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Repdigits as Sums of Four Tribonacci Numbers

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Abstract: In this paper, we show that 66666 is the largest repdigit expressible as the sum of four tribonacci numbers. We used Binet's formula, Baker's theory, and a reduction method during the proving procedure. We also used the periodic properties of tribonacci number modulo 9 to deal with three individual cases.

Keywords: tribonacci numbers; repdigits; diophantine equations; Binet's formula; linear form in logarithms

MSC: 11B39; 11J86

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1. Introduction

A palindromic number is a number that has reflectional symmetry across a vertical axis. A repdigit is a palindromic number N that only has one distinct digit when it is represented in base 10. That is, N has the form

$$d\left(\frac{10^l - 1}{9}\right),$$

for some positive integers d and l with $1 \leq d \leq 9$ and $l \geq 1$.

The problem of finding all repdigits that are perfect powers was posed by Obláth [1] and settled in 1999 by Bugeaud and Mignotte [2]. In 2000, Luca [3] proved that the largest repdigits in Fibonacci and Lucas sequences were $F_{10} = 55$ and $L_5 = 11$. Afterward, finding repdigits in recurrence sequences has drawn much attention in the literature. For instance, Dıaz-Alvarado and Luca [4] proved that the largest Fibonacci number that can be represented as a sum of two repdigits is $F_{20} = 6765 = 6666 + 99$. Luca [5] answered the question of finding all repdigits as sums of three Fibonacci numbers. Luca, Normenyo, and Togbé [6] obtained all repdigits expressible as sums of three Lucas numbers. They also solved a similar problem involving three Pell numbers in [7]. Ddamulira [8] identified all repdigits as sums of three balancing numbers. In [9], the authors confirmed a conjecture by Luca [5]. More precisely, they determined all repdigits expressible as sums of four Fibonacci or Lucas numbers. Afterward, in [10], the authors obtained analogous results for Pell numbers. Keskin and Erduvan [11] tackled the same problem with four balancing numbers.

Recently, Ddamulira [12] found all repdigits as sums of three Padovan numbers. As far as we know, this is the only reference involving repdigits that are expressible as sums of more than two numbers in a high-order recurrence sequence. Compared to the second-order recurrence sequence, it is difficult to solve a similar question with a high-order recurrence sequence. Therefore, it is interesting to find all repdigits that are sums of four numbers in some other third-order recurrence sequence.

In this paper, we investigate the presence of repdigits as sums of four tribonacci numbers. More precisely, we prove the following theorem.

Theorem 1. *All non-negative integer solutions* $(m_1, m_2, m_3, m_4, N, d, \ell)$ *of the Diophantine equation*

$$N = T_{m_1} + T_{m_2} + T_{m_3} + T_{m_4} = d\left(\frac{10^l - 1}{9}\right) \quad \textit{with} \quad d \in \{1, \cdots, 9\} \tag{1}$$

have

$$N \in \left\{ \begin{array}{l} 4,5,6,7,8,9,11,22,33,44,55,66,77,88,99, \\ 111,333,555,666,999,2222,3333,66666 \end{array} \right\}.$$

Our paper is organized as follows. In the following section, we recall some important results that are useful for the proof of our main theorem. We use them in Section 3 to prove Theorem 1. During the proof, first, we use Baker's method several times to obtain a bound of m_1, which is too large to conduct a brute force search. We then apply the reduction method of de Weger several times to find a very low bound for m_1, which enables us to run a simple computer program in Mathematica to find the small solutions. It is worth mentioning that the reduction method is invalid in three cases during the computations. For these cases, we use periodic properties of $\{T_n\} \pmod 9$ to reach the contradictions.

2. Auxiliary Results

We use a definition of tribonacci numbers (see [13]) with little difference from the common one that starts from $0, 0, 1$. Let $\{T_n\}_{n \geq 0}$ be the tribonacci sequence satisfying the recurrence relation $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ with initial conditions $T_0 = 0$ and $T_1 = T_2 = 1$. The first few terms of this sequence are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 10609, 19513 \ldots.$$

Its characteristic equation, $z^3 - z^2 - z - 1 = 0$, has one real root α and two complex roots β and $\gamma = \overline{\beta}$. In 1982, Spickerman [14] found the following "Binet-like" formula:

$$T_n = a\alpha^n + b\beta^n + c\gamma^n, \quad \text{for all} \quad n \geq 0, \tag{2}$$

where

$$a = \frac{1}{-\alpha^2 + 4\alpha - 1}, \; b = \frac{1}{-\beta^2 + 4\beta - 1}, \; c = \frac{1}{-\gamma^2 + 4\gamma - 1} = \overline{b}.$$

Numerically, we have $1.83 \leq \alpha \leq 1.84$, $0.73 \leq |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} \leq 0.74$, $0.33 \leq a \leq 0.34$ and $0.25 \leq |b| = |c| \leq 0.26$. It follows that the complex conjugate roots β and γ have little influence on the right side of Equation (2), setting

$$e(n) := T_n - a\alpha^n = b\beta^n + c\gamma^n \quad \text{then} \quad |e(n)| \leq \frac{1}{\alpha^{\frac{n}{2}}} \tag{3}$$

holds for all $n \geq 1$. In addition, it is known that:

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{4}$$

Let $\eta^{(1)} = \eta$ be an algebraic number of degree d with a minimal primitive polynomial

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the positive integer a_0 is the leading coefficient and $\eta^{(i)} (i = 2, \ldots, d)$ are the conjugates of η. The logarithmic height of η is given by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\left(\max\left\{ \left|\eta^{(i)}\right|, 1 \right\} \right) \right). \tag{5}$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following properties of $h(\eta)$ will be used in the next section.

Lemma 1 ([15]). *Let a and b be algebraic numbers. Then,*

$$\begin{aligned} h(a \pm b) &\leq h(a) + h(b) + \log 2, \\ h(ab^{\pm}) &\leq h(a) + h(b), \\ h(a^r) &= |r| h(a) \quad (r \in \mathbb{Z}). \end{aligned}$$

We recall a variation of a result of Matveev [16] due to Bugeaud, Mignotte, and Siksek [17], which helps us to give an upper bound of m_1.

Lemma 2 ([17], Theorem 9.4). *Let $\eta_1, \dots, \eta_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree $D_{\mathbb{K}}$, $b_1, \dots, b_t$ be nonzero integers and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2 (1 + \log D_{\mathbb{K}})(1 + \log B) A_1 A_2 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad \textit{for all} \quad i = 1, \dots, t.$$

To reduce the bound we obtain from Lemma 2, we introduce a modified version of the Baker and Davenport reduction method that appears in [18]. Let $v_1, v_2, \beta \in \mathbb{R}$, and let $x_1, x_2 \in \mathbb{Z}$ be unknown. Let

$$\Lambda = \beta + x_1 v_1 + x_2 v_2. \tag{6}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0 be a (large) positive constant. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{7}$$

$$X \leq X_0. \tag{8}$$

When $\beta = 0$ in (6), we have

$$\Lambda = x_1 v_1 + x_2 v_2.$$

Put $v = -v_1/v_2$. Let the continued fraction expansion of v be given by

$$[a_0, a_1, a_2, \dots],$$

and let the k-th convergent of v be p_k/q_k for $k = 0, 1, 2, \dots$. We may assume without loss of generality that $|v_1| < |v_2|$ and that $x_1 > 0$. We have the following result.

Lemma 3 ([18], Lemma 3.1). *(1) If (7) and (8) hold for x_1, x_2 with $X \geq X^*$, then $(-x_2, x_1) = (p_k, q_k)$ for an index k that satisfies*

$$k \leq -1 + \frac{\log(1 + X_0\sqrt{5})}{\log(\frac{1+\sqrt{5}}{2})} := Y_0. \tag{9}$$

Moreover, the partial quotient a_{k+1} satisfies

$$a_{k+1} > -2 + \frac{|v_2| \exp(\delta q_k)}{c q_k}. \tag{10}$$

(2) *If for some k with $q_k \geq X^*$, we have*

$$a_{k+1} > \frac{|v_2| \exp(\delta q_k)}{c q_k}, \tag{11}$$

then (7) *holds for* $(-x_2, x_1) = (p_k, q_k)$.

Lemma 4 ([18], Lemma 3.2)**.** *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (7) *and* (8) *hold for x_1, x_2 and $\beta = 0$, then*

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|v_2|}\right). \tag{12}$$

When $\beta v_1 v_2 \neq 0$ in (6), put $v = v_1/v_2$ and $\psi = \beta/v_2$. Then we have

$$\frac{\Lambda}{v_2} = \psi - x_1 v + x_2.$$

Let p/q be a convergent of v with $q > X_0$. For a real number x, we define

$$\|x\| = \min\{|x-n|, n \in \mathbb{Z}\}$$

be the distance from x to the nearest integer. We have the following Davenport lemma.

Lemma 5 ([18], Lemma 3.3)**.** *Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solution of (7) *and* (8) *satisfy*

$$Y < \frac{1}{\delta} \log\left(\frac{q^2 c}{|v_2| X_0}\right).$$

Finally, the following result of Le [19] will help us to deal with some inequalities involving logarithms.

Lemma 6 ([19])**.** *Let $f(y) \in \mathbb{R}[x]$ be a polynomial with degree n and $f^{(m)}(y)$ be its m-th derivative. If there is a real number x_0 satisfying*

$$x_0 > \max\Big(0, f(\log x_0), f^{(1)}(\log x_0), \cdots, f^{(n)}(\log x_0)\Big),$$

then $x - f(\log x) > 0$ when $x \geq x_0$.

3. Proof of Theorem 1

3.1. Bounding the Variables

We assume that $m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0$. From (1) and (4), we have

$$\alpha^{m_1-2} \leq T_{m_1} \leq T_{m_1} + T_{m_2} + T_{m_3} + T_{m_4} = d\left(\frac{10^\ell - 1}{9}\right) \leq 10^\ell \tag{13}$$

and

$$10^{\ell-1} \leq d\left(\frac{10^\ell - 1}{9}\right) \leq T_{m_1} + T_{m_2} + T_{m_3} + T_{m_4} \leq 4T_{m_1} < \alpha^{m_1+2}, \tag{14}$$

where we use $4 < \alpha^3$. Thus,

$$\ell \geq (m_1 - 2)\frac{\log \alpha}{\log 10} \tag{15}$$

and

$$\ell - 1 \leq (m_1 + 2)\frac{\log \alpha}{\log 10}. \tag{16}$$

Since

$$\frac{1}{6} < \frac{\log \alpha}{\log 10} = 0.264649\ldots < \frac{1}{3},$$

we have

$$\frac{m_1 - 2}{6} < \ell < \frac{m_1 + 5}{3}$$

by (15) and (16). Therefore, if $m_1 \leq 330$, then $1 \leq \ell \leq 111$. Running a Mathematica program in the range $0 \leq m_4 \leq m_3 \leq m_2 \leq m_1 \leq 330$, $1 \leq d \leq 9$, and $1 \leq \ell \leq 111$, we find no other solutions except those listed in Theorem 1.

From now on, we assume that $m_1 > 330$. By using (3), Equation (1) can be written as

$$a\alpha^{m_1} + e(m_1) + a\alpha^{m_2} + e(m_2) + a\alpha^{m_3} + e(m_3) = d\left(\frac{10^\ell - 1}{9}\right). \tag{17}$$

We then consider (17) in four different cases as follows.

3.2. Case 1

Identity (17) is equivalent to

$$a\alpha^{m_1} - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - a(\alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4}) - e(m_1) - e(m_2) - e(m_3) - e(m_4). \tag{18}$$

Taking the absolute value on both sides of (18), estimate the size of the right-hand side of the equality. We have

$$\begin{aligned}
\left| a\alpha^{m_1} - \frac{d \cdot 10^\ell}{9} \right| &\leq \frac{d}{9} + a(\alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4}) + |e(m_1)| + |e(m_2)| + |e(m_3)| + |e(m_4)| \\
&< 1 + 3a\alpha^{m_2} + 4\alpha^{-m_4/2} \\
&< 3a + 3a\alpha^{m_2} + 12a\alpha^{(2m_2 - m_4)/2} \\
&< 18a\alpha^{m_2}.
\end{aligned} \tag{19}$$

To apply Lemma 2. We divide (through (19)) by $a\alpha^{m_1}$ to obtain

$$\left| 10^\ell \cdot \alpha^{-m_1}\left(\frac{d}{9a}\right) - 1 \right| < \frac{18}{\alpha^{m_1 - m_2}}. \tag{20}$$

Then, let

$$\Lambda_1 := 10^\ell \cdot \alpha^{-m_1}\left(\frac{d}{9a}\right) - 1. \tag{21}$$

It is sufficient to check that $\Lambda_1 \neq 0$. Suppose that $\Lambda_1 = 0$, then we have

$$\frac{10^\ell \cdot d}{9} = a\alpha^{m_1}. \tag{22}$$

Now, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. Since $\Lambda_1 = 0$, we have $\sigma(\Lambda_1) = 0$. Thus, conjugating the relation (22) under σ, and taking absolute values on both sides, we have

$$\frac{10^\ell \cdot d}{9} = |\sigma(a\alpha^{m_1})| = b|\beta|^{m_1} < |b| < \frac{1}{2},$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_1 \neq 0$.

Then, we apply Lemma 2 with the data

$$\eta_1 := 10, \quad \eta_2 := \alpha, \quad \eta_3 := \frac{d}{9a}, \quad b_1 := \ell, \quad b_2 := -m_1, \quad b_3 := 1, \quad t := 3.$$

The minimal polynomial of a is $44x^3 - 2x - 1$ and has roots a, b, c. Since $|b| = |c| < |a| < 1$, we have

$$h(a) = \frac{1}{3}\log 44.$$

Let $\mathbb{K} := \mathbb{Q}(\alpha)$. Then $D_{\mathbb{K}} = 3$ because $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$. Since $\max\{\ell, m_1, 1\} \leq m_1$, we take $B := m_1$. Further, the minimal polynomial of α over $\mathbb{Z}$ is $x^3 - x^2 - x - 1$ has roots α, β, γ with $1.83 \leq \alpha \leq 1.84$ and $|\beta| = |\gamma| < 1$. Thus, we have $h(\alpha) = \frac{1}{3}\log\alpha$. Since $h(10) = \log 10$ and

$$h(\eta_3) \leq h(d) + h(9) + h(a) \leq 4\log 3 + \frac{1}{3}\log 44 < 6\log 3,$$

we take $A_1 := 3\log 10$, $A_2 := \log\alpha$ and $A_3 := 18\log 3$. Then, from Lemma 2, (21) is bounded below by

$$\begin{aligned}\log|\Lambda_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2(1+\log 3)(1+\log m_1)(3\log 10)(\log\alpha)(18\log 3)\\ &> -7.39 \times 10^{14}(\log m_1)\log\alpha.\end{aligned}$$

Combining the above inequality with (20), we have

$$m_1 - m_2 \leq 7.41 \times 10^{14}\log m_1.$$

3.3. Case 2

Identity (17) is also equivalent to

$$a(\alpha^{m_1} + \alpha^{m_2}) - \frac{d \cdot 10^\ell}{9} = -\frac{d}{9} - a(\alpha^{m_3} + \alpha^{m_4}) - e(m_1) - e(m_2) - e(m_3) - e(m_4).$$

Thus, it follows that

$$\begin{aligned}\left|a(\alpha^{m_1} + \alpha^{m_2}) - \frac{d \cdot 10^\ell}{9}\right| &\leq \frac{d}{9} + a(\alpha^{m_3} + \alpha^{m_4}) + |e(m_1)| + |e(m_2)| + |e(m_3)| + |e(m_4)|\\ &< 1 + 2a\alpha^{m_3} + 4\alpha^{-m_4/2}\\ &< 3a + 2a\alpha^{m_3} + 12a\alpha^{(2m_3-m_4)/2}\\ &< 17a\alpha^{m_3}.\end{aligned} \tag{23}$$

As before, we divide both sides of (23) by $a(\alpha^{m_1} + \alpha^{m_2})$ to have

$$\left|10^\ell\alpha^{-m_2}\left(\frac{d}{9a(1+\alpha^{m_1-m_2})}\right) - 1\right| < \frac{17\alpha^{m_3-m_1}}{1+\alpha^{m_2-m_1}},$$

which yields

$$\left|10^\ell\alpha^{-m_2}\left(\frac{d}{9a(1+\alpha^{m_1-m_2})}\right) - 1\right| < \frac{17}{\alpha^{m_1-m_3}}. \tag{24}$$

Let

$$\Lambda_2 := 10^{\ell}\alpha^{-m_2}\left(\frac{d}{9a(1+\alpha^{m_1-m_2})}\right) - 1. \tag{25}$$

Again, we need to check that $\Lambda_2 \neq 0$. Suppose that $\Lambda_2 = 0$, then

$$a(\alpha^{m_1} + \alpha^{m_2}) = \frac{10^{\ell}\cdot d}{9}. \tag{26}$$

We consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha,\beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. We have $\sigma(\Lambda_2) = 0$ because $\Lambda_2 = 0$. Thus, conjugating the relation (26) under σ, and taking the absolute values on both sides, we have

$$\frac{10^{\ell}\cdot d}{9} = |\sigma(a(\alpha^{m_1}+\alpha^{m_2}))| = |b|(|\beta|^{m_1}+|\beta|^{m_2}) < 2|b| < 1,$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_2 \neq 0$. Hence, we apply Lemma 2 with the data

$$\eta_1 := 10, \quad \eta_2 := \alpha, \quad \eta_3 := \frac{d}{9a(1+\alpha^{m_1-m_2})},$$
$$b_1 := \ell, \quad b_2 := -m_2, \quad b_3 := 1, \quad t := 3.$$

Let $\mathbb{K} := \mathbb{Q}(\alpha)$. Then $D_{\mathbb{K}} = 3$ because $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$. Since $\max\{\ell, m_1, 1\} \leq m_1$, we take $B := m_1$. Since

$$\begin{aligned} h(\eta_3) &\leq h(d) + h(9) + h(a) + h(1+\alpha^{m_1-m_2}) \\ &\leq 4\log 3 + \frac{1}{3}\log 44 + (m_1 - m_2)\log\alpha + \log 2 \\ &< 4.52\times 10^{14}\log m_1, \end{aligned}$$

we can take $A_1 := 3\log 10$, $A_2 := \log\alpha$ and $A_3 := 1.36\times 10^{15}\log m_1$. So, Lemma 2 reveals that (25) is bounded below by

$$\begin{aligned} \log|\Lambda_2| >& -1.4\times 30^6\times 3^{4.5}\times 3^2(1+\log 3)(1+\log m_1)(3\log 10)(\log\alpha) \\ &\times 1.36\times 10^{15}\log m_1 \\ >& -5.09\times 10^{28}(\log m_1)^2\log\alpha. \end{aligned}$$

Combining the above inequality with (24), we have

$$m_1 - m_3 \leq 5.11\times 10^{28}(\log m_1)^2.$$

3.4. Case 3

Rewriting (17) as below:

$$a(\alpha^{m_1} + a\alpha^{m_2} + \alpha^{m_3}) - \frac{d\cdot 10^{\ell}}{9} = -\frac{d}{9} - a\alpha^{m_4} - e(m_1) - e(m_2) - e(m_3) - e(m_4).$$

Thus, it follows that

$$\begin{aligned} \left|a(\alpha^{m_1} + \alpha^{m_2} + a\alpha^{m_3}) - \frac{d\cdot 10^{\ell}}{9}\right| &\leq \frac{d}{9} + a\alpha^{m_4} + |e(m_1)| + |e(m_2)| + |e(m_3)| + |e(m_4)| \\ &< 1 + a\alpha^{m_4} + 4\alpha^{-\frac{m_4}{2}} \\ &< 3a + a\alpha^{m_4} + 12a\alpha^{-\frac{m_4}{2}} \\ &< 16a\alpha^{m_4}, \end{aligned} \tag{27}$$

Dividing through (27) by $a(\alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3})$, we have

$$\left|10^{\ell}\alpha^{-m_3}\left(\frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})}\right)-1\right| < \frac{16\alpha^{m_4-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}}.$$

This means that

$$\left|10^{\ell}\alpha^{-m_3}\left(\frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})}\right)-1\right| < \frac{16}{\alpha^{m_1-m_4}}. \tag{28}$$

Thus,we put

$$\Lambda_3 := 10^{\ell}\alpha^{-m_3}\left(\frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})}\right)-1. \tag{29}$$

Then, we need to verify that $\Lambda_3 \neq 0$. Suppose that $\Lambda_3 = 0$, then we have

$$a(\alpha^{m_1}+\alpha^{m_2}+\alpha^{m_3}) = \frac{10^{\ell}\cdot d}{9}. \tag{30}$$

To see that this is not true, we again consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha,\beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. Since $\Lambda_3 = 0$, we have $\sigma(\Lambda_3) := 0$. Thus, conjugating the relation (30) under σ, and taking absolute values on both sides, we have

$$\frac{10^{\ell}\cdot d}{9} = |\sigma(a(\alpha^{m_1}+\alpha^{m_2}\alpha^{m_3}))| = |b|(|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}) < 3|b| < \frac{3}{2},$$

which is false for $\ell \geq 1$ and $d \geq 1$. Therefore, $\Lambda_3 \neq 0$. So, we apply Lemma 2 with the data

$$\eta_1 := 10, \quad \eta_2 := \alpha, \quad \eta_3 := \frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})},$$
$$b_1 := \ell, \quad b_2 := -m_3, \quad b_3 := 1, \quad t := 3.$$

Since $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$, we take the field $\mathbb{K} := \mathbb{Q}(\alpha)$ with degree $D_{\mathbb{K}} := 3$. Since $\max\{\ell, m_1, 1\} \leq m_1$, we take $B := m_1$. Further,

$$\begin{aligned} h(\eta_3) &\leq h(d) + h(9) + h(a) + h\big(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3}\big) \\ &\leq 4\log 3 + \frac{1}{3}\log 44 + (m_1-m_3)\log\alpha + (m_2-m_3)\log\alpha + 2\log 2 \\ &\leq 4\log 3 + \frac{1}{3}\log 44 + ((m_1-m_2)+2(m_2-m_3))\log\alpha + 2\log 2 \\ &< 9.36\times 10^{28}(\log m_1)^2. \end{aligned}$$

Thus, we can take $A_1 := 3\log 10$, $A_2 := \log\alpha$, and $A_3 := 2.81\times 10^{29}(\log m_1)^2$. According to Lemma 2, (29) is bounded below by

$$\begin{aligned} \log|\Lambda_2| >& -1.4\times 30^6\times 3^{4.5}\times 3^2(1+\log 3)(1+\log m_1)(3\log 10)(\log\alpha) \\ &\times 2.81\times 10^{29}(\log m_1)^2 \\ >& -1.05\times 10^{43}(\log m_1)^3\log\alpha. \end{aligned}$$

By comparing the above inequality with the right-hand side of (28), we have

$$m_1 - m_4 \leq 1.07\times 10^{43}(\log m_1)^3.$$

3.5. Case 4

Equation (17) is equivalent to

$$a(\alpha_1^m + \alpha_2^m + \alpha_3^m + \alpha_4^m) - \frac{d \cdot 10^\ell}{9} = -\frac{1}{9} - e(m_1) - e(m_2) - e(m_3) - e(m_4). \tag{31}$$

Taking the absolute value on both sides, (31) shows that

$$\begin{aligned} \left| a(\alpha_1^m + \alpha_2^m + \alpha_3^m + \alpha_4^m) - \frac{d \cdot 10^\ell}{9} \right| &\leq \frac{1}{9} + |e(m_1)| + |e(m_2)| + |e(m_3)| + |e(m_4)| \\ &< 1 + 4\alpha^{-\frac{m_4}{2}} \\ &< 3a + 12a\alpha^{-\frac{m_4}{2}} \\ &< 15a. \end{aligned} \tag{32}$$

Dividing both sides by $a(\alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4})$, we have

$$\left| 10^\ell \alpha^{-m_4} \left(\frac{d}{9a(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4})} \right) - 1 \right| < \frac{15a\alpha^{-m_1}}{a(\alpha^{m_2-m_1} + \alpha^{m_3-m_1}\alpha^{m_4-m_1} + 1)},$$

which implies

$$\left| 10^\ell \alpha^{-m_4} \left(\frac{d}{9a(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4})} \right) - 1 \right| < \frac{15}{\alpha^{m_1}}. \tag{33}$$

Put

$$\Lambda_4 := 10^\ell \alpha^{-m_4} \left(\frac{d}{9a(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4})} \right) - 1.$$

As before, one can justify that $\Lambda_4 \neq 0$. Suppose that $\Lambda_4 = 0$, then we have

$$a(\alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4}) = \frac{10^\ell \cdot d}{9}. \tag{34}$$

To see that this is not true, we consider the $\mathbb{Q}$-automorphism σ of the Galois extension $\mathbb{Q}(\alpha, \beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) := \beta$ and $\sigma(\beta) := \alpha$. Now, since $\Lambda_4 = 0$, we have $\sigma(\Lambda_4) := 0$. Thus, conjugating the relation (34) under σ, and taking absolute values on both sides, we have

$$\frac{10^\ell \cdot d}{9} = |\sigma(a(\alpha^{m_1} + \alpha^{m_2}\alpha^{m_3} + \alpha^{m_4}))| = |b|(|\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4}) < 4|b| < 2,$$

which is false for $\ell \geq 2$ and $d \geq 1$. Therefore, $\Lambda_4 \neq 0$.

Thus, we apply Lemma 2 with the data

$$\eta_1 := 10, \quad \eta_2 := \alpha, \quad \eta_3 := \frac{d}{9a(1 + \alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4})},$$
$$b_1 := \ell, \quad b_2 := -m_4, \quad b_3 := 1, \quad t := 3.$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\alpha)$. We can take the field $\mathbb{K} := \mathbb{Q}(\alpha)$ with degree $D_{\mathbb{K}} := 3$. As $\max\{\ell, m_1, 1\} \leq m_1$, we can also take $B := m_1$. Further,

$$\begin{aligned} h(\eta_3) &\leq h(d) + h(9) + h(a) + h(1 + \alpha^{m_1 - m_4} + \alpha^{m_2 - m_4} + \alpha^{m_3 - m_4}) \\ &\leq 4\log 3 + \frac{1}{3}\log 17 + (m_1 - m_4)\log\alpha + (m_2 - m_4)\log\alpha + (m_3 - m_4)\log\alpha + 3\log 2 \\ &\leq 4\log 3 + \frac{1}{3}\log 17 + ((m_1 - m_2) + 2(m_2 - m_3)\log\alpha + 3(m_3 - m_4))\log\alpha + 3\log 2 \\ &< 3.92 \times 10^{43}(\log m_1)^3. \end{aligned}$$

Thus, we choose $A_1 := 3\log 10$, $A_2 := \log\alpha$ and $A_3 := 1.18 \times 10^{44}(\log m_1)^3$. From Lemma 2, we obtain

$$\begin{aligned} \log|\Lambda_4| > &- 1.4 \times 30^6 \times 3^{4.5} \times 3^2(1 + \log 3)(1 + \log m_1)(3\log 10)(\log\alpha) \\ &\times 1.18 \times 10^{44}(\log m_1)^3 \\ > &- 4.41 \times 10^{57}(\log m_1)^4 \log\alpha, \end{aligned}$$

which, combined with (33), gives us

$$m_1 \leq 4.43 \times 10^{57}(\log m_1)^4.$$

From Lemma 6, we can choose $f(x) := 4.43 \times 10^{57}x^4$. Then, we obtain $m_1 \leq 2.42 \times 10^{66}$. We record the above as the following Lemma.

Lemma 7. *Let $(m_1, m_2, m_3, m_4, N, d, \ell)$ be the nonnegative integer solutions to the Diophantine Equation (1) with $m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0, 1 \leq d \leq 9$, and $\ell \geq 2$. Then we have*

$$\ell < m_1 \leq 2.42 \times 10^{66}.$$

4. Reducing The Bounds

The bounds obtained in Lemma 7 are too large to carry out meaningful computations on the computer. Thus, we need to reduce these bounds. To do so, we return to (20), (24), (28) and (33) apply Lemma 5 via the following procedure.

4.1. Step 1

First, let

$$\tau_1 := \ell\log 10 - m_1\log\alpha + \log\left(\frac{d}{9a}\right), \quad 1 \leq d \leq 9. \tag{35}$$

For technical reasons, we assume that $m_1 - m_2 \geq 20$ for the moment and go to (19). We will obtain a bound of $m_1 - m_2$ larger than 20. Thus, we can get rid of this condition in both cases. Note that $e^{\tau_1} - 1 = \Lambda_1 \neq 0$. Thus, $\tau_1 \neq 0$. If $\tau_1 > 0$, then

$$0 < \tau_1 < e^{\tau_1} - 1 = |\Lambda_1| < \frac{18}{\alpha^{m_1 - m_2}}.$$

If $\tau_1 < 0$, by (13), we have

$$\left|e^{-\tau_1} - 1\right| = \left|\frac{9a\alpha^{m_1}}{d10^\ell} - 1\right| < \frac{9 \cdot 18a\alpha^{m_2}}{d10^\ell} \leq \frac{\alpha^{m_2+7}}{\alpha^{m_1-2}} \leq \frac{1}{\alpha^{m_1-m_2-9}} \leq \frac{1}{\alpha^{11}} < \frac{1}{2}.$$

Hence, $e^{-\tau_1} < 2$. Thus, we have

$$0 < |\tau_1| < e^{|\tau_1|} - 1 = e^{|\tau_1|}|\Lambda_1| < \frac{36}{\alpha^{m_1 - m_2}}.$$

Therefore, in both cases, we have

$$0 < |\tau_1| = \left| \ell \log 10 - m_1 \log \alpha + \log\left(\frac{d}{9a}\right) \right| < \frac{36}{\alpha^{m_1 - m_2}}.$$

This means that

$$|\tau_1| < 36\alpha^{m_2 - m_1} < \alpha^{m_2 - m_1 + 6} < \alpha^{6.1} \exp(-0.609(m_1 - m_2)),$$

with

$$X = \max\{m_1, n\} = m_1 \leq 2.42 \times 10^{66}.$$

Dividing through (35) by $\log 10$, we have

$$\frac{\tau_1}{\log 10} = \frac{\log(\frac{d}{9a})}{\log 10} - m_1 \frac{\log \alpha}{\log 10} + n. \tag{36}$$

Then, we put

$$c_1 := \alpha^{6.1}, \quad \delta := 0.609, \quad X_0 := 2.42 \times 10^{66}, \quad \psi := \frac{\log(\frac{d}{9a})}{\log 10}, \quad Y := m_1 - m_2,$$
$$v := \frac{\log \alpha}{\log 10}, \quad v_1 := -\log \alpha, \quad v_2 := \log 10, \quad \beta := \log\left(\frac{d}{9a}\right).$$

We now apply Lemma 5 on (36). A quick computer search in Mathematica reveals that the convergent

$$\frac{p_{130}}{q_{130}} = \frac{177923488364632912571628513817306063449056536854979651306447664686 3}{672298743659444088707203774386355809118749920368794556178412479832 6}$$

of τ is such that $q_{130} > X_0$. Therefore, we find that $q = q_{134}$ satisfies the hypothesis of Lemma 5 for $d = 1, \ldots, 9$. Applying Lemma 5, we have $m_1 - m_2 \leq 273$.

4.2. Step 2

Next, we put

$$\tau_2 := \ell \log 10 - m_2 \log \alpha + \log\left(\frac{d}{9a(1 + \alpha^{m_1 - m_2})}\right), \quad 1 \leq d \leq 9. \tag{37}$$

For technical reasons, as before, we assume that $m_1 - m_3 \geq 20$ for the moment and go to (23). We will obtain a bound of $m_1 - m_3$ larger than 20. Thus, we can get rid of this condition in both cases. Note that $e^{\tau_2} - 1 = \Lambda_2 \neq 0$. Thus, $\tau_2 \neq 0$. If $\tau_2 > 0$, then

$$0 < \tau_2 < e^{\tau_2} - 1 = |\Lambda_2| < \frac{17}{\alpha^{m_1 - m_3}}.$$

If $\tau_2 < 0$, we have

$$\left|e^{-\tau_2} - 1\right| = \left| \frac{9a(\alpha^{m_1} + \alpha^{m_2})}{d10^{\ell}} - 1 \right| < \frac{9 \cdot 17a\alpha^{m_3}}{d10^{\ell}} \leq \frac{\alpha^{m_3 + 7}}{\alpha^{m_1 - 2}} \leq \frac{1}{\alpha^{m_1 - m_3 - 9}} \leq \frac{1}{\alpha^{11}} < \frac{1}{2}.$$

Hence, $e^{-\tau_2} < 2$. Thus, we have

$$0 < |\tau_2| < e^{|\tau_2|} - 1 = e^{-\tau_2}|\Lambda_2| < \frac{34}{\alpha^{m_1 - m_3}}.$$

Therefore, in both cases, we have

$$0 < |\tau_2| = \left|\ell \log 10 - m_2 \log \alpha + \log\left(\frac{d}{9a(1+\alpha^{m_1-m_2})}\right)\right| < \frac{36}{\alpha^{m_1-m_3}}.$$

This means that

$$|\tau_2| < \alpha^{m_3-m_1+6} < \alpha^{6.1} \exp(-0.609(m_1-m_3)).$$

Dividing through (37) by $\log 10$, we have

$$\frac{\tau_2}{\log 10} = \frac{1}{\log 10} \log\left(\frac{d}{9a(\alpha^{m_1-m_2}+1)}\right) - m_2 \frac{\log \alpha}{\log 10} + \ell. \tag{38}$$

Thus,we put

$$\begin{gathered} c_1 := \alpha^{6.1}, \quad \delta := 0.609, \quad X_0 := 2.42 \times 10^{66}, \\ Y := m_1 - m_3, \quad v := \frac{\log \alpha}{\log 10}, \quad v_1 := -\log \alpha, \quad v_2 := \log 10, \\ \psi := \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right), \quad \beta := \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right). \end{gathered}$$

We now apply Lemma 5 on (38). We found that $q = q_{138}$ satisfies the hypothesis of Lemma 5 for $d = 1, \ldots, 9$. Thus, we have $m_1 - m_3 \le 298$. Hence, $m_3 \ge 32$ by the assumption $m_1 \ge 330$.

4.3. Step 3

Now, we put

$$\tau_3 := \ell \log 10 - m_3 \log \alpha + \log\left(\frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})}\right), \quad 1 \le d \le 9. \tag{39}$$

For technical reasons, we assume that $m_1 - m_4 \ge 20$ for the moment and go to (27). We will obtain a bound of $m_1 - m_4$ larger than 20. Thus, we can remove this condition in both cases. Note that $e^{\tau_3} - 1 = \Lambda_3 \neq 0$. Thus, $\tau_3 \neq 0$. If $\tau_3 > 0$, then

$$0 < \tau_3 < e^{\tau_3} - 1 = |\Lambda_3| < \frac{16}{\alpha^{m_4-m_1}}.$$

If $\tau_3 < 0$, we have

$$\left|e^{-\tau_3} - 1\right| = \left|\frac{9a\left(\alpha_1^m + \alpha_2^m + \alpha_3^m\right)}{d10^\ell} - 1\right| < \frac{9 \cdot 16a\alpha^{m_4}}{d10^\ell} \le \frac{\alpha^{m_4+7}}{\alpha^{m_1-2}} \le \frac{1}{\alpha^{m_1-m_4-9}} \le \frac{1}{\alpha^{11}} < \frac{1}{2}.$$

Hence, $e^{-\tau_3} < 2$. Thus, we have

$$0 < |\tau_3| < e^{|\tau_3|} - 1 = e^{-\tau_3}|\Lambda_3| < \frac{32}{\alpha^{m_1-m_4}}.$$

Therefore, in both cases, we have

$$0 < |\tau_3| = \left|\ell \log 10 - m_3 \log \alpha + \log\left(\frac{d}{9a(1+\alpha^{m_1-m_3}+\alpha^{m_2-m_3})}\right)\right| < \frac{36}{\alpha^{m_1-m_4}}.$$

This means that

$$|\tau_3| < \alpha^{m_4-m_1+6} < \alpha^{6.1} \exp(-0.609(m_1-m_4)).$$

Dividing through (39) by $\log 10$, we have

$$\frac{\tau_3}{\log 10} = \frac{1}{\log 10}\log\left(\frac{d}{9a(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right) - m_3\frac{\log\alpha}{\log 10} + \ell$$

Thus, we can take

$$c_1 := \alpha^{6.1}, \quad \delta := 0.609, \quad X_0 := 4.1\times 10^{66},$$
$$Y := m_1 - m_4, \quad v := \frac{\log\alpha}{\log 10}, \quad v_1 := -\log\alpha, \quad v_2 := \log 10,$$
$$\psi := \frac{1}{\log 10}\log\left(\frac{d}{9a(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right), \quad \beta := \log\left(\frac{d}{9a(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right).$$

We find that $q = q_{143}$ satisfies the hypothesis of Lemma 5 for $1\le d\le 9$, $0\le m_2-m_3\le m_1-m_3\le 298$. Applying Lemma 5, we have $m_1-m_4\le 313$ and, hence, $m_4\ge 17$.

4.4. Step 4

Lastly, we put

$$\tau_4 := \ell\log 10 - m_4\log\alpha + \log\left(\frac{d}{9a(1+\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4})}\right) \quad 1\le d\le 9. \tag{40}$$

We use the original assumption that $m_1 > 330$ and go to (32). Note that $e^{\tau_4}-1 = \Lambda_4 \neq 0$. Thus, $\tau_4\neq 0$. If $\tau_4 > 0$, then

$$0 < \tau_4 < e^{\tau_4} - 1 = |\Lambda_4| < \frac{15}{\alpha^{m_1}}.$$

If $\tau_4 < 0$, we have

$$\left|e^{-\tau_4}-1\right| = \left|\frac{9a(\alpha^{m_1}+\alpha^{m_2}+\alpha^{m_3}+\alpha^{m_4})}{d10^{\ell}} - 1\right| < \frac{9\cdot 15a}{d10^{\ell}} \le \frac{\alpha^7}{\alpha^{m_1-2}} \le \frac{1}{\alpha^{m_1-9}} \le \frac{1}{\alpha^{11}} < \frac{1}{2}.$$

Hence, $e^{-\tau_4} < 2$. Thus, we have

$$0 < |\tau_4| < e^{|\tau_4|} - 1 = e^{-\tau_4}|\Lambda_4| < \frac{30}{\alpha^{m_1}}.$$

Therefore,in both cases, we have

$$0 < |\tau_4| = \left|\ell\log 10 - m_4\log\alpha + \log\left(\frac{d}{9a(1+\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4})}\right)\right| < \frac{36}{\alpha^{m_1}}.$$

This means that

$$|\tau_4| < \alpha^{-m_1+6} < \alpha^{6.1}\exp(-0.609 m_1).$$

Dividing through (40) by $\log 10$, we have

$$\frac{\tau_4}{\log 10} = \frac{1}{\log 10}\log\left(\frac{d}{9a(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right) - m_4\frac{\log\alpha}{\log 10} + \ell$$

Thus, we can take

$$
\begin{aligned}
c_1 &:= \alpha^{6.1}, \quad \delta := 0.609, \quad X_0 := 4.1 \times 10^{66}, \\
Y &:= m_1, \quad v := \frac{\log\alpha}{\log 10}, \quad v_1 = -\log\alpha, \quad v_2 = \log 10, \\
\psi &:= \frac{1}{\log 10}\log\left(\frac{d}{9a(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right), \\
\beta &:= \log\left(\frac{d}{9a(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right).
\end{aligned}
$$

We find that $q = q_{147}$ satisfies the hypothesis of Lemma 5 for $1 \le d \le 9$, $0 \le m_3 - m_4 \le m_2 - m_4 \le m_1 - m_4 \le 313$ except for three special cases $(d, m_1 - m_4, m_2 - m_4, m_3 - m_4) = (9,3,0,1), (9,6,5,1), (9,6,4,4)$. Applying Lemma 5, we have $m_1 \le 322$, which contradicts the assumption that $m_1 > 330$.

Now, we consider the three cases

$$
(d, m_1 - m_4, m_2 - m_4, m_3 - m_4) = (9,3,1,0), (9,6,5,1), (9,6,4,4).
$$

Obviously, $d(10^m - 1)/9 \equiv 0 (\bmod\ 9)$ when $d = 9$. It is easy to see that the period of tribonacci numbers modulo 9 is 39. Since

$$
\begin{aligned}
T_n + T_n + T_{n+1} + T_{n+3} &\not\equiv 0 (\bmod\ 9), \\
T_n + T_{n+1} + T_{n+5} + T_{n+6} &\not\equiv 0 (\bmod\ 9), \\
T_n + T_{n+4} + T_{n+4} + T_{n+6} &\not\equiv 0 (\bmod\ 9)
\end{aligned}
$$

for $1 \le n \le 39$, there is no solution to Equation (1) in the above cases. □

5. Conclusions

In this paper, we completely solved the diophantine Equation (1). More precisely, we found that 66666 is the largest repdigit expressible as the sum of four tribonacci numbers. Our method is based on Baker's method. We first gave a larger upper bound of m_1. Then, the reduction method reduced such a bound to an applicable one. During the reduction procedure, the periodic properties of $\{T_n\}(\bmod\ 9)$ were used to deal with three individual cases.

It is worth mentioning that our method could be applied in b-repdigits. For each b, we may give a large bound. It would be a challenge to find all solutions for every b. We may not find a unified bound of b. It likely has infinite solutions.

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