symmetry

Article

Δ–Gronwall–Bellman–Pachpatte Dynamic Inequalities and Their Applications on Time Scales

Ahmed A. El-Deeb [1,*], Dumitru Baleanu [2,3] and Jan Awrejcewicz [4,*]

1 Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo 11884, Egypt
2 Institute of Space Science, 077125 Magurele, Romania
3 Department of Mathematics, Cankaya University, Ankara 06530, Turkey
4 Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland
* Correspondence: ahmedeldeeb@azhar.edu.eg (A.A.E.-D.); jan.awrejcewicz@p.lodz.pl (J.A.)

Abstract: In this article, with the help of Leibniz integral rule on time scales, we prove some new dynamic inequalities of Gronwall–Bellman–Pachpatte-type on time scales. These inequalities can be used as handy tools to study the qualitative and quantitative properties of solutions of the initial boundary value problem for partial delay dynamic equation.

Keywords: Gronwall's inequality; dynamic inequality; time scales; Leibniz integral rule on time scales

Citation: El-Deeb, A.A.; Baleanu, D.; Awrejcewicz, J. Δ–Gronwall–Bellman–Pachpatte Dynamic Inequalities and Their Applications on Time Scales. *Symmetry* **2022**, *14*, 1804. https://doi.org/10.3390/sym14091804

Academic Editors: Wei-Shih Du and Sergei D. Odintsov

Received: 6 July 2022
Accepted: 10 August 2022
Published: 31 August 2022

1. Introduction

In [1], the authors discussed the following results:

$$\begin{aligned}\Gamma(\Theta(\ell,t)) &\leq a(\ell,t)+\int_0^{\theta(\ell)}\int_0^{\vartheta(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)\varpi(\Theta(\varsigma,\eta))\\&\quad+\int_0^{\varsigma}\Im_2(\chi,\eta)\varpi(\Theta(\chi,\eta))d\chi]d\eta d\varsigma,\end{aligned}$$

$$\begin{aligned}\Gamma(\Theta(\ell,t)) &\leq a(\ell,t)+\int_0^{\theta(\ell)}\int_0^{\vartheta(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)\varpi(\Theta(\varsigma,\eta))\eta(\Theta(\varsigma,\eta))\\&\quad+\int_0^{\varsigma}\Im_2(\chi,\eta)\varpi(\Theta(\chi,\eta))d\chi]d\eta d\varsigma,\end{aligned}$$

and

$$\begin{aligned}\Gamma(\Theta(\ell,t)) &\leq a(\ell,t)+\int_0^{\theta(\ell)}\int_0^{\vartheta(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\varpi(\Theta(\varsigma,\eta))\\&\quad+\int_0^{\varsigma}\Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\varpi(\Theta(\chi,\eta))d\chi]d\eta d\varsigma,\end{aligned}$$

where Θ, f, $\Im \in C(I_1\times I_2,\mathbb{R}_+)$, $a\in C(\zeta,\mathbb{R}_+)$ are nondecreasing functions, $I_1, I_2 \in \mathbb{R}$, $\theta \in C^1(I_1,I_1), \vartheta \in C^1(I_2,I_2)$ are nondecreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(\zeta,\mathbb{R}_+)$, and Γ, ζ, $\varpi \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\{\Gamma,\zeta,\varpi\}(\Theta) > 0$ for $\Theta > 0$, and $\lim_{\Theta\to+\infty}\Gamma(\Theta)=+\infty$.

Recently, Gronwall–Bellman-type inequalities, which have several applications in the qualitative and quantitative behavior, have been developed by many mathematicians and several refinements and extensions have been made to the previous results; we refer the reader to the works of [2–13].

Time scales calculus with the objective to unify discrete and continuous analysis was introduced by S. Hilger [14]. For additional subtleties on time scales, we refer the reader to the books by Bohner and Peterson [15,16].

Theorem 1 ([16])**.** *Suppose Π on $[a,b]$, is ∇-integrable then so is $|\Pi|$, and*

$$\left|\int_a^b \Pi(\eta)\nabla\eta\right| \le \int_a^b |\Pi(\eta)|\nabla\eta.$$

Theorem 2 ([11] Leibniz Integral Rule on Time Scales)**.** *In the following by $\Lambda^{\Delta}(\varrho,\varsigma)$ we mean the delta derivative of $\Lambda(\varrho,\varsigma)$ with respect to ϱ. Similarly, $\Lambda^{\nabla}(\varrho,\varsigma)$ is understood. If Λ , Λ^{Δ} and Λ^{∇} are continuous, and $u,h:\mathbb{T}\to\mathbb{T}$ are delta differentiable functions, then the following formulas holds $\forall\varrho\in\mathbb{T}^{\kappa}$.*

(i) $$\left[\int_{u(\varrho)}^{h(\varrho)}\Lambda(\varrho,\varsigma)\Delta\varsigma\right]^{\Delta} = \int_{u(\varrho)}^{h(\varrho)}\Lambda^{\Delta}(\varrho,\varsigma)\Delta\varsigma + h^{\Delta}(\varrho)\Lambda(\sigma(\varrho),h(\varrho)) - u^{\Delta}(\varrho)\Lambda(\sigma(\varrho),u(\varrho));$$

(ii) $$\left[\int_{u(\varrho)}^{h(\varrho)}\Lambda(\varrho,\varsigma)\Delta\varsigma\right]^{\nabla} = \int_{u(\varrho)}^{h(\varrho)}\Lambda^{\nabla}(\varrho,\varsigma)\Delta\varsigma + h^{\nabla}(\varrho)\Lambda(\rho(\varrho),h(\varrho)) - u^{\nabla}(\varrho)\Lambda(\rho(\varrho),u(\varrho));$$

(iii) $$\left[\int_{u(\varrho)}^{h(\varrho)}\Lambda(\varrho,\varsigma)\nabla\varsigma\right]^{\Delta} = \int_{u(\varrho)}^{h(\varrho)}\Lambda^{\Delta}(\varrho,\varsigma)\nabla\varsigma + h^{\Delta}(\varrho)\Lambda(\sigma(\varrho),h(\varrho)) - u^{\Delta}(\varrho)\Lambda(\sigma(\varrho),u(\varrho));$$

(iv) $$\left[\int_{u(\varrho)}^{h(\varrho)}\Lambda(\varrho,\varsigma)\nabla\varsigma\right]^{\nabla} = \int_{u(\varrho)}^{h(\varrho)}\Lambda^{\nabla}(\varrho,\varsigma)\nabla\varsigma + h^{\nabla}(\varrho)\Lambda(\rho(\varrho),h(\varrho)) - u^{\nabla}(\varrho)\Lambda(\rho(\varrho),u(\varrho)).$$

In this article, by employing the results of Theorems 2, we establish the delayed time scale case of the inequalities proved in [1]. Further, the results that are proved in this paper extend some known results in [17–19].

2. Main Results

We start with the following basic lemma:

Lemma 1. *Suppose $\mathbb{T}_1$, $\mathbb{T}_2$ are two times scales and $a\in C(\Omega=\mathbb{T}_1\times\mathbb{T}_2,\mathbb{R}_+)$ is nondecreasing with respect to $(\ell,t)\in\Omega$. Assume that $\Im$, Θ, $f\in C_{rd}(\Omega,\mathbb{R}_+)$, $\tau_1\in C^1_{rd}(\mathbb{T}_1,\mathbb{T}_1)$ and $\tau_2\in C^1_{rd}(\mathbb{T}_2,\mathbb{T}_2)$ be nondecreasing functions with $\tau_1(\ell)\le\ell$ on $\mathbb{T}_1$, $\tau_2(t)\le t$ on $\mathbb{T}_2$. Furthermore, suppose Γ, $\zeta\in C(\mathbb{R}_+,\mathbb{R}_+)$ are nondecreasing functions with $\{\Gamma,\zeta\}(\Theta)>0$ for $\Theta>0$, and $\lim\limits_{\Theta\to+\infty}\Gamma(\Theta)=+\infty$. If $\Theta(\ell,t)$ satisfies*

$$\Gamma(\Theta(\ell,t)) \le a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)}\Im(\varsigma,\eta)f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\Delta\eta\Delta\varsigma, \tag{1}$$

for $(\ell,t)\in\Omega$, then

$$\Theta(\ell,t) \le \Gamma^{-1}\left\{G^{-1}G(a(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)}\Im(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right], \tag{2}$$

for $0\le\ell\le\ell_1, 0\le t\le t_1$, where

$$G(v)=\int_{v_0}^{v}\frac{\Delta\varsigma}{\zeta(\Gamma^{-1}(\varsigma))}, v\ge v_0>0,\ G(+\infty)=\int_{v_0}^{+\infty}\frac{\Delta\varsigma}{\zeta(\Gamma^{-1}(\varsigma))}=+\infty, \tag{3}$$

and $(\ell_1,t_1)\in\Omega$ is chosen so that

$$\left(G(a(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right)\in\mathrm{Dom}\left(G^{-1}\right).$$

Proof. First, we assume that $a(\ell,t) > 0$. Fixing an arbitrary $(\ell_0,t_0) \in \Omega$, we define a positive and nondecreasing function $\varphi(\ell,t)$ by

$$\varphi(\ell,t) = a(\ell_0,t_0) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \zeta(\Theta(\varsigma,\eta)) \Delta\eta\Delta\varsigma \tag{4}$$

for $0 \le \ell \le \ell_0 \le \ell_1, 0 \le t \le t_0 \le t_1$, then $\varphi(\ell_0,t) = \varphi(\ell,t_0) = a(\ell_0,t_0)$ and

$$\Theta(\ell,t) \le \Gamma^{-1}(\varphi(\ell,t)). \tag{5}$$

Taking Δ-derivative for (4) with employing Theorem 2(iv), we have

$$\begin{aligned} \varphi^{\Delta_\ell}(\ell,t) &= \tau_1^\Delta(\ell) \int_{t_0}^{\tau_2(t)} \Im(\tau_1(\ell),\eta) f(\tau_1(\ell),\eta) \zeta(\Theta(\tau_1(\ell),\eta)) \Delta\eta \\ &\le \tau_1^\Delta(\ell) \int_{t_0}^{\tau_2(t)} \Im(\tau_1(\ell),\eta) f(\tau_1(\ell),\eta) \zeta\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\eta))\Big) \Delta\eta \\ &\le \zeta\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\tau_2(t)))\Big) \tau_1^\Delta(\ell) \int_{t_0}^{\tau_2(t)} \Im(\tau_1(\ell),\eta) f(\tau_1(\ell),\eta) \Delta\eta. \end{aligned} \tag{6}$$

The inequality (6) can be written in the form

$$\frac{\varphi^{\Delta_\ell}(\ell,t)}{\zeta(\Gamma^{-1}(\varphi(\ell,t)))} \le \tau_1^\Delta(\ell) \int_{t_0}^{\tau_2(t)} \Im(\tau_1(\ell),\eta) f(\tau_1(\ell),\eta) \Delta\eta. \tag{7}$$

Taking Δ-integral for Inequality (7), obtains

$$\begin{aligned} G(\varphi(\ell,t)) &\le G(\varphi(\ell_0,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma \\ &\le G(a(\ell_0,t_0)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma. \end{aligned}$$

Since $(\ell_0,t_0) \in \Omega$ is chosen arbitrary,

$$\varphi(\ell,t) \le G^{-1}\left[G(a(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right]. \tag{8}$$

From (8) and (5) we obtain the desired result (2). We carry out the above procedure with $\epsilon > 0$ instead of $a(\ell,t)$ when $a(\ell,t) = 0$ and subsequently let $\epsilon \to 0$. □

Remark 1. *If we take $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$ and $t_0 = 0$ in Lemma 1, then, inequality (1) becomes the inequality obtained in ([1] Lemma 2.1).*

Theorem 3. *Let Θ, a, f, τ_1 and τ_2 be as in Lemma 1. Let $\Im_1, \Im_2 \in C_{rd}(\Omega,\mathbb{R}_+)$. If $\Theta(\ell,t)$ satisfies*

$$\begin{aligned} \Gamma(\Theta(\ell,t)) \le\; & a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta)) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\Delta\chi]\Delta\eta\Delta\varsigma, \end{aligned} \tag{9}$$

for $(\ell,t) \in \Omega$, then

$$\Theta(\ell,t) \le \Gamma^{-1}\left\{G^{-1}\left(p(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right)\right\} \tag{10}$$

for $0 \leq \ell \leq \ell_1, 0 \leq t \leq t_1$, *where G is defined by (3) and*

$$p(\ell,t) = G(a(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi \right) \Delta\eta\Delta\varsigma \tag{11}$$

and $(\ell_1, t_1) \in \Omega$ *is chosen so that*

$$\left(p(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right) \in \mathrm{Dom}\left(G^{-1}\right).$$

Proof. By the same steps of the proof of Lemma 1 we can obtain (10), with suitable changes. □

Remark 2. *If we take* $\Im_2(\ell,t) = 0$, *then Theorem 3 reduces to Lemma 1.*

Corollary 1. *Let the functions* Θ, f, $\Im_1$, $\Im_2$, a, τ_1 *and* τ_2 *be as in Theorem 3. Further suppose that* $q > p > 0$ *are constants. If* $\Theta(\ell,t)$ *satisfies*

$$\begin{aligned} \Theta^q(\ell,t) \quad &\leq \quad a(\ell,t) + \frac{q}{q-p} \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\Theta^p(\varsigma,\eta) \\ &\qquad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Theta^p(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma, \end{aligned} \tag{12}$$

for $(\ell,t) \in \Omega$, *then*

$$\Theta(\ell,t) \leq \left\{ p(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right\}^{\frac{1}{q-p}}, \tag{13}$$

where

$$p(\ell,t) = (a(\ell,t))^{\frac{q-p}{q}} + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi \right) \Delta\eta\Delta\varsigma.$$

Proof. In Theorem 3, by letting $\Gamma(\Theta) = \Theta^q, \zeta(\Theta) = \Theta^p$ we have

$$G(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta(\Gamma^{-1}(\varsigma))} = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varsigma^{\frac{p}{q}}} \geq \frac{q}{q-p} \left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}} \right), v \geq v_0 > 0$$

and

$$G^{-1}(v) \geq \left\{ v_0^{\frac{q-p}{q}} + \frac{q-p}{q} v \right\}^{\frac{1}{q-p}},$$

we obtain the inequality (13). □

Theorem 4. *Suppose* $\Gamma, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with* $\{\Gamma, \zeta, \varpi\}(\Theta) > 0$ *for* $\Theta > 0$, $\Theta(\ell,t)$ *and with the conditions of Theorem 3, satisfies*

$$\begin{aligned} \Gamma(\Theta(\ell,t)) \quad &\leq \quad a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\varpi(\Theta(\varsigma,\eta)) \\ &\qquad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\Delta\chi]\Delta\eta\Delta\varsigma, \end{aligned} \tag{14}$$

for $(\ell,t) \in \Omega$, *then*

$$\Theta(\ell,t) \leq \Gamma^{-1}\left\{ G^{-1}\left(F^{-1}\left[F(p(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right] \right) \right\} \tag{15}$$

for $0 \leq \ell \leq \ell_1, 0 \leq t \leq t_1$, *where* G *and* p *are as in* (3), (11), *respectively, and*

$$F(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varpi(\Gamma^{-1}(G^{-1}(\varsigma)))}, v \geq v_0 > 0, \qquad F(+\infty) = +\infty, \tag{16}$$

and $(\ell_1, t_1) \in \Omega$ *is chosen so that*

$$\left[F(p(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\ell,t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\varphi(\ell,t)$ by

$$\begin{aligned} \varphi(\ell,t) &= a(\ell_0,t_0) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\varpi(\Theta(\varsigma,\eta)) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\Delta\chi]\Delta\eta\Delta\varsigma, \end{aligned} \tag{17}$$

for $0 \leq \ell \leq \ell_0 \leq \ell_1, 0 \leq t \leq t_0 \leq t_1$, then $\varphi(\ell_0,t) = \varphi(\ell,t_0) = a(\ell_0,t_0)$ and

$$\Theta(\ell,t) \leq \Gamma^{-1}(\varphi(\ell,t)). \tag{18}$$

Taking Δ-derivative for (17) with employing Theorem 2 (*i*), gives

$$\begin{aligned} \varphi^{\Delta_\ell}(\ell,t) &= \tau_1^{\Delta}(\ell) \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)[f(\tau_1(\ell),\eta)\zeta(\Theta(\tau_1(\ell),\eta))\varpi(\Theta(\tau_1(\ell),\eta)) \\ &\quad + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\Delta\chi]\Delta\eta \\ &\leq \tau_1^{\Delta}(\ell) \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)\Big[f(\tau_1(\ell),\eta)\zeta\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\eta))\Big)\varpi\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\eta))\Big) \\ &\quad + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi,\eta)\zeta\Big(\Gamma^{-1}(\varphi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta \\ &\leq \tau_1^{\Delta}(\ell).\zeta\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\tau_2(t)))\Big) \times \\ &\quad \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)\Big[f(\tau_1(\ell),\eta)\varpi\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\eta))\Big) + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta. \end{aligned} \tag{19}$$

From (19), we have

$$\begin{aligned} \frac{\varphi^{\Delta_\ell}(\ell,t)}{\zeta(\Gamma^{-1}(\varphi(\ell,t)))} &\leq \tau_1^{\Delta}(\ell) \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)\Big[f(\tau_1(\ell),\eta)\varpi\Big(\Gamma^{-1}(\varphi(\tau_1(\ell),\eta))\Big) \\ &\quad + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta. \end{aligned} \tag{20}$$

Taking Δ-integral for (20), gives

$$\begin{aligned} G(\varphi(\ell,t)) &\leq G(\varphi(\ell_0,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma \\ &\leq G(a(\ell_0,t_0)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma. \end{aligned}$$

Since $(\ell_0, t_0) \in \Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$G(\varphi(\ell,t)) \leq p(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \varpi\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \Delta\eta\Delta\varsigma. \tag{21}$$

Since $p(\ell, t)$ is a nondecreasing function, an application of Lemma 1 to (21) gives us

$$\varphi(\ell,t) \leq G^{-1}\left(F^{-1}\left[F(p(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right]\right). \tag{22}$$

From (18) and (22) we obtain the desired inequality (15).

Now, we take the case $a(\ell,t) = 0$ for some $(\ell,t) \in \Omega$. Let $a_\epsilon(\ell,t) = a(\ell,t) + \epsilon$, for all $(\ell,t) \in \Omega$, where $\epsilon > 0$ is arbitrary, then $a_\epsilon(\ell,t) > 0$ and $a_\epsilon(\ell,t) \in C(\Omega, \mathbb{R}_+)$ be nondecreasing with respect to $(\ell,t) \in \Omega$. We carry out the above procedure with $a_\epsilon(\ell,t) > 0$ instead of $a(\ell,t)$, and we get

$$\Theta(\ell,t) \leq \Gamma^{-1}\left\{G^{-1}\left(F^{-1}\left[F(p_\epsilon(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right]\right)\right\}$$

where

$$p_\epsilon(\ell,t) = G(a_\epsilon(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta) \Delta\chi\right) \Delta\eta\Delta\varsigma.$$

Letting $\epsilon \to 0^+$, we obtain (15). The proof is complete. □

Remark 3. *If we take $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$ and $t_0 = 0$ in Theorem 4, then, inequality (14) becomes the inequality obtained in ([1], Theorem 2.2(A_2)) .*

Corollary 2. *Let the functions Θ, a, f, $\Im_1$, $\Im_2$, τ_1 and τ_2 be as in Theorem 3. Further suppose that q, p and r are constants with $p > 0$, $r > 0$ and $q > p + r$. If $\Theta(\ell,t)$ satisfies*

$$\begin{aligned} \Theta^q(\ell,t) \quad \leq \quad & a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\Theta^p(\varsigma,\eta)\Theta^r(\varsigma,\eta) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Theta^p(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma, \end{aligned} \tag{23}$$

for $(\ell,t) \in \Omega$, then

$$\Theta(\ell,t) \leq \left\{[p(\ell,t)]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q} \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta\Delta\varsigma\right\}^{\frac{1}{q-p-r}}, \tag{24}$$

where

$$p(\ell,t) = (a(\ell,t))^{\frac{q-p}{q}} + \frac{q-p}{q} \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta) \Delta\chi\right) \Delta\eta\Delta\varsigma.$$

Proof. An application of Theorem 4 with $\Gamma(\Theta) = \Theta^q$, $\zeta(\Theta) = \Theta^p$, and $\varpi(\Theta) = \Theta^r$ yields the desired inequality (24). □

Theorem 5. *Suppose $\Gamma, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\Gamma, \zeta, \varpi\}(\Theta) > 0$ for $\Theta > 0$, $\Theta(\ell,t)$ and with the conditions of Theorem 3. If $\Theta(\ell,t)$ satisfies*

$$\begin{aligned} \Gamma(\Theta(\ell,t)) \quad \leq \quad & a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\varpi(\Theta(\varsigma,\eta)) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\varpi(\Theta(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{25}$$

for $(\ell, t) \in \Omega$, *then*

$$\Theta(\ell,t) \le \Gamma^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right)\right\} \tag{26}$$

for $0 \le \ell \le \ell_1$, $0 \le t \le t_1$ *where*

$$p_0(\ell,t) = F(G(a(\ell,t))) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma$$

and $(\ell_1, t_1) \in \Omega$ *is chosen so that*

$$\left[p_0(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\ell,t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\varphi(\ell,t)$ by

$$\begin{aligned}\varphi(\ell,t) \quad = \quad & a(\ell_0,t_0) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta))\varpi(\Theta(\varsigma,\eta)) \\ & + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\zeta(\Theta(\chi,\eta))\varpi(\Theta(\chi,\eta))\Delta\chi]\Delta\eta\Delta\varsigma\end{aligned}$$

for $0 \le \ell \le \ell_0 \le \ell_1$, $0 \le t \le t_0 \le t_1$, then $\varphi(\ell_0,t) = \varphi(\ell,t_0) = a(\ell_0,t_0)$, and

$$\Theta(\ell,t) \le \Gamma^{-1}(\varphi(\ell,t)). \tag{27}$$

By the same steps as the proof of Theorem 4, we obtain

$$\begin{aligned}\varphi(\ell,t) \quad \le \quad & G^{-1}\Big\{G(a(\ell_0,t_0)) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \\ & + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta\Delta\varsigma\Big\}.\end{aligned}$$

We define a nonnegative and nondecreasing function $v(\ell,t)$ by

$$\begin{aligned}v(\ell,t) \quad = \quad & G(a(\ell_0,t_0)) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Big[\Big[f(\varsigma,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big)\Big] \\ & + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\varpi\Big(\Gamma^{-1}(\varphi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta\Delta\varsigma\end{aligned}$$

then $v(\ell_0,t) = v(\ell,t_0) = G(a(\ell_0,t_0))$,

$$\varphi(\ell,t) \le G^{-1}[v(\ell,t)] \tag{28}$$

and then

$$\begin{aligned}v^{\Delta\ell}(\ell,t) \quad \le \quad & \tau_1^{\Delta}(\ell)\int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)[f(\tau_1(\ell),\eta)\varpi\Big(\Gamma^{-1}\Big(G^{-1}(v(\tau_1(\ell),t))\Big)\Big) \\ & + \int_{\ell_0}^{\tau_1(\ell)}\Im_2(\chi,\eta)\varpi\Big(\Gamma^{-1}\Big(G^{-1}(v(\chi,t))\Big)\Big)\Delta\chi]\Delta\eta \\ \le \quad & \tau_1^{\Delta}(\ell)\varpi\Big(\Gamma^{-1}\Big(G^{-1}(v(\tau_1(\ell),\tau_2(t)))\Big)\Big)\int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)[f(\tau_1(\ell),\eta) \\ & + \int_{\ell_0}^{\tau_1(\ell)}\Im_2(\chi,\eta)\Delta\chi]\Delta\eta,\end{aligned}$$

or

$$\frac{v^{\Delta\ell}(\ell,t)}{\varpi(\Gamma^{-1}(G^{-1}(v(\ell,t))))} \le \tau_1^{\Delta}(\ell)\int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell),\eta)[f(\tau_1(\ell),\eta) + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi,\eta)\Delta\chi]\Delta\eta.$$

Taking Δ-integral for the above inequality, gives

$$F(v(\ell,t)) \le F(v(\ell_0,t)) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\left[f(\varsigma,\eta) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\Delta\varsigma,$$

or

$$v(\ell,t) \le F^{-1}\left\{F(G(a(\ell_0,t_0))) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma\right\}. \tag{29}$$

From (27)–(29), and since $(\ell_0,t_0) \in \Omega$ is chosen arbitrarily, we obtain the desired inequality (26). If $a(\ell,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\ell,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 4. *If we take $\mathbb{T} = \mathbb{R}$ and $\ell_0 = 0$ and $t_0 = 0$ in Theorem 5, then, inequality (25) becomes the inequality obtained in ([1], Theorem 2.2(A_3)).*

Corollary 3. *Let the functions Θ, a, f, $\Im_1$, $\Im_2$, τ_1 and τ_2 be as in Theorem 3. Further suppose that q, p and r are constants with $p > 0$, $r > 0$ and $q > p + r$. If $\Theta(\ell,t)$ satisfies*

$$\Theta^q(\ell,t) \le a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\Theta^p(\varsigma,\eta)\Theta^r(\varsigma,\eta) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Theta^p(\chi,\eta)\Theta^r(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma, \tag{30}$$

for $(\ell,t) \in \Omega$, then

$$\Theta(\ell,t) \le \left\{p_0(\ell,t) + \frac{q-p-r}{q}\int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right\}^{\frac{1}{q-p-r}}, \tag{31}$$

where

$$p_0(\ell,t) = (a(\ell,t))^{\frac{q-p-r}{q}} + \frac{q-p-r}{q}\int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma.$$

Proof. An application of Theorem 5 with $\Gamma(\Theta) = \Theta^q, \zeta(\Theta) = \Theta^p$, and $\varpi(\Theta) = \Theta^r$ yields the desired inequality (31). □

Theorem 6. *Suppose $\Gamma, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\Gamma,\zeta,\varpi\}(\Theta) > 0$ for $\Theta > 0$, $\Theta(\ell,t)$ and with the conditions of Theorem 3. If $\Theta(\ell,t)$ satisfies*

$$\Gamma(\Theta(\ell,t)) \le a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\varpi(\Theta(\varsigma,\eta)) \times \left[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta)) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\Delta\varsigma, \tag{32}$$

for $(\ell, t) \in \Omega$, *then*

$$\Theta(\ell, t) \leq \Gamma^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\ell, t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \Delta\varsigma\right]\right)\right\}, \tag{33}$$

for $0 \leq \ell \leq \ell_2, 0 \leq t \leq t_2$, *where*

$$G_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varpi(\Gamma^{-1}(\varsigma))}, v \geq v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{\varpi(\Gamma^{-1}(\varsigma))} = +\infty \tag{34}$$

$$F_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta\left[\Gamma^{-1}\left(G_1^{-1}(\varsigma)\right)\right]}, v \geq v_0 > 0, F_1(+\infty) = +\infty \tag{35}$$

$$p_1(\ell, t) = G_1(a(\ell, t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta)\Delta\chi\right)\Delta\eta\Delta\varsigma \tag{36}$$

and $(\ell_2, t_2) \in \Omega$ *is chosen so that*

$$\left[F_1(p_1(\ell, t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \Delta\varsigma\right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. Suppose that $a(\ell, t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\varphi(\ell, t)$ by

$$\begin{aligned} \varphi(\ell, t) \quad = \quad & a(\ell_0, t_0) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta)\varpi(\Theta(\varsigma, \eta))[f(\varsigma, \eta)\zeta(\Theta(\varsigma, \eta)) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned}$$

for $0 \leq \ell \leq \ell_0 \leq \ell_2, 0 \leq t \leq t_0 \leq t_2$, then $\varphi(\ell_0, t) = \varphi(\ell, t_0) = a(\ell_0, t_0)$,

$$\Theta(\ell, t) \leq \Gamma^{-1}(\varphi(\ell, t)) \tag{37}$$

and

$$\begin{aligned} \varphi^{\Delta_\ell}(\ell, t) \quad \leq \quad & \tau_1^{\Delta}(\ell) \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell), \eta)\eta\left[\Gamma^{-1}(\varphi(\tau_1(\ell), \eta))\right]\left[f(\tau_1(\ell), \eta)\zeta\left(\Gamma^{-1}(\varphi(\tau_1(\ell), \eta))\right)\right. \\ & \left. + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi, \eta)\Delta\chi\right]\Delta\eta \\ \leq \quad & \tau_1^{\Delta}(\ell)\eta\left[\Gamma^{-1}(\varphi(\tau_1(\ell), \tau_2(t)))\right] \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell), \eta)\left[f(\tau_1(\ell), \eta)\zeta\left(\Gamma^{-1}(\varphi(\tau_1(\ell), \eta))\right)\right. \\ & \left. + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi, \eta)\Delta\chi\right]\Delta\eta, \end{aligned}$$

then

$$\begin{aligned} \frac{\varphi^{\Delta_\ell}(\ell, t)}{\eta[\Gamma^{-1}(\varphi(\ell, t))]} \quad \leq \quad & \tau_1^{\Delta}(\ell) \int_{t_0}^{\tau_2(t)} \Im_1(\tau_1(\ell), \eta)[f(\tau_1(\ell), \eta)\zeta\left(\Gamma^{-1}(\varphi(\tau_1(\ell), \eta))\right) \\ & + \int_{\ell_0}^{\tau_1(\ell)} \Im_2(\chi, \eta)\Delta\chi]\Delta\eta. \end{aligned}$$

Taking Δ-integral for the above inequality, gives

$$\begin{aligned} G_1(\varphi(\ell,t)) &\leq G_1(\varphi(0,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma \end{aligned}$$

then

$$\begin{aligned} G_1(\varphi(\ell,t)) &\leq G_1(a(\ell_0,t_0)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi]\Delta\eta\Delta\varsigma. \end{aligned}$$

Since $(\ell_0,t_0) \in \Omega$ is chosen arbitrary, the last inequality can be restated as

$$G_1(\varphi(\ell,t)) \leq p_1(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\zeta\Big(\Gamma^{-1}(\varphi(\varsigma,\eta))\Big)\Delta\eta\Delta\varsigma \tag{38}$$

It is easy to observe that $p_1(\ell,t)$ is positive and nondecreasing function for all $(\ell,t) \in \Omega$, then an application of Lemma 1 to (38) yields the inequality

$$\varphi(\ell,t) \leq G_1^{-1}\bigg(F_1^{-1}\bigg[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\bigg]\bigg). \tag{39}$$

From (39) and (37) we get the desired inequality (33).

If $a(\ell,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\ell,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 5. *If we take $\mathbb{T} = \mathbb{R}$ and $\ell_0 = 0$ and $t_0 = 0$ in Theorem 6, then, inequality (33) becomes the inequality obtained in ([1], Theorem 2.7).*

Theorem 7. *Suppose $\Gamma, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\Gamma, \zeta, \varpi\}(\Theta) > 0$ for $\Theta > 0$, $\Theta(\ell,t)$ and with the conditions of Theorem 3 and let p be a nonnegative constant. If $\Theta(\ell,t)$ satisfies*

$$\begin{aligned} \Gamma(\Theta(\ell,t)) &\leq a(\ell,t) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)\Theta^p(\varsigma,\eta) \times \\ &\quad \bigg[f(\varsigma,\eta)\zeta(\Theta(\varsigma,\eta)) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\bigg]\Delta\eta\Delta\varsigma, \end{aligned} \tag{40}$$

for $(\ell,t) \in \Omega$, then

$$\Theta(\ell,t) \leq \Gamma^{-1}\bigg\{G_1^{-1}\bigg(F_1^{-1}\bigg[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\bigg]\bigg)\bigg\}, \tag{41}$$

for $0 \leq \ell \leq \ell_2, 0 \leq t \leq t_2$, where

$$G_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{[\Gamma^{-1}(\varsigma)]^p}, v \geq v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{[\Gamma^{-1}(\varsigma)]^p} = +\infty, \tag{42}$$

and F_1, p_1 are as in Theorem 6 and $(\ell_2,t_2) \in \Omega$ is chosen so that

$$\bigg[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\bigg] \in \mathrm{Dom}\Big(F_1^{-1}\Big).$$

Proof. An application of Theorem 6, with $\varpi(\Theta) = \Theta^p$ yields the desired inequality (41). □

Remark 6. *Taking $\mathbb{T} = \mathbb{R}$. The inequality established in Theorem 7 generalizes ([19], Theorem 1) (with $p = 1$, $a(\ell, t) = b(\ell) + c(t)$, $\ell_0 = 0$, $t_0 = 0$, $\Im_1(\varsigma, \eta) f(\varsigma, \eta) = h(\varsigma, \eta)$, and $\Im_1(\varsigma, \eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) = g(\varsigma, \eta)$).*

Corollary 4. *Suppose $\Gamma, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\Gamma, \zeta, \varpi\}(\Theta) > 0$ for $\Theta > 0$, $\Theta(\ell, t)$ and with the conditions of Theorem 3 and let p be a nonnegative constant, and $q > p > 0$ be constants. If $\Theta(\ell, t)$ satisfies*

$$\begin{aligned} \Theta^q(\ell, t) \quad \leq \quad & a(\ell, t) + \frac{p}{p-q} \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta) \Theta^p(\varsigma, \eta) \times \\ & \left[f(\varsigma, \eta) \zeta(\Theta(\varsigma, \eta)) + \int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right] \Delta\eta \Delta\varsigma \end{aligned} \tag{43}$$

for $(\ell, t) \in \Omega$, then

$$\Theta(\ell, t) \leq \left\{ F_1^{-1} \left[F_1(p_1(\ell, t)) + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \Delta\varsigma \right] \right\}^{\frac{1}{q-p}} \tag{44}$$

for $0 \leq \ell \leq \ell_2$, $0 \leq t \leq t_2$, where

$$p_1(\ell, t) = [a(\ell, t)]^{\frac{q-p}{q}} + \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \Im_1(\varsigma, \eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) \Delta\eta \Delta\varsigma,$$

and F_1 is defined in Theorem 7.

Proof. An application of Theorem 7 with $\Gamma(\Theta(\ell, t)) = \Theta^p$ to (43) yields the inequality (44); to save space we omit the details. □

Remark 7. *Taking $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$, $t_0 = 0$, $a(\ell, t) = b(\ell) + c(t)$, $\Im_1(\varsigma, \eta) f(\varsigma, \eta) = h(\varsigma, \eta)$, and $\Im_1(\varsigma, \eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) = g(\varsigma, \eta)$ in Corollary 4 we obtain ([20], Theorem 1).*

Remark 8. *Taking $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$, $t_0 = 0$, $a(\ell, t) = c^{\frac{p}{p-q}}$, $\Im_1(\varsigma, \eta) f(\varsigma, \eta) = h(\eta)$, and $\Im_1(\varsigma, \eta) \left(\int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) = g(\eta)$ and keeping t fixed in Corollary 4, we obtain ([21], Theorem 2.1).*

3. Application

In the following, we discus the boundedness of the solutions of the initial boundary value problem for partial delay dynamic equation, which maybe describe environmental phenomena, physical and engineering sciences, of the form:

$$(\varphi^q)^{\Delta_\ell \Delta_t}(\ell, t) = A\left(\ell, t, \varphi(\ell - h_1(\ell), t - h_2(t)), \int_{\ell_0}^{\ell} B(\varsigma, t, \varphi(\varsigma - h_1(\varsigma), t)) \Delta\varsigma \right) \tag{45}$$

$$\varphi(\ell, t_0) = a_1(\ell), \varphi(\ell_0, t) = a_2(t), a_1(\ell_0) = a_{t_0}(0) = 0$$

for $(\ell, t) \in \Omega$, where $\varphi, b \in C(\Omega, \mathbb{R}_+), A \in C(\Omega \times R^2, R), B \in C(\zeta \times R, R)$ and $h_1 \in C^1_{rd}(\mathbb{T}_1, \mathbb{R}_+), h_2 \in C^1_{rd}(\mathbb{T}_2, \mathbb{R}_+)$ are nondecreasing functions such that $h_1(\ell) \leq \ell$ on $\mathbb{T}_1$, $h_2(t) \leq t$ on $\mathbb{T}_2$, and $h_1^\Delta(\ell) < 1, h_2^\Delta(t) < 1$.

Theorem 8. *Assume that the functions b, A, B in (45) satisfy the conditions*

$$|a_1(\ell) + a_2(t)| \leq a(\ell, t) \tag{46}$$

$$|A(\varsigma, \eta, \varphi, \Theta)| \leq \frac{q}{q-p} \Im_1(\varsigma, \eta) [f(\varsigma, \eta) |\varphi|^p + |\Theta|] \tag{47}$$

$$|B(\chi, \eta, \varphi)| \leq \Im_2(\chi, \eta) |\varphi|^p, \tag{48}$$

where $a(\ell,t), \Im_1(\varsigma,\eta), f(\varsigma,\eta)$, and $\Im_2(\chi,\eta)$ are as in Theorem 3, $q > p > 0$ are constants. If $\varphi(\ell,t)$ satisfies (45), then

$$|\varphi(\ell,t)| \leq \left\{ p(\ell,t) + M_1 M_2 \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \bar{\Im}_1(\varsigma,\eta)\bar{f}(\varsigma,\eta)\Delta\eta\Delta\varsigma \right\}^{\frac{1}{q-p}}, \tag{49}$$

where

$$\begin{aligned} p(\ell,t) &= (a(\ell,t))^{\frac{q-p}{q}} \\ &+ M_1 M_2 \int_{\ell_0}^{\tau_1(\ell)} \int_{t_0}^{\tau_2(t)} \bar{\Im}_1(\varsigma,\eta)\left(M_1 \int_{\ell_0}^{\varsigma} \bar{\Im}_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma \end{aligned}$$

and

$$M_1 = \underset{\ell\in I_1}{Max}\frac{1}{1-h_1^{\Delta}(\ell)}, \qquad M_2 = \underset{t\in I_2}{Max}\frac{1}{1-h_2^{\Delta}(t)}$$

and $\bar{\Im}_1(\gamma,\xi) = \Im_1(\gamma + h_1(\varsigma), \xi + h_2(\eta)), \bar{\Im}_2(\mu,\xi) = \Im_2(\mu, \xi + h_2(\eta)), \bar{f}(\gamma,\xi) = f(\gamma + h_1 (\varsigma), \xi + h_2(\eta))$.

Proof. If $\varphi(\ell,t)$ is any solution of (45), then

$$\begin{aligned} &\varphi^q(\ell,t) = a_1(\ell) + a_2(t) \\ &+ \int_{\ell_0}^{\ell}\int_{t_0}^{t} A\left(\varsigma,\eta,\varphi(\varsigma - h_1(\varsigma), \eta - h_2(\eta)), \int_{\ell_0}^{\varsigma} B(\chi,\eta,\varphi(\chi - h_1(\chi),\eta))\Delta\chi\right)\Delta\eta\Delta\varsigma. \end{aligned} \tag{50}$$

Using the conditions (46)–(48) in (50) we obtain

$$\begin{aligned} |\varphi(\ell,t)|^q &\leq a(\ell,t) + \frac{q-p}{q}\int_{\ell_0}^{\ell}\int_{t_0}^{t} \Im_1(\varsigma,\eta)\left[f(\varsigma,\eta)|\varphi(\varsigma - h_1(\varsigma), \eta - h_2(\eta))|^p\right. \\ &\left.+ \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)|\varphi(\chi,\eta)|^p\Delta\chi\right]\Delta\eta\Delta\varsigma. \end{aligned} \tag{51}$$

Now making a change of variables on the right side of (51), $\varsigma - h_1(\varsigma) = \gamma, \eta - h_2(\eta) = \xi$, $\ell - h_1(\ell) = \tau_1(\ell)$ for $\ell \in \mathbb{T}_1, t - h_2(t) = \tau_2(t)$ for $t \in \mathbb{T}_2$ we obtain the inequality

$$\begin{aligned} |\varphi(\ell,t)|^q &\leq a(\ell,t) + \frac{q-p}{q}M_1M_2\int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \bar{\Im}_1(\gamma,\xi)\left[\bar{f}(\gamma,\xi)|\varphi(\gamma,\xi)|^p\right. \\ &\left.+M_1\int_{\ell_0}^{\gamma} \bar{\Im}_2(\mu,\xi)|\varphi(\mu,\eta)|^p\Delta\mu\right]\Delta\xi\Delta\gamma. \end{aligned} \tag{52}$$

We can rewrite the inequality (52) as follows:

$$\begin{aligned} |\varphi(\ell,t)|^q &\leq a(\ell,t) + \frac{q-p}{q}M_1M_2\int_{\ell_0}^{\tau_1(\ell)}\int_{t_0}^{\tau_2(t)} \bar{\Im}_1(\varsigma,\eta)\left[\bar{f}(\varsigma,\eta)|\varphi(\varsigma,\eta)|^p\right. \\ &\left.+M_1\int_{\ell_0}^{\varsigma} \bar{\Im}_2(\chi,\eta)|\varphi(\chi,\eta)|^p\Delta\chi\right]\Delta\eta\Delta\varsigma. \end{aligned} \tag{53}$$

As an application of Corollary 1 to (53) with $\Theta(\ell,t) = |\varphi(\ell,t)|$ we obtain the desired inequality (49). □

4. Conclusions

Using the Leibniz integral rule on time scales, we examined additional generalisations of the integral retarded inequality presented in [1] and generalised a few of those inequalities to a generic time scale. We also looked at the qualitative characteristics of various different dynamic equations' time-scale solutions. Furthermore, in the future, we think to extend these results in other directions by using (q,ω)-Hahn difference operator.

Author Contributions: Conceptualization, A.A.E.-D., D.B. and J.A.; formal analysis, A.A.E.-D., D.B. and J.A.; investigation, A.A.E.-D., D.B. and J.A.; writing—original draft preparation, A.A.E.-D., D.B. and J.A.; writing—review and editing, A.A.E.-D., D.B. and J.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Boudeliou, A.; Khellaf, H. On some delay nonlinear integral inequalities in two independent variables. *J. Inequalities Appl.* **2015**, *2015*, 313. [CrossRef]
2. Abdeldaim, A.; El-Deeb, A.A.; Agarwal, P.; El-Sennary, H.A. On some dynamic inequalities of Steffensen type on time scales. *Math. Methods Appl. Sci.* **2018**, *41*, 4737–4753. [CrossRef]
3. Agarwal, R.; O'Regan, D.; Saker, S. *Dynamic Inequalities on Time Scales*; Springer: Cham, Switzerland, 2014.
4. Akdemir, A.O.; Butt, S.I.; Nadeem, M.; Ragusa, M.A. New general variants of chebyshev type inequalities via generalized fractional integral operators. *Mathematics* **2021**, *9*, 122. [CrossRef]
5. Akin-Bohner, E.; Bohner, M.; Akin, F. Pachpatte inequalities on time scales. *JIPAM J. Inequal. Pure Appl. Math.* **2005**, *6*, 23.
6. Bohner, M.; Matthews, T. The Grüss inequality on time scales. C*ommun. Math. Anal.* **2007**, *3*, 1–8.
7. Bohner, M.; Matthews, T. Ostrowski inequalities on time scales. *JIPAM J. Inequal. Pure Appl. Math.* **2008**, *9*, 8.
8. Dinu, C. Hermite-Hadamard inequality on time scales. *J. Inequal. Appl.* **2008**, *24*, 287947. [CrossRef]
9. El-Deeb, A.A. Some Gronwall-bellman type inequalities on time scales for Volterra-Fredholm dynamic integral equations. *J. Egypt. Math. Soc.* **2018**, *26*, 1–17. [CrossRef]
10. El-Deeb, A.A.; Xu, H.; Abdeldaim, A.; Wang, G. Some dynamic inequalities on time scales and their applications. *Adv. Difference Equ.* **2019**, *19*, 130. [CrossRef]
11. El-Deeb, A.A.; Rashid, S. On some new double dynamic inequalities associated with leibniz integral rule on time scales. *Adv. Differ. Equ.* **2021**, *2021*, 125. [CrossRef]
12. Kh, F.M.; El-Deeb, A.A.; Abdeldaim, A.; Khan, Z.A. On some generalizations of dynamic Opial-type inequalities on time scales. *Adv. Differ. Eq.* **2019**, *2019*, 323. [CrossRef]
13. Tian, Y.; El-Deeb, A.A.; Meng, F. Some nonlinear delay Volterra-Fredholm type dynamic integral inequalities on time scales. *Discrete Dyn. Nat. Soc.* **2018**, *8*, 5841985. [CrossRef]
14. Hilger, S. Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. Thesis, Universitat Wurzburg, Wurzburg, Germany, 1988.
15. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhauser Boston, Inc.: Boston, MA, USA, 2001.
16. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhauser: Boston, MA, USA, 2003.
17. Ferreira, R.A.C.; Torres, D.F.M. Generalized retarded integral inequalities. *Appl. Math. Lett.* **2009**, *22*, 876–881. [CrossRef]
18. Ma, Q.; Pecaric, J. Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 393–407. [CrossRef]
19. Tian, Y.; Fan, M.; Meng, F. A generalization of retarded integral inequalities in two independent variables and their applications. *Appl. Math. Comput.* **2013**, *221*, 239–248. [CrossRef]
20. Xu, R.; Sun, Y.G. On retarded integral inequalities in two independent variables and their applications. *Appl. Math. Comput.* **2006**, *182*, 1260–1266. [CrossRef]
21. Sun, Y.G. On retarded integral inequalities and their applications. *J. Math. Anal. Appl.* **2005**, *301*, 265–275. [CrossRef]