbf{w}_x$, so $\int_{\mathbb{R}}(\mathbf{b}\mathbf{b}_x^2+\mathbf{b}_x^2v_x^2+\mathbf{b}^2u_x^2+\mathbf{w}_x^2v_x^2)dx\leq C\sup_{x\in\mathbb{R}}(\mathbf{b}_x^2+\mathbf{w}_x^2+u_x^2)$ have the same estimation. The other terms on the right-hand side of (47) could be estimated easily.

Putting (48) and (49) into (47) and choosing ϵ to be suitably small yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)dx+\int_{\mathbb{R}}(u_{xx}^2+\mathbf{b}_{xx}^2+\mathbf{w}_{xx}^2)dx \\
\leq\ & C\int_{\mathbb{R}}(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)dx+C+C\int_{\mathbb{R}}\theta_x^2dx \\
\leq\ & C\int_{\mathbb{R}}(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)dx+C+C_1\int_{\mathbb{R}}\theta^b\theta_x^2dx.
\end{aligned}\tag{50}$$

Next, motivated by [16], we integrate $(2)_2$ multiplied by $(\theta-2)_+\triangleq\sup\{\theta-2,0\}$ over $\mathbb{R}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\left(\int_{\mathbb{R}}(\theta-2)_+^2dx\right)_t+\int_{(\theta>2)(t)}\frac{\theta^b\theta_x^2}{v}dx \\
=\ & -\int_{\mathbb{R}}\frac{\theta}{v}u_x(\theta-2)_+dx+\int_{\mathbb{R}}\frac{u_x^2}{v}(\theta-2)_+dx+\int_{\mathbb{R}}\frac{\mathbf{b}_x^2}{v}(\theta-2)_+dx+\int_{\mathbb{R}}\frac{\mathbf{w}_x^2}{v}(\theta-2)_+dx \\
\leq\ & C\sup_{x\in\mathbb{R}}\theta\left(\int_{\mathbb{R}}(\theta-2)_+^2dx+\int_{\mathbb{R}}(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)dx\right).
\end{aligned}\tag{51}$$

Noticing that

$$\begin{aligned}\int_{\mathbb{R}} \theta^b \theta_x^2 dx &\le \int_{(\theta>2)(t)} \theta^b \theta_x^2 dx + \int_{(\theta\le 2)(t)} \theta^b \theta_x^2 dx \\ &\le C \int_{(\theta>2)(t)} \frac{\theta^b \theta_x^2}{v} dx + C \int_{(\theta\le 2)(t)} \frac{\theta^{b-2} \theta_x^2}{v} dx.\end{aligned}$$

We deduce from (51) that

$$\begin{aligned}&\frac{1}{2}\left(\int_{\mathbb{R}} (\theta-2)_+^2 dx\right)_t + C_2 \int_{\mathbb{R}} \theta^b \theta_x^2 dx \\ \le\ & C + C\int_{\mathbb{R}} \frac{\theta^{b-2}\theta_x^2}{v} + C\sup_{x\in\mathbb{R}} \theta \int_{\mathbb{R}} (\theta-2)_+^2 dx + C \sup_{x\in\mathbb{R}} \theta \int_{\mathbb{R}} (u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2) dx.\end{aligned}$$

Adding this to (50) together with Gronwall's inequality gives

$$\sup_{0\le t\le T} \int_{\mathbb{R}} (u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2 + (\theta-2)_+^2) dx + \int_{Q_T} (u_{xx}^2 + \mathbf{b}_{xx}^2 + \mathbf{w}_{xx}^2 + \theta^b \theta_x^2) dxdt \le C. \tag{52}$$

Taking the $L^2(\mathbb{R})$ norm of both sides of $(2)_3$ and $(2)_4$ yields that the $L^2(\mathbb{R})$ norm of $\mathbf{b}_t$ and $\mathbf{w}_t$ are bounded. This completes the proof of Lemma 7. □

Lemma 8. *There exists a positive constant C such that*

$$\sup_{0\le t\le T} \int_{\mathbb{R}} \theta_x^2 dx + \int_{Q_T} (\theta_t^2 + \theta_{xx}^2) dxdt \le C. \tag{53}$$

Proof. Multiplying $(2)_5$ by $\theta^b\theta_t$, integrating over $\mathbb{R}$, and integration by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}} \theta^b \theta_t^2 dx + \int_{\mathbb{R}} \frac{\theta^{b+1}\theta_t u_x}{v} dxdt \\ =\ & \int_{\mathbb{R}} \left(\frac{(u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2)\theta^b\theta_t}{v}\right) dx + \int_{\mathbb{R}} \theta^b\theta_t \left(\frac{\theta^b\theta_x}{v}\right)_x dx \\ =\ & -\int_{\mathbb{R}} (\theta^b\theta_t)_x \frac{\theta^b\theta_x}{v} dx + \int_{\mathbb{R}} \left(\frac{(u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2)\theta^b\theta_t}{v}\right) dx \\ =\ & -\frac{1}{2}\int_{\mathbb{R}} \frac{((\theta^b\theta_x)^2)_t}{v} dx + \int_{\mathbb{R}} \left(\frac{(u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2)\theta^b\theta_t}{v}\right) dx \\ =\ & -\frac{1}{2}\left(\int_{\mathbb{R}} \frac{(\theta^b\theta_x)^2}{v} dx\right)_t - \frac{1}{2}\int_{\mathbb{R}} \frac{(\theta^b\theta_x)^2 u_x}{v^2} dx + \int_{\mathbb{R}} \left(\frac{(u_x^2 + \mathbf{b}_x^2 + \mathbf{w}_x^2)\theta^b\theta_t}{v}\right) dx,\end{aligned} \tag{54}$$

which gives

$$
\begin{aligned}
&\int_{\mathbb{R}} \theta^b \theta_t^2 dx + \frac{1}{2}\left(\int_{\mathbb{R}} \frac{(\theta^b\theta_x)^2}{v} dx\right)_t \\
= & -\frac{1}{2}\int_{\mathbb{R}} \frac{(\theta^b\theta_x)^2 u_x}{v^2} dx - \int_{\mathbb{R}} \frac{\theta^{b+1}\theta_t u_x}{v} dxdt + \int_{\mathbb{R}} \left(\frac{(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)\theta^b\theta_t}{v}\right) dx \\
\leq & \; C\sup_{\mathbb{R}}(|u_x|\theta^{b/2}) \int_{\mathbb{R}} (\theta^{3b/4}\theta_x)^2 dx + \frac{1}{2}\int_{\mathbb{R}} \theta^b\theta_t^2 dx \\
& + C\int_{\mathbb{R}} \theta^{b+2} u_x^2 dx + C\int_{\mathbb{R}} \theta^b (u_x^4 + \mathbf{b}_x^4 + \mathbf{w}_x^4) dx \\
\leq & \; C\int_{\mathbb{R}} \theta^{2b}\theta_x^2 dx \int_{\mathbb{R}} \theta^b\theta_x^2 dx + \frac{1}{2}\int_{\mathbb{R}} \theta^b\theta_t^2 dx + C\sup_{\mathbb{R}}(\theta^{b+2} + \theta^b(u_x^2+\mathbf{b}_x^2+\mathbf{w}_x^2)) + C \\
\leq & \; C\int_{\mathbb{R}} \theta^{2b}\theta_x^2 dx \int_{\mathbb{R}} \theta^b\theta_x^2 dx + \frac{1}{2}\int_{\mathbb{R}} \theta^b\theta_t^2 dx + C\sup_{\mathbb{R}}(\theta^{2b+2} + (u_x^4+\mathbf{b}_x^4+\mathbf{w}_x^4)) + C.
\end{aligned} \tag{55}
$$

Next, it follows from (49) and (52) that

$$
\int_0^T \sup_{x\in\mathbb{R}}(u_x^4+\mathbf{b}_x^4+\mathbf{w}_x^4)dt \leq C,
$$

which together with (55), the Gronwall inequality, and (53) leads to

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}} (\theta^b\theta_x)^2 dx + \int_0^T\int_{\mathbb{R}} \theta^b\theta_t^2 dxdt \leq C, \tag{56}
$$

where we have used

$$
\sup_{x\in\mathbb{R}} \theta^{2b+2} \leq C + C\int_{\mathbb{R}} (\theta^b\theta_x)^2 dx. \tag{57}
$$

Combining (57) with (56) implies that for all $(x,t) \in \mathbb{R}\times[0,T]$,

$$
\theta(x,t) \leq C. \tag{58}
$$

Then, combining the bounds of $\theta(x,t)$ from below and (56) leads to

$$
\sup_{0\leq t\leq T}\int_{\mathbb{R}} \theta_x^2 dx + \int_0^T\int_{\mathbb{R}} \theta_t^2 dxdt \leq C. \tag{59}
$$

Finally, it follows from $(2)_5$ that

$$
\frac{\theta^b\theta_{xx}}{v} = \theta_t + \frac{\theta u_x}{v} - \frac{u_x^2}{v} - \frac{b\theta^{b-1}\theta_x^2}{v} + \frac{\theta^b\theta_x v_x}{v^2} - \frac{\mathbf{b}_x^2+\mathbf{w}_x^2}{v},
$$

which together with (30), (49), (52), (58) and (59) gives

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}} \theta_{xx}^2 dxdt & \leq C\int_0^T\int_{\mathbb{R}} (\theta_t^2 + u_x^2 + u_x^4 + \mathbf{b}_x^4 + \mathbf{w}_x^4 + \theta_x^4 + \theta_x^2 v_x^2) dxdxdt \\
& \leq C + C\int_0^T \sup_{\mathbb{R}}(\theta_x^2 + u_x^2) dt \\
& \leq C + C\int_0^T\int_{\mathbb{R}} \theta_x^2 dxdt + \frac{1}{2}\int_0^T\int_{\mathbb{R}} \theta_{xx}^2 dxdt.
\end{aligned}
$$

Combining this with (53) and (59), the proof of Lemma 8 is finished. □

This finishes the proof to Theorem 1. The pointwise bounds of $v(x,t)$ and $\theta(x,t)$ from below and above are proved in Lemmas 2–4 and 8. So the other estimates in Theorem 1 can be obtained by a standard energy method.

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