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Nonlocal Pseudo-Parabolic Equation with Memory Term and Conical Singularity: Global Existence and Blowup

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Abstract: Considered herein is the initial-boundary value problem for a semilinear parabolic equation with a memory term and non-local source $w_t - \Delta_{\mathbb{B}} w - \Delta_{\mathbb{B}} w_t + \int_0^t g(t-\tau)\Delta_{\mathbb{B}} w(\tau)d\tau = |w|^{p-1}w - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx'$ on a manifold with conical singularity, where the Fuchsian type Laplace operator $\Delta_{\mathbb{B}}$ is an asymmetry elliptic operator with conical degeneration on the boundary $x_1 = 0$. Firstly, we discuss the symmetrical structure of invariant sets with the help of potential well theory. Then, the problem can be decomposed into two symmetric cases: if $w_0 \in W$ and $\Pi(w_0) > 0$, the global existence for the weak solutions will be discussed by a series of energy estimates under some appropriate assumptions on the relaxation function, initial data and the symmetric structure of invariant sets. On the contrary, if $w_0 \in V$ and $\Pi(w_0) < 0$, the nonexistence of global solutions, i.e., the solutions blow up in finite time, is obtained by using the convexity technique.

Keywords: pseudo-parabolic equation; non-local source; cone Sobolev spaces; blow-up

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1. Introduction

In this paper, the author studied the initial boundary value problem for the following semilinear parabolic equation with non-local source and conical singularity

$$\begin{aligned} & w_t - \Delta_{\mathbb{B}} w - \Delta_{\mathbb{B}} w_t + \int_0^t g(t-\tau)\Delta_{\mathbb{B}} w(\tau)d\tau \\ & \quad = |w|^{p-1}w - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx',\ x \in \mathrm{int}\mathbb{B}, t > 0, \end{aligned} \tag{1}$$

$$\nabla_{\mathbb{B}} w \cdot \nu = 0,\ \ x \in \partial\mathbb{B}, t \geq 0, \tag{2}$$

$$w(x,0) = w_0(x),\ \ x \in \mathrm{int}\mathbb{B}, \tag{3}$$

where the initial data $w_0(x) \in \tilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \setminus \{0\}$, where $\mathbb{B} = [0,1) \times X$, $\partial\mathbb{B} = \{0\} \times X$ and X is an $(n-1)$-dimensional closed compact manifold. ν is the unit normal vector pointing toward the exterior of $\mathbb{B}$. We also assume that the volume $|\mathbb{B}| = \int_{\mathbb{B}} \frac{dx_1}{x_1}dx' < +\infty$. Moreover, the Fuchsian-type Laplace operator $\Delta_{\mathbb{B}}$ in (1) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$ and is an asymmetry elliptic operator with conical degeneration on the boundary $x_1 = 0$, and the divergence operator $\mathrm{div}_{\mathbb{B}}$ is defined by $x_1\partial_{x_1} + \partial_{x_2} + \cdots + \partial_{x_n}$. The corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$. In the neighbourhood of $\partial\mathbb{B}$, we will use the coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1, x' \in X$. The function g represents the relaxation function (or kernel of the memory term). The problem (1)–(3) can be decomposed into two symmetric cases: if $w_0 \in W$, then $w \in W$. On the contrary, if $w_0 \in V$, we have $w \in V$.

This type of equation describes a variety of important physical processes, such as the analysis of heat conduction in materials with memory and viscous flow in materials with memory, and arises in the model of phenomena in population dynamics, biological sciences [1] and nuclear sciences [2]. In recent years, the nonlinear heat equations and thin-film equations with a nonlocal source $|w|^{p-1}w - \frac{1}{|\Omega|}\int_\Omega |w|^{p-1}w dx$ have attracted many authors' attention (see [1,3–16] and papers cited therein). Not only were the existence and uniqueness results obtained, but some other properties of solutions, such as blow-up, asymptotic behavior and regularity, were also investigated. For example, Soufi, Jazar and Monneau [3] considered the initial boundary value problem for the following semilinear parabolic equation:

$$w_t = \Delta w + |w|^p - \frac{1}{|\Omega|}\int_\Omega |w|^p dx. \tag{4}$$

They constructed a symmetrical situation: for the case $1 < p \leq 2$, Soufi et al. [3] obtained a blow-up criterion by using the maximum principle. For the case $p > 2$, Jazar and Kiwan in [4] established the blow-up result in finite time with the initial energy being non-positive. Qu et al. [7,8] considered the p-Laplace equation

$$w_t - \operatorname{div}(|\nabla w|^{p-2}\nabla w) = |w|^q - \frac{1}{|\Omega|}\int_\Omega |w|^q dx,\ (x,t) \in \Omega \times (0,T) \tag{5}$$

with a nonlinear source. For Equation (5), the authors obtained the nonexistence of global sign-changing weak solutions in the case of a slow diffusive type ($p > 2$). At the same time, the fast diffusive type ($1 < p < 2$) was also studied. More recently, Guo et al. [9] established a non-extinction result for the changing sign solutions with negative initial energy. Their results gave an answer to Equation (5), unsolved in [8] for $0 < q \leq p-1$. For more works on the above problems, we refer the reader to [10–16] and references therein.

Another interesting type of model is the evolution equation with conical singularity (see [17–27]). Chen et al. established some classic inequalities on the cone Sobolev spaces in [17,18]. On this basis, they obtained the existence and blow-up results using potential well methods for the following equation on a manifold with conical singularity [19],

$$w_t - \Delta_{\mathbb{B}} w = w|w|^{p-1}, x \in \operatorname{int}\mathbb{B}, t > 0. \tag{6}$$

Later, Li et al. [20] studied the global existence and finite time blow-up of weak solutions for a class of semilinear pseudo-parabolic equation with conical singularity.

In the absence of memory term ($g \equiv 0$), the model (1) is reduced to a nonlocal semilinear equation with damping terms $\Delta_{\mathbb{B}} w_t$, which appears in the study of thermodynamics, hydrodynamics, filtration theory, etc. (see [28,29]). Regarding the qualitative properties for parabolic Equation (1) without g, many authors have focused attention on this equation for quite a long time. Di and Shang [21] considered the nonlocal nonlinear parabolic equation

$$w_t - \Delta_{\mathbb{B}} w - \Delta_{\mathbb{B}} w_t = w|w|^{p-1} - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1} w \frac{dx_1}{x_1} dx',\ x \in \operatorname{int}\mathbb{B}, t > 0 \tag{7}$$

with conical degeneration. The authors studied global existence, nonexistence and general decay of the solutions by constructing a modified method of the potential well.

Regarding the works mentioned above, we remark that for the nonlocal semilinear pseudo-parabolic equation with conical degeneration, most experts have been concerned with the global well-posedness of initial-boundary value problems without the kernel of the memory term g (see [20,21]). However, to the best of our knowledge, there is little information involving the global existence and blow-up phenomenon of the above problems with the memory term g on a manifold with conical singularity. Majdoub and Mliki in [30] considered local existence and uniqueness for the nonlinear integro-differential equations of parabolic type under the effect of an additive fractional Brownian noise with Hurst parameter $H > \max\{1/2, N/4\}$. El-Borai et al. [31] studied the existence, uniqueness and stability of solutions for the fractional parabolic integro-partial differential equations without any restrictions on the characteristic forms when the Hurst parameter of the fractional Brownian motion is less than half. Hence, the goal of the present work is to study the global existence and blow-up phenomenon for the initial-value problem (1)–(3).

In practical applications, compared with the case ($g(t) = 0$), the problem (1)–(3) can describe some physical phenomena more accurately. Naturally, we want to know what will happen to the qualitative properties of the solutions for the problem (1)–(3), and in particular whether the appearance of the memory term $g(t)$ will have and influence on the blow-up results of reference [21]. This question is a very interesting and eye-opening. In mathematics studies, the memory term $\int_0^t g(t-\tau)\Delta_{\mathbb{B}} w(\tau) d\tau$, damping term, non-local source $|w|^{p-1}w - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1} w \frac{dx_1}{x_1} dx'$ and conical singularity simultaneously appear in the initial-boundary value problem (1)–(3), which causes some difficulties in the method of the proof when we consider the qualitative theory of the solutions. In particular, the interactions among the above terms mean that it requires a rather delicate analysis. Thus, we need to utilize some new skills and methods to overcome these above difficulties. In doing so, the first intention of this paper is to prove the global existence of the solutions with the number of a priori estimates by the combination of potential well and monotonicity-compactness methods. Another goal in this paper is to investigate the finite time blow-up phenomena of the solutions by means of the perturbed energy method and integro-differential inequalities.

This article is organized as follows. In Section 2, we recall the cone Sobolev spaces, introduce some function spaces and important lemmas and state the main results of this paper. In Section 3, we give some properties associated with the potential wells and the symmetric structure of invariant sets to the problem (1)–(3), which is useful in the process of our main results. In Section 4, we give the proofs for the results of global existence and finite time blowup for our problems. Finally, the main results are summarized and we briefly illustrate the results of the paper with one example.

2. Preliminaries and Main Results

In this section, we will recall the cone Sobolev spaces and some basic notations, concepts and lemmas.

Definition 1 ([17])**.** Let $\mathbb{B} = [0,1) \times X$ be the stretched manifold of the manifold B with conical singularity. Then, the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ for $m \in \mathbb{N}, \gamma \in \mathbb{R}$ and $1 < p < \infty$, is defined as

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{v \in W_{loc}^{m,p}(\mathrm{int}\mathbb{B}) | \omega v \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\},$$

for any cut-off function ω, supported by a collar neighborhood of $(0,1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) = [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1-\omega] W_0^{m,p}(\mathrm{int}\mathbb{B}),$$

where $X^\wedge = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base X, $W_0^{m,p}(\mathrm{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\mathrm{int}\mathbb{B})$ in Sobolev spaces and $W^{m,p}(\bar{X})$ when $\bar{X}$ is a closed compact C^∞ manifold of dimension n that containing B as a submanifold with boundary.

Definition 2 ([17])**.** *Let $\mathbb{B} = [0,1) \times X$. Then $v(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < \infty$ and $\gamma \in \mathbb{R}$ if*

$$\|v(x)\|_{L_p^\gamma(\mathbb{B})}^p = \int_{\mathbb{B}} x_1^n |x_1^{-\gamma} v(x)|^p \frac{dx_1}{x_1} dx' < +\infty.$$

Lemma 1 ([18], Hölder's inequality)**.** *Let* $u(x) \in L_p^{\frac{n}{p}}(\mathbb{B}), v(x) \in L_q^{\frac{n}{q}}(\mathbb{B})$ *with* $p, q \in (1, +\infty)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_{\mathbb{B}} |u(x)v(x)| \frac{dx_1}{x_1} dx' \leq (\int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx')^{\frac{1}{p}} (\int_{\mathbb{B}} |v(x)|^q \frac{dx_1}{x_1} dx')^{\frac{1}{q}}. \tag{8}$$

For convenience, we denote

$$(u, v)_2 = \int_{\mathbb{B}} u(x)v(x) \frac{dx_1}{x_1} dx', \|u\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} = \int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx', \kappa = \int_0^t g(\tau) d\tau.$$

$$(g \circ \nabla_{\mathbb{B}} u)(t) = \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} u(t) - \nabla_{\mathbb{B}} u(\tau)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau.$$

$$\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) := \{u(x) \in \mathcal{H}_2^{1,\frac{n}{2}}(\mathbb{B}) | \nabla_{\mathbb{B}} u \cdot \nu = 0 \ on \ \partial\mathbb{B}\}$$

with the norm

$$\|u\|^2_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} = \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

The space $\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ with the norm $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$ is a Banach space, where the norm $\|u\|_{\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}$ is equivalent to the norm $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$.

Lemma 2. *Let* $u(x), v(x) \in \tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. *Then,*

$$\int_{\mathbb{B}} v \Delta_{\mathbb{B}} u \frac{dx_1}{x_1} dx' = -\int_{\mathbb{B}} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \frac{dx_1}{x_1} dx'. \tag{9}$$

Proof. See the Appendix A. □

Lemma 3 ([19], Poincaré inequality)**.** *Let* $\mathbb{B} = [0, 1) \times X$ *be a bounded subspace in* $\mathbb{R}^n_+$ *with* $X \subset \mathbb{R}^{n-1}$, *and* $1 < p < +\infty, \gamma \in \mathbb{R}$. *If* $u(x) \in \tilde{\mathcal{H}}_{p,0}^{1,\gamma}(\mathbb{B})$, *then*

$$\|u(x)\|_{L_p^\gamma(\mathbb{B})} \leq c_* \|\nabla_{\mathbb{B}} u(x)\|_{L_p^\gamma(\mathbb{B})}, \tag{10}$$

where $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$ *and the constant* c_* *depends only on* $\mathbb{B}$.

Lemma 4 ([19])**.** *For* $1 < p < \frac{2n}{n-2}$, *the embedding* $\tilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \tilde{\mathcal{H}}_{p,0}^{0,\frac{n}{p}}(\mathbb{B})$ *is continuous.*

Moreover, we give the following assumptions to the problems (1)–(3).
(A_1), so the constant p satisfies

$$1 < p < +\infty, if \ n = 1, 2;$$
$$1 < p < \frac{n+2}{n-2} = p^*, if \ n \geq 3,$$

where p^* is the critical Sobolev exponent.
(A_2) The relaxation function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g'(\tau) \leq 0, \ \ 1 - \kappa > 1 - \int_0^\infty g(\tau) d\tau = r > 0, \tag{11}$$

and

$$\int_0^\infty g(\tau)d\tau < \frac{(p+1)(p-1)}{(p+1)(p-1)+1}. \tag{12}$$

Now, we give the weak solutions of problems (1)–(3) as follows:

Definition 3. *The function $w(x,t) \in L^\infty(0,T;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $w_t(x,t) \in L^2(0,T;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ is called a weak solution of (1)–(3), if $w(x,0) = w_0(x) \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \setminus \{0\}$ and $w(x,t)$ satisfies*

$$\begin{aligned}
&(w_t,v)_2 + (\nabla_{\mathbb{B}}w, \nabla_{\mathbb{B}}v)_2 + (\nabla_{\mathbb{B}}w_t, \nabla_{\mathbb{B}}v)_2 - \left(\int_0^t g(t-\tau)\nabla_{\mathbb{B}}w(\tau)d\tau, \nabla_{\mathbb{B}}v\right)_2 \\
&\quad = \left(|w|^{p-1}w - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx', v\right)_2,
\end{aligned} \tag{13}$$

for any $v \in L^2(0,T;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$.

Considering the non-local source $|w|^{p-1}w - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx'$ of problems (1)–(3), it is easy to obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{B}} w\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} w_t\frac{dx_1}{x_1}dx' \\
&= \int_{\mathbb{B}} \Big[\Delta_{\mathbb{B}}w + \Delta_{\mathbb{B}}w_t + |w|^{p-1}w - \int_0^t g(t-\tau)\Delta_{\mathbb{B}}w(\tau)d\tau \\
&\quad - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx'\Big]\frac{dx_1}{x_1}dx' \\
&= \int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}w\cdot\nu\frac{dx_1}{x_1}dx' + \frac{d}{dt}\int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}w\cdot\nu\frac{dx_1}{x_1}dx' \\
&\quad - \int_0^t g(t-\tau)\int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}w\cdot\nu\frac{dx_1}{x_1}dx'd\tau + \int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx' \\
&\quad - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |w|^{p-1}w\frac{dx_1}{x_1}dx'\int_{\mathbb{B}}\frac{dx_1}{x_1}dx' \\
&= 0
\end{aligned} \tag{14}$$

From the above equation, the function $\int_{\mathbb{B}} w\frac{dx_1}{x_1}dx'$ is a constant for all $t \in [0,T)$, which means that

$$\Pi(w_0) = \int_{\mathbb{B}} w\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} w_0\frac{dx_1}{x_1}dx'. \tag{15}$$

Next, we introduce the following functionals and potential well sets on the cone Sobolev space $\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$:

$$\mathcal{J}(w(t)) = \frac{1}{2}(1-\kappa)\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}, \tag{16}$$

$$\mathcal{I}(w(t)) = (1-\kappa)\|\nabla_{\mathbb{B}}w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}, \tag{17}$$

$$\begin{aligned}\mathcal{E}(t) &= \int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + \frac{1}{2}(g \circ \nabla_{\mathbb{B}} w)(t) + \mathcal{J}(w(t)) \\ &= \int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + \frac{1}{2}(g \circ \nabla_{\mathbb{B}} w)(t) \\ &\quad + \frac{1}{2}(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\end{aligned} \tag{18}$$

defined on $\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. The Nehari manifold is defined as

$$\mathcal{N} = \{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}(w) = 0, \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0\}. \tag{19}$$

$$d = \inf\{\sup_{\gamma \geq 0} \mathcal{J}(\gamma w), w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})), \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0\}, \tag{20}$$

$$W = \{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}(w) > 0, \mathcal{J}(w) < d\} \cup \{0\}. \tag{21}$$

$$V = \{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}(w) < 0, \mathcal{J}(w) < d\}. \tag{22}$$

One has $0 < d = \inf_{w \in \mathcal{N}} \mathcal{J}(w)$.

Lemma 5. *Assume that* $(\mathrm{A}_1) - (\mathrm{A}_2)$ *hold. Let* $w(x,t)$ *be the solution of the problem* (1)–(3). *Then, the energy functional* $\mathcal{E}(t)$ *defined by* (18) *is non-increasing, that is,*

$$\mathcal{E}'(t) \leq 0. \tag{23}$$

Proof. See Appendix A. □

Now, for $\delta > 0$, we define some modified functionals and potential well sets as follows:

$$\mathcal{J}_\delta(w) = \frac{\delta}{2}(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}, \tag{24}$$

$$\mathcal{I}_\delta(w) = \delta(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}. \tag{25}$$

The Functions $\mathcal{J}_\delta(w), \mathcal{I}_\delta(w)$ are also associated with the integral kernel function $g(\tau)$.

$$\mathcal{N}_\delta = \left\{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}_\delta(w) = 0, \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0\right\}. \tag{26}$$

$$\Gamma(\delta) = \left(\frac{(1-\kappa)\delta}{C_*^{p+1}}\right)^{\frac{1}{p-1}}, \tag{27}$$

where $C_* = \sup\left\{\frac{\|w\|_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})}}\right\}$. For $0 < \delta < \frac{p+1}{2}$, we define

$$W_\delta = \{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}_\delta(w) > 0, \mathcal{J}_\delta(w) < d_\delta\} \cup \{0\}, \tag{28}$$

$$V_\delta = \{w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) | \mathcal{I}_\delta(w) < 0, \mathcal{J}_\delta(w) < d_\delta\}. \tag{29}$$

Remark 1. *The potential depth is given by*

$$d_\delta = \inf\{\sup_{\gamma \geq 0} \mathcal{J}_\delta(\gamma w), w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}), \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0\}.$$

Fixing $w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ with $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$, it follows from the next Lemma 6 that there exists a unique positive constant γ^* satisfying $\gamma^* w \in \mathcal{N}_\delta$, and $\mathcal{J}_\delta(\gamma w)$ takes the maximum at $\gamma = \gamma^*$. Hence, the potential depth d_δ is also be defined as

$$d_\delta = \inf_{w \in \mathcal{N}_\delta} \mathcal{J}_\delta(w).$$

We are now in a positive to state our main results as follows.

Theorem 1 (Global existence)**.** *Let p and g satisfy (A_1)- (A_2). Suppose that $w_0 \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, $w_0 \in W$ and $\Pi(w_0) > 0$. Then, problems (1)–(3) show a global weak solution*

$$w(x,t) \in L^\infty(0,\infty; \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})) \text{ with } w_t(x,t) \in L^2(0,\infty; \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$$

and $w(t) \in W$ for $0 \leq t < \infty$.

Theorem 2 (Finite time blow-up)**.** *Let the assumptions (A_1)-(A_2) hold. Suppose that $\mathcal{E}(w_0) < d$, $w_0 \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, $w_0 \in V$ and $\Pi(w_0) < 0$. Then, the weak solution $w(x,t)$ of the problems (1)–(3) blows up in finite time, that is, there exists a $T^* \in (0,\infty)$ such that*

$$\lim_{t \to T^{*-}} \int_0^t \|w(\tau)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau = +\infty.$$

3. Properties of Potential Wells and Symmetric Structure of Invariant Sets

In this section, we will give some properties about the potential wells defined above. In particular, Lemmas 6–8 are similar to the results of [20].

Lemma 6. *Assume $w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, and $\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$. Then:*

(1) $\lim_{\gamma \to 0} \mathcal{J}(\gamma w) = 0$. $\lim_{\gamma \to +\infty} \mathcal{J}(\gamma w) = -\infty$.

(2) *There exists a unique $\gamma^* = \gamma^*(w)$, such that $\frac{d}{d\gamma}\mathcal{J}(\gamma w)|_{\gamma=\gamma^*} = 0$.*

(3) *$\mathcal{J}(\gamma w)$ is increasing on $0 \leq \gamma \leq \gamma^*$, decreasing on $\gamma^* \leq \gamma \leq \infty$ and takes the maximum at $\gamma = \gamma^*$.*

(4) *$\mathcal{I}(\gamma w) > 0$ for $0 < \gamma < \gamma^*$, $\mathcal{I}(\gamma w) < 0$ for $\gamma^* < \gamma < \infty$, and $\mathcal{I}(\gamma^* w) = 0$.*

Lemma 7. *Let $w \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. Then:*

(1) *$0 < \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < \Gamma(\delta)$, then $\mathcal{I}_\delta(w) > 0$. In particular, if $0 < \|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < \Gamma(1)$, then $\mathcal{I}(w) > 0$.*

(2) *If $\mathcal{I}_\delta(w) < 0$, then $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \Gamma(\delta)$. In particular, if $\mathcal{I}(w) < 0$, then $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \Gamma(1)$.*

(3) *If $\mathcal{I}_\delta(w) = 0$ and $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$, then $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq \Gamma(\delta)$.*

(4) *If $\mathcal{I}_\delta(w) = 0$ and $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$, then $\mathcal{J}(w) > 0$ for $0 < \delta < \frac{p+1}{2}$, $\mathcal{J}(w) = 0$ for $\delta = \frac{p+1}{2}$, $\mathcal{J}(w) < 0$ for $\delta > \frac{p+1}{2}$.*

Now, we show the properties of potential wells $d(\delta)$ in the following lemmas.

Lemma 8. *$d(\delta)$ satisfies the following properties:*
(1) *$d(\delta) \geq A(\delta)\Gamma^2(\delta)$, for $A(\delta) = \left(\frac{1}{2} - \frac{1}{p+1}\right)(1-\kappa)\delta$ and $0 < \delta < \frac{p+1}{2}$. Moreover, we have*

$$d \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1-\kappa}{C_*^2}\right)^{\frac{p+1}{p-1}} > \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{r}{C_*^2}\right)^{\frac{p+1}{p-1}}. \tag{30}$$

(2) *$\lim\limits_{\delta\to 0} d(\delta) = 0$, $d\left(\frac{p+1}{2}\right) = 0$, and $d(\delta) < 0$ for $\delta > \frac{p+1}{2}$.*
(3) *$d(\delta)$ is increasing on $0 < \delta \leq 1$, decreasing on $1 \leq \delta \leq \frac{p+1}{2}$ and takes the maximum $d(1) = d$.*

Lemma 9. *Let $w \in \widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $\|\nabla_{\mathbb{B}} w\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$. Assume that $0 < \mathcal{J}(w) < d$ and $\delta_1 < \delta_2$ are two roots of equation $\mathcal{J}_\delta(w) = d(\delta)$. Then, the sign of $\mathcal{I}_\delta(w)$ is unchangeable for $\delta_1 < \delta < \delta_2$.*

Proof. Assuming that the sign of $\mathcal{I}_\delta(w)$ is changeable for $\delta_1 < \delta < \delta_2$, then we choose $\tilde{\delta} \in (\delta_1, \delta_2)$ and $\mathcal{I}_{\tilde{\delta}}(w) = 0$. Thus, by the definition of $\mathcal{N}_\delta$, we can obtain that $w \in \mathcal{N}_{\tilde{\delta}}$. Thus, we have $\mathcal{J}_\delta(w) \geq d(\tilde{\delta})$. By using Lemma 8 (3), $d(\tilde{\delta}) > d(\delta_1) = d(\delta_2) = \mathcal{J}_\delta(w)$, which contradicts with $\mathcal{J}_\delta(w) \geq d(\tilde{\delta})$. □

Lemma 10. *Suppose that $w_0 \in V$, then we have*

$$w(t) \in V,\ \forall t \in [0, T). \tag{31}$$

$$d < \left(\frac{1}{2(1-\kappa)} - \frac{1}{p+1}\right)\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}, \forall t \in [0, T). \tag{32}$$

Proof. See the Appendix A. □

Now, we give the symmetric structure of invariant sets corresponding to the problem (1)–(3).

Lemma 11. *Let $w(x,t)$ be the weak solutions of problems (1)–(3). Assume that $w \in \widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $0 < e < d$, δ_1, δ_2 satisfy $d(\delta) = e$ and $\delta_1 < \delta_2$. T is the maximal existence time.*
(1) *If $\mathcal{J}(w_0) = e$ and $\mathcal{I}(w_0) > 0$, then $w(x,t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*
(2) *If $\mathcal{J}(w_0) = e$ and $\mathcal{I}(w_0) < 0$, then $w(x,t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*

4. Proofs of the Main Results

In this section, we prove the main results by making use of the family of potential wells introduced above.

Proof of Theorem 1. Let $\{\psi_j(x)\}$ be the eigenfunctions of the Laplace operator subject to Neumann boundary value condition

$$\begin{cases} -\Delta_{\mathbb{B}}\psi_j = \lambda_j\psi_j, x \in \mathrm{int}\mathbb{B}, \\ \nabla_{\mathbb{B}}\psi_j \cdot \nu = 0, \quad x \in \partial\mathbb{B}. \end{cases} \tag{33}$$

The eigenfunctions $\{\psi_j(x)\}$ are orthogonal in $\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $L_2^{\frac{n}{2}}(\mathbb{B})$ scalar product. Now, we construct the following approximate solution $w_m(x,t)$ of problems (1)–(3):

$$w_m(x,t) = \sum_{j=1}^{m} \alpha_{jm}(t)\psi_j(x), m = 1, 2, \ldots,$$

which satisfies

$$\begin{cases} (w_{mt}, \psi_j)_2 + (\nabla_{\mathbb{B}} w_m, \nabla_{\mathbb{B}} \psi_j)_2 + (\nabla_{\mathbb{B}} w_{mt}, \nabla_{\mathbb{B}} \psi_j)_2 \\ \quad - \Big(\int_0^t g(t-\tau) \nabla_{\mathbb{B}} w_m(\tau) d\tau, \nabla_{\mathbb{B}} \psi_j \Big)_2 = (|w_m|^{p-1} w_m \\ \quad - \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} |w_m|^{p-1} w_m \frac{dx_1}{x_1} dx', \psi_j)_2, s = 1, 2, \ldots. \\ u_m(x,0) = \sum_{j=1}^{m} (w_0, \psi_j)_2 \psi_j(x). \end{cases} \tag{34}$$

It is easy to obtain $w_m(x,0) = w_{0m} \to w_0 \; in \; \widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ as $m \to +\infty$. This gives an initial value problem of an ordinary differential equation system

$$\begin{cases} \dot{\alpha}_{jm}(t) + \lambda_j \alpha_{jm}(t) + \lambda_j \dot{\alpha}_{jm}(t) - \lambda_j \int_0^t g(t-\tau) \alpha_{jm}(\tau) d\tau \\ \quad = (|w_m|^{p-1} w_m - \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} |w_m|^{p-1} w_m \frac{dx_1}{x_1} dx', \psi_j)_2, \\ \alpha_{jm}(0) = (w_0, \psi_j)_2. \end{cases} \tag{35}$$

It is easy to find that the above problem admits a local solution.Next, we show that the sign-changing weak solution $w(x,t)$ of problem (1)–(3) can be approximated by the function $w_m(x,t)$. Multiplying (34) by $\alpha'_{sm}(t)$, summing for s, and integrating from 0 to t, we obtain

$$\int_0^t \|w_{m\tau}\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} d\tau + \mathcal{J}(w_m) + \frac{1}{2}(g \circ \nabla_{\mathbb{B}} w_m)(t) = \mathcal{J}(w_m(0)). \tag{36}$$

By (34), we can find $\mathcal{J}(w_m(0)) \to \mathcal{J}(w_0)$; then, for sufficiently large m, we have

$$\int_0^t \|w_{m\tau}\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} d\tau + \mathcal{J}(w_m) + \frac{1}{2}(g \circ \nabla_{\mathbb{B}} w_m)(t) < d. \tag{37}$$

From (37) and Lemma 11, we can find $w_m(t) \in W$ for $0 \le t < \infty$ and sufficiently large m. Hence, by (37) and the definition of $\mathcal{J}(w)$, we obtain

$$\int_0^t \|w_{m\tau}\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} d\tau + \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} w_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} \mathcal{I}(w_m) < d, \tag{38}$$

for sufficiently large m and $0 \le t < \infty$. From the definition of W, we have $\mathcal{I}(w_m) > 0$, which yields

$$\int_0^t \|w_{m\tau}\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} d\tau + \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} w_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < d, \; 0 \le t < \infty, \tag{39}$$

for sufficiently large m. Then,

$$\|\nabla_{\mathbb{B}} w_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < \frac{2(p+1)}{p-1} d, \; 0 \le t < \infty, \tag{40}$$

$$\int_0^t \|w_{m\tau}\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} d\tau < d, \; 0 \le t < \infty, \tag{41}$$

$$\begin{aligned}&\int_{\mathbb{B}}|w_m|^{p-1}u_m\Big|^{\frac{p+1}{p}}\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}}|w_m|^{p+1}\frac{dx_1}{x_1}dx' = \|w_m\|^{p+1}_{L^{\frac{n}{p+1}}_{p+1}(\mathbb{B})}\\ \leq\ & C_*^{p+1}\|\nabla_{\mathbb{B}}w_m\|^{p+1}_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq C_*^{p+1}\Big(\frac{2(p+1)}{p-1}d\Big)^{\frac{p+1}{2}}.\end{aligned} \tag{42}$$

Therefore, there exists a w and a subsequence still denotes $\{w_m\}$ for which $m \to \infty$, such that

$$w_m \to w \text{ in } L^\infty(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})) \text{ weakly star and a.e. in } int\mathbb{B}\times[0,\infty),$$

$$w_{mt} \to w_t \text{ in } L^2(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})) \text{ weakly},$$

$$|w_m|^{p-1}w_m \to |w|^{p-1}w \text{ in } L^\infty(0,\infty;L^{\frac{np}{p+1}}_{\frac{p+1}{p}}(\mathbb{B})) \text{ weakly star}.$$

In (34), we fixed s, letting $m \to \infty$. Then, we have

$$\begin{aligned}&(w_t,\psi_j)_2 + (\nabla_{\mathbb{B}}w,\nabla_{\mathbb{B}}\psi_j)_2 + (\nabla_{\mathbb{B}}w_t,\nabla_{\mathbb{B}}\psi_j)_2\\ &\quad-\Big(\int_0^t g(t-\tau)\nabla_{\mathbb{B}}w(\tau)d\tau,\nabla_{\mathbb{B}}\psi_j\Big)_2 = (|w|^{p-1}w\\ &\quad-\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}}|w|^{p-1}w\frac{dx_1}{x_1}dx',\psi_j)_2,\end{aligned} \tag{43}$$

and

$$\begin{aligned}&(w_t,v)_2 + (\nabla_{\mathbb{B}}w,\nabla_{\mathbb{B}}v)_2 + (\nabla_{\mathbb{B}}w_t,\nabla_{\mathbb{B}}v)_2\\ &\quad-\Big(\int_0^t g(t-\tau)\nabla_{\mathbb{B}}w(\tau)d\tau,\nabla_{\mathbb{B}}v\Big)_2 = (|w|^{p-1}w-\\ &\quad\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}}|w|^{p-1}w\frac{dx_1}{x_1}dx',v)_2, \forall v\in L^2(0,T;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})).\end{aligned} \tag{44}$$

From (34), we obtain $w(x,0) = w_0(x)$ in $\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}), t \in (0,T)$. By density, we find that $w(x,t)\in L^\infty(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ (with $w_t(x,t)\in L^2(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$) is a global weak solution of the problems (1)–(3) with $\mathcal{I}(w)\geq 0$ and $\mathcal{J}(w) < d$ for $0 < t < \infty$. The whole proof of this theorem is completed. □

Proof of Theorem 2. Assume by contradiction that the solution $w(x,t)$ is global. Then, we consider $\Psi : [0,T] \to \mathbb{R}^+$ defined by

$$\Psi(t) = \int_0^t \|w(\tau)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}d\tau + (T-t)\|w_0\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}, \tag{45}$$

We see that $\Psi(t) > 0$ for all $t\in[0,T]$. Furthermore,

$$\begin{aligned}\Psi'(t) &= \|w(t)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} - \|w_0\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\\ &= 2\int_0^t (w(\tau),w_\tau(\tau))_2 d\tau + 2\int_0^t(\nabla_{\mathbb{B}}w(\tau),\nabla_{\mathbb{B}}w_\tau(\tau))_2 d\tau,\end{aligned} \tag{46}$$

and

$$\Psi''(t) = 2\int_{\mathbb{B}} ww_t\frac{dx_1}{x_1}dx' + 2\int_{\mathbb{B}}\nabla_{\mathbb{B}}w\nabla_{\mathbb{B}}w_t\frac{dx_1}{x_1}dx'. \tag{47}$$

Replacing v by w in (13), we obtain that

$$\begin{aligned}(w_t, w)_2 + (\nabla_{\mathbb{B}} w, \nabla_{\mathbb{B}} w_t)_2 = &-\int_{\mathbb{B}}\int_0^t g(t-\tau)\Delta w(\tau)d\tau w(t)\frac{dx_1}{x_1}dx' \\ &- \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} - \frac{\Pi(w_0)}{|\mathbb{B}|}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}.\end{aligned} \tag{48}$$

This implies

$$\begin{aligned}\Psi''(t) = &-2\int_{\mathbb{B}}\int_0^t g(t-\tau)\Delta w(\tau)d\tau w(t)\frac{dx_1}{x_1}dx' - 2\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &+ 2\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} - 2\frac{\Pi(w_0)}{|\mathbb{B}|}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\frac{dx_1}{x_1}dx'.\end{aligned} \tag{49}$$

Therefore, we have

$$\begin{aligned}&\Psi(t)\Psi''(t) - \frac{p+3}{4}\Psi'(t)^2 \\ &= 2\Psi(t)\Big[-\int_{\mathbb{B}}\int_0^t g(t-\tau)\Delta w(\tau)d\tau w(t)\frac{dx_1}{x_1}dx' \\ &\quad - \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} - \frac{\Pi(w_0)}{|\mathbb{B}|}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\frac{dx_1}{x_1}dx'\Big] \\ &\quad - \frac{p+3}{4}\Big[2\int_0^t (w(\tau), w_\tau(\tau))_2 d\tau \\ &\quad + 2\int_0^t (\nabla_{\mathbb{B}} w(\tau), \nabla_{\mathbb{B}} w_\tau(\tau))_2 d\tau\Big]^2 \\ &= 2\Psi(t)\Big[-\int_{\mathbb{B}}\int_0^t g(t-\tau)\Delta w(\tau)d\tau w(t)\frac{dx_1}{x_1}dx' - \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\ &\quad + \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} - \frac{\Pi(u_0)}{|\mathbb{B}|}\|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})}\frac{dx_1}{x_1}dx'\Big] \\ &\quad + (p+3)\Big\{H(t) - \Big[\Psi(t) - (T-t)\|w_0\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}\Big] \\ &\quad \cdot \int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}\Big\},\end{aligned} \tag{50}$$

where

$$\begin{aligned}H(t) = &\Big(\int_0^t \|w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau + \int_0^t \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau\Big) \\ &\cdot \Big(\int_0^t \|w_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau + \int_0^t \|\nabla_{\mathbb{B}} w_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau\Big) \\ &- \Big[\int_0^t (w(\tau), w_\tau(\tau))_2 d\tau + \int_0^t (\nabla_{\mathbb{B}} w(\tau), \nabla_{\mathbb{B}} w_\tau(\tau))_2 d\tau\Big]^2.\end{aligned} \tag{51}$$

Applying the Schwarz's inequalities, we have from (51) that $H(t) \geq 0$. Moreover, combining (50) and (51), we obtain

$$\Psi(t)\Psi''(t) - \frac{p+3}{4}\Psi'(t)^2 \geq \Psi(t)G(t), \tag{52}$$

where

$$
\begin{aligned}
G(t) = & \, 2\int_{\mathbb{B}} w w_t \frac{dx_1}{x_1} dx' + 2\int_{\mathbb{B}} \nabla_{\mathbb{B}} w \nabla_{\mathbb{B}} w_t \frac{dx_1}{x_1} dx' \\
& - (p+3)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau.
\end{aligned} \tag{53}
$$

Making use of (1), (18) and (53), we deduce from $\Pi(w_0) < 0$ that

$$
\begin{aligned}
G(t) \geq & \, 2\int_{\mathbb{B}} |w|^{p+1} \frac{dx_1}{x_1} dx' - 2\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1} dx' - \frac{2\Pi(w_0)}{|\mathbb{B}|} \|w\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} \\
& - 2\int_{\mathbb{B}} \int_0^t g(t-\tau) \Delta_{\mathbb{B}} w(\tau) d\tau w(t) \frac{dx_1}{x_1} dx' \\
& - (p+3)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau.
\end{aligned} \tag{54}
$$

For the forth on the right (54), we obtain

$$
\begin{aligned}
& - \int_{\mathbb{B}} \int_0^t g(t-\tau) \Delta_{\mathbb{B}} w(\tau) d\tau w(t) \frac{dx_1}{x_1} dx' \\
= & \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} w(\tau) \nabla_{\mathbb{B}} w(t) \frac{dx_1}{x_1} dx' d\tau \\
= & \int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} w(t) \nabla_{\mathbb{B}} [w(\tau) - w(t)] \frac{dx_1}{x_1} dx' d\tau \\
& + \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau.
\end{aligned} \tag{55}
$$

By (54) and (55), we find

$$
\begin{aligned}
G(t) \geq & -(p+3)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + 2\int_{\mathbb{B}} |w|^{p+1} \frac{dx_1}{x_1} dx' \\
& - 2\Big(1 - \int_0^t g(t-\tau) d\tau\Big) \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& + 2\int_0^t g(t-\tau) \int_{\mathbb{B}} \nabla_{\mathbb{B}} w(t) \nabla_{\mathbb{B}} [w(\tau) - w(t)] \frac{dx_1}{x_1} dx' d\tau \\
\geq & -(p+3)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + 2\int_{\mathbb{B}} |w|^{p+1} \frac{dx_1}{x_1} dx' \\
& - 2\Big(1 - \int_0^t g(t-\tau) d\tau\Big) \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& - 2\Big[\frac{p+1}{2} \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau \\
& + \frac{1}{2(p+1)} \int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau\Big] \\
= & -2(p+1)\Big[\frac{1}{2}\int_0^t g(t-\tau) \|\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau \\
& + \frac{1}{2}\Big(1 - \int_0^t g(t-\tau) d\tau\Big) \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1} \|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\Big] \\
& + (p-1)\Big(1 - \int_0^t g(t-\tau) d\tau\Big) \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& - (p+3)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau - \frac{1}{p+1}\int_0^t g(\tau) d\tau \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.
\end{aligned} \tag{56}
$$

Using Lemma 5, we obtain

$$\mathcal{E}(t) \leq \mathcal{E}(0).$$

Thus,

$$\begin{aligned}
G(t) \geq & -2(p+1)\mathcal{E}(0) + (p-1)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau \\
& + (p-1)\Big(1 - \int_0^t g(t-\tau)d\tau\Big)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& - \frac{1}{p+1}\int_0^t g(\tau)d\tau \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
= & \, 2(p+1)\Big\{\frac{p-1}{2(p+1)}\Big[\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& - \frac{1}{(p+1)^2}\int_0^t g(\tau)d\tau \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\Big] - \mathcal{E}(0)\Big\} \\
& + (p-1)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau \\
= & \, 2(p+1)\Big\{\frac{p-1}{2(p+1)}\Big[\Big(1 - \int_0^t g(\tau)d\tau\Big)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
& - \frac{1}{(p+1)(p-1)}\int_0^t g(\tau)d\tau \|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\Big] - \mathcal{E}(0)\Big\} \\
& + (p-1)\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau.
\end{aligned} \tag{57}$$

If $\mathcal{E}(0) < 0$, using the assumption of (A_2), it follows that

$$G(t) > \theta, \tag{58}$$

where $\theta > 0$. If $0 < \mathcal{E}(0) < d$, using the assumption of (A_2), we have

$$0 < 1 - \int_0^t g(\tau)d\tau - \frac{1}{(p+1)(p-1)}\int_0^t g(\tau)d\tau \leq r.$$

By the Lemma 10, we see that

$$d < \Big(\frac{1}{2} - \frac{1-\kappa}{p+1}\Big)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

This implies $G(t) > \theta$ for θ is a positive constant.

From what has been discussed above, we have

$$\Psi(t)\Psi''(t) - \frac{p+3}{4}\Psi'(t)^2 \geq \Psi(t)\theta. \tag{59}$$

By (45), there exists a positive constant $\lambda > 0$ such that

$$\Psi(t) \geq \lambda, \quad for \;\; t \in [0, T).$$

Then, we deduce that

$$\Psi(t)\Psi''(t) - \frac{p+3}{4}\Psi'(t)^2 \geq \lambda\theta. \tag{60}$$

Then,

$$
\begin{aligned}
&\left(\Psi^{-\frac{p-1}{4}}(t)\right)''\\
&=\Big(-\frac{p-1}{4}\Big)\Psi^{-\frac{p-1}{4}-2}(t)\Big[\Psi(t)\Psi''(t)-\frac{p+3}{4}\Psi'(t)^2\Big]\\
&\leq\Big(-\frac{p-1}{4}\Big)\lambda\theta\Psi^{-\frac{p+7}{4}}(t)<0.
\end{aligned}
\tag{61}
$$

Then, since a concave function must lie below any tangent line, we can see that

$$
\Psi^{-\frac{p-1}{4}}(t)\leq\Psi^{-\frac{p-1}{4}}(0)+\Big[\Psi^{-\frac{p-1}{4}}(0)\Big]'t \tag{62}
$$

or

$$
\Psi(t)\geq\Psi^{\frac{p+3}{p-1}}(t)\Big[\Psi(0)-\frac{p-1}{4}\Psi'(0)t\Big]^{-\frac{p-1}{4}}. \tag{63}
$$

We choose T large enough such that

$$
T\geq\frac{4\Psi(0)}{(p-1)\Psi'(0)}.
$$

Thus, from the last above inequality, it follows that the interval of existence of solutions $w(x,t)$ must be contained in $\Big[0,\frac{4\Psi(0)}{(p-1)\Psi'(0)}\Big]$. Hence, there exists $T^*\in[0,T]$, such that

$$
\Psi(t)\to+\infty\ as\ t\to T^{*-}
$$

i.e.,

$$
\int_0^t\|w(\tau)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}d\tau\to+\infty\ as\ t\to T^{*-}.
$$

This contradicts our assumption.This completes the proof of this theorem. □

5. Conclusions

In this work, we consider the initial boundary value problem for a class of pseudo-parabolic equations with power nonlinearity and nonlocal source on a manifold with conical singularity. Some new results of global existence, blow-up and blow-up time under the condition of $\mathcal{J}(w_0)<d$ are obtained. The blow-up results of problems (1)–(3) with arbitrary initial energy will be the direction of further research. From Theorems 1 and 2, we can moreover obtain the following exact conditions for the global existence of solutions for problems (1)–(3):

Let $w_0\in\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $\mathcal{J}(w_0)<d$. Then, the sign of $\mathcal{I}(w_0)$ plays a critical role in the solutions of problems (1)–(3), namely

(1) When $\mathcal{I}(w_0)\geq0$, the problem (1)–(3) admits a global weak solution

$$
w(x,t)\in L^\infty(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))\text{ with }w_t(x,t)\in L^2(0,\infty;\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})).
$$

(2) When $\mathcal{I}(w_0)<0$, there is no global weak solution for problems (1)–(3), such that the solution of problems (1)–(3) blows up in finite time in the sense of

$$
\lim_{t\to T^{*-}}\int_0^t\|w(\tau)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}d\tau=+\infty.
$$

Example 1. *As an example, with $g(t)=b/(1+t)^2(0<b<\frac{6}{7})$ and $p=2$, conditions (A_1) and (A_2) are satisfied. For the initial boundary value problems (1)–(3), we take specific functions*

$w_0(x) = -\sin x$ and let $\mathbb{B} = [0,1)$; obviously, $w_0 \in \widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. After some simple calculation, it implies $\|w_{0x}\|^2_{L^2(\mathbb{B})} = \frac{1}{2}(1+\frac{\sin 2}{2})$, $\|w_0\|^3_{L^3(\mathbb{B})} = \frac{1}{3}(\sin^2 1+4)$, $\Pi(w_0) = -2 < 0$ and

$$\begin{aligned}\mathcal{E}(w_0) &= \frac{1}{2}(1-\kappa)\|\nabla_{\mathbb{B}} w_0\|^2_{L^2(\mathbb{B})} - \frac{1}{3}\|w_0\|^3_{L^3(\mathbb{B})} \\ &= \frac{1}{4}(1-\kappa)(1+\frac{\sin 2}{2}) - \frac{1}{9}(\sin^2 1+4),\end{aligned} \tag{64}$$

$$\begin{aligned}\mathcal{I}(w_0) &= (1-\kappa)\|\nabla_{\mathbb{B}} w_0\|^2_{L^2(\mathbb{B})} - \|w_0\|^3_{L^3(\mathbb{B})} \\ &= \frac{1}{2}(1-\kappa)(1+\frac{\sin 2}{2}) - \frac{1}{3}(\sin^2 1+4).\end{aligned} \tag{65}$$

Hence, we see from (64), (65) that $\mathcal{E}(w_0) < 0$ and $\mathcal{I}(w_0) < 0$. Then, the conditions of Theorem 2 are satisfied. Hence, there exists a $T^* \in (0,\infty)$ such that $\int_0^t \|w(\tau)\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau \to \infty$ as $t \to t \to T^{*-}$.

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Appendix A

Proof of Lemma 2. Here, we first suppose $u(x), v(x) \in C_0^\infty(\mathbb{B})$. From the definition of $\Delta_{\mathbb{B}}$, it follows that

$$\begin{aligned}&\int_{\mathbb{B}} v\Delta_{\mathbb{B}} u \frac{dx_1}{x_1} dx' \\ =& \int_{\mathbb{B}} x_1\partial_{x_1}(x_1\partial_{x_1}u)\cdot v\frac{dx_1}{x_1}dx' + \int_{\mathbb{B}}(\partial^2_{x_2}u+\cdots+\partial^2_{x_n}u)\cdot v\frac{dx_1}{x_1}dx' \\ =& \int_{\mathbb{B}} \partial_{x_1}(x_1\partial_{x_1}u)\cdot v dx + \int_{\mathbb{B}}(\partial^2_{x_2}u+\cdots+\partial^2_{x_n}u)\cdot v\frac{dx_1}{x_1}dx' \\ =& \int_{\mathbb{B}} \operatorname{div}\big(x_1\partial_{x_1}u, \frac{\partial_{x_2}u}{x_1},\cdots,\frac{\partial_{x_n}u}{x_1}\big)\cdot v dx \\ =& -\int_{\mathbb{B}} \big(x_1\partial_{x_1}u, \frac{\partial_{x_2}u}{x_1},\cdots,\frac{\partial_{x_n}u}{x_1}\big)\cdot \nabla v dx \\ =& -\int_{\mathbb{B}} \big(x_1^2\partial_{x_1}u, \partial_{x_2}u,\cdots,\partial_{x_n}u\big)\cdot \nabla v \frac{dx_1}{x_1}dx' \\ =& -\int_{\mathbb{B}} \big(x_1\partial_{x_1}u, \partial_{x_2}u,\cdots,\partial_{x_n}u\big)\cdot \big(x_1\partial_{x_1}v, \partial_{x_2}v,\cdots,\partial_{x_n}v\big) \frac{dx_1}{x_1}dx' \\ =& -\int_{\mathbb{B}} \nabla_{\mathbb{B}}u\cdot\nabla_{\mathbb{B}}v\frac{dx_1}{x_1}dx'.\end{aligned} \tag{A1}$$

Finally, since $C_0^\infty(\mathbb{B})$ is dense in $\tilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, the equation above holds in case of $u(x), v(x) \in \tilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. □

Proof of Lemma 5. Replacing v by w_t in (13), it is easy to obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1} dx' \\ &\quad - \frac{1}{p+1}\int_{\mathbb{B}} |w|^{p+1}\frac{dx_1}{x_1}dx'\Big] \\ &\quad - \int_0^t g(t-\tau)\int_{\mathbb{B}} \nabla_{\mathbb{B}} w_t(t)\nabla_{\mathbb{B}} w(\tau)\frac{dx_1}{x_1}dx'd\tau = 0. \end{aligned} \tag{A2}$$

For the last term on the left side of (A2), it follows that

$$\begin{aligned} &\int_0^t g(t-\tau)\int_{\mathbb{B}} \nabla_{\mathbb{B}} w_t(t)\nabla_{\mathbb{B}} w(\tau)\frac{dx_1}{x_1}dx'd\tau \\ &= \int_0^t g(t-\tau)\int_{\mathbb{B}} \nabla_{\mathbb{B}} w_t(t)\big[\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)\big]\frac{dx_1}{x_1}dx'd\tau \\ &\quad + \int_0^t g(t-\tau)\int_{\mathbb{B}} \nabla_{\mathbb{B}} w_t(t)\nabla_{\mathbb{B}} w(t)\frac{dx_1}{x_1}dx'd\tau \\ &= -\frac{1}{2}\int_0^t g(t-\tau)\Big[\frac{d}{dt}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'\Big]d\tau \\ &\quad + \frac{1}{2}\int_0^t g(\tau)\Big[\frac{d}{dt}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'\Big]d\tau \\ &= -\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(t-\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau\Big] \\ &\quad + \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau\Big] \\ &\quad + \frac{1}{2}\int_0^t g'(t-\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau \\ &\quad - \frac{1}{2}g(t)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'. \end{aligned} \tag{A3}$$

Inserting (A3) into (A2), we obtain

$$\begin{aligned} &\frac{d}{dt}\Big[\int_0^t \|w_\tau\|^2_{\widetilde{\mathcal{H}}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})} d\tau + \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w|^2 \frac{dx_1}{x_1} dx' \\ &\quad - \frac{1}{p+1}\int_{\mathbb{B}} |w|^{p+1}\frac{dx_1}{x_1}dx'\Big] \\ &\quad + \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(t-\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau\Big] \\ &\quad - \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau\Big] \\ &\quad = \frac{1}{2}\int_0^t g'(t-\tau)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(\tau) - \nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx'd\tau \\ &\quad - \frac{1}{2}g(t)\int_{\mathbb{B}} |\nabla_{\mathbb{B}} w(t)|^2\frac{dx_1}{x_1}dx' \leq 0 \end{aligned} \tag{A4}$$

for a regular solution. The proof of the Lemma 5 is completed. □

Proof of Lemma 10. Since $w_0 \in V$, now we will prove that $w(t) \in V$ for all $t \in [0, T)$. Assume that there exists $t_0 \in [0, T)$ such that $w(t_0) \notin V$. Then, we have

$$(1-\kappa)\|\nabla_{\mathbb{B}} w(t_0)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq \|w(t_0)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

By the continuity of $w(t)$, there exists at least one $s \in (0, t_0]$ such that

$$(1-\kappa)\|\nabla_{\mathbb{B}} w(s)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \|w(s)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

Let

$$t^\star = \inf\left\{s \in (0, t_0] : (1-\kappa)\|\nabla_{\mathbb{B}} w(s)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \|w(s)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}\right\}.$$

In particular, the regularity of $w(t)$ implies that $t^\star \in (0, t_0]$. Then, we have

$$(1-\kappa)\|\nabla_{\mathbb{B}} w(t^\star)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \|w(t^\star)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

and $w(t) \in V$, for all $t \in [0, t^\star)$. Next, two cases can be considered.
First case: $\|\nabla_{\mathbb{B}} w(t^\star)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$.
In this case, by the continuity of $w(t)$, we have

$$\lim_{t \to t^{\star-}} \|\nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0. \tag{A5}$$

On the other hand, the fact that $w(t) \in V$, for all $t \in [0, t^\star)$ implies that $\|\nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$,

$$(1-\kappa)\|\nabla_{\mathbb{B}} w(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < \|w(t)\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}, t \in [0, t^\star). \tag{A6}$$

By the definition of C_*, we find

$$\begin{aligned}
\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} &= \frac{\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}}{\|\nabla_{\mathbb{B}} w\|^{p+1}_{L_2^{\frac{n}{2}}(\mathbb{B})}} \|\nabla_{\mathbb{B}} w\|^{p+1}_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\leq C_*^{p+1} \|\nabla_{\mathbb{B}} w\|^{p+1}_{L_2^{\frac{n}{2}}(\mathbb{B})}.
\end{aligned} \tag{A7}$$

Then, by (A6), (A7), we have

$$\lim_{t \to t^{\star-}} \|\nabla_{\mathbb{B}} w(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > \left(\frac{1-\kappa}{C_*^{p+1}}\right)^{\frac{1}{p-1}}.$$

This contradicts (A5).
Second case: $\|\nabla_{\mathbb{B}} w(t^\star)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$.
By using (20), we have $\mathcal{J}(w(t^\star)) \geq d$, which contradicts the fact that $\mathcal{J}(w(t)) \leq \mathcal{J}(w_0) < d$. Hence, in either case, we conclude that $w(t) \in V$, for all $t \in [0, T)$. Since

$$\mathcal{J}(\gamma w) = \frac{1}{2}\gamma^2(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{\gamma^{p+1}}{p+1}\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

We obtain

$$\frac{d}{d\gamma}\mathcal{J}(\gamma w) = \gamma(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \gamma^p\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

$$\frac{d^2}{d\gamma^2}\mathcal{J}(\gamma w) = (1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - p\gamma^{p-1}\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.$$

Let $\frac{d}{d\gamma}\mathcal{J}(\gamma w) = 0$, which implies

$$\bar{\gamma}_1 = 0, \bar{\gamma}_2 = \left(\frac{(1-\kappa)\|\nabla_{\mathbb{B}} w\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}}\right)^{\frac{1}{p-1}}.$$

An elemeatary calculation shows

$$\frac{d^2}{d\gamma^2}\mathcal{J}(\bar{\gamma}_1 w) > 0, \frac{d^2}{d\gamma^2}\mathcal{J}(\bar{\gamma}_2 w) < 0.$$

So, we have

$$\sup_{\gamma\geq 0}\mathcal{J}(\gamma w) = \mathcal{J}(\bar{\gamma}_2 w) = \frac{p-1}{2(p+1)}\frac{\|\nabla_{\mathbb{B}} w\|^{\frac{2(p+1)}{p-1}}_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\|w\|^{\frac{2(p+1)}{p-1}}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}}.$$

By $\mathcal{I}(w) < 0$, we have

$$\begin{aligned} d \leq \sup_{\gamma\geq 0}\mathcal{J}(\gamma w) = \mathcal{J}(\bar{\gamma}_2 w) &= \left(\frac{1}{2} - \frac{1-\kappa}{p+1}\right)\frac{\left[(1-\kappa)\|\nabla_{\mathbb{B}} w\|^{p+1}\right]^{\frac{2}{p-1}}_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\|w\|^{\frac{2(p+1)}{p-1}}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}} \\ &< \left(\frac{1}{2(1-\kappa)} - \frac{1}{p+1}\right)\|w\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}. \end{aligned} \tag{A8}$$

□

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