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Well-Posedness for Nonlinear Parabolic Stochastic Differential Equations with Nonlinear Robin Conditions

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Abstract: In this paper, we present the existence and uniqueness of strong probabilistic solutions for nonlinear parabolic Stochastic Partial Differential Equations (SPDEs) with nonlinear Robin boundary conditions in a domain with holes. On the boundary of the holes, a nonlinear Robin condition is imposed, while a homogeneous Dirichlet condition is prescribed on the exterior boundary. The coefficient matrix is assumed to be symmetric, while the nonlinear random forces are assumed to satisfy some types of regularities. We use Galerkin's approximation method, probabilistic compactness results and some results from stochastic calculus.

Keywords: stochastic PDES; Robin boundary conditions; existence and uniqueness; perforated domains

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1. Introduction

The groundbreaking contributions of Bensoussan and Temam [1,2] throughout the 1970s can be credited with launching the mathematically rigorous investigation of Stochastic Partial Differential Equations (SPDEs). The theses of Viot [3] and Pardoux [4] came after the Bensoussan and Temam results. Viot uses the compactness approach to address significant types of nonlinear SPDEs within the context of weak probabilistic solutions in infinite dimensions. Using the monotonicity principle, Pardoux established a rather general theory of strong solutions for nonlinear equations. The foundational research of Krylov and Rozovskii [5], which focuses on general investigations on theories of strong probabilistic solutions employing compactness and traditional monotonicity approaches, further extends this theory, see also [6–8], for more interesting results in the theoretical framework. Recently, there are several results answering interesting questions for the existence and uniqueness of stochastic models in applications [9–15]. The investigation of the asymptotic behavior for composite materials and flow of fluid in fixed and porous media has been very well established (from mathematical analysis point of view) by the so called homogenization theory, see, for instance, [16–18] and the references therein, for homogenization of deterministic partial differential equations. We also mention that there are few results in the stochastic setting, see [19–22]. However, one of the most important principles on which the theory of homogenization depends is the well posedness of the governing equation. Despite the fact that the problem considered in this paper models well motivated physical problems such as the random effect on climatization or some chemical reactions, see [18,23]. Up to the author's knowledge, homogenization results for these types of models have not been investigated, let alone existence and uniqueness results. From this point of view, the findings of this paper are new, and open the door for many problems in the homogenization and asymptotic analysis for SPDEs. The aim of this paper is to give some existence and uniqueness results for strong probabilistic solutions of nonlinear stochastic equations in the perforated domain $D \subset \mathbb{R}^n$. More precisely, we consider the following problem:

$$(I)\begin{cases} dv - \operatorname{div} A\nabla v dt = F_1(v)dt + F_2(t,x,v)dB & in \quad (D\backslash S)\times(0,T) \\ A\nabla v\cdot\hat{n} + h(v) = g \quad on \quad \partial S\times(0,T) \\ v = 0 \quad on \quad (\partial D\backslash\partial S)\times(0,T), \\ v(x,0) = v_0(x), \quad in \quad D; \end{cases}$$

where D is an open bounded domain in $\mathbb{R}^n$, S is a connected subset included in D with Lipschitz boundary, $\hat{n}$ is the unit outward normal to the set S, $T \in (0,\infty)$, $B = (B(t))$ $(t \in [0,T])$ is an m-dimensional standard Wiener process, i.e., $t\times\Omega \mapsto B(t,\omega)$ $(t\in(0,T)) = (B_1(t,\omega),\cdots,B_m(t,\omega))$ such that $B_k(t,\omega), k = 1,\cdots,m$ are one dimensional standard Wiener processes that are identically distributed and pairwise independent, and B is defined on a given filtered complete probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{0\le t\le T})$. The model considered in this paper is related to homogenization theory on material science and engineering. Further, the model is physically motivated by the fact that several composites of the thermal conductivity depend, in a nonlinear way, on the temperature itself. Moreover, the nonlinear Robin boundary conditions appear in several physical models such as climatization or some chemical reactions, see [18,23]. Let us consider the following assumptions:

(A_1) The matrix $A(x) = (a_{i,j}(x)_{1\le i,j\le n}$ is an $n\times n$, symmetric, bounded and:

$$(A\eta,\eta) \ge \beta|\eta|^2, \quad \beta > 0 \quad \text{and } \eta\in\mathbb{R}^n \quad \text{(coercivity)}.$$

(A_2) $F_1 \in L_2(0,T;L_2(D))$ such that $\parallel F_1(t,v) \parallel \le C\Big(1+ \parallel v \parallel_{L_2(D)}\Big)$,

(A_3) $F_2(t,x,v)$ is an m-dimensional vector function whose components:

$F_{2j}(t,x,v)$ satisfy the following conditions:

- $F_{2j}(t,x,v)$ is measurable with respect to x for almost all $t\in(0,T)$ and for all $u\in L_2(D)$;
- $F_{2j}(t,x,v)$ is continuous with respect to u for almost all $(t,x)\in(0,T)\times D$, and there exists a positive constant C independent of t and x such that $\parallel F_{2j}(t,x,v) \parallel_{L_2(D)} \le C\Big(1+ \parallel v \parallel_{L_2(D)}\Big)$;
- $F_{2j}(t,x,v)$ satisfies Lipschitz condition with respect to the L_2 metric.

The differential $F_2 dB$ is understood in the sense of Itô.

(A_4) $h : \mathbb{R}\times(0,T)\to\mathbb{R}$, such that:

- $(\mathbb{I}_1)$ h is continuously differentiable function in the first argument $t\in[0,T]$;
- $(\mathbb{I}_2)$ h is monotonously non-decreasing in the first argument;
- $(\mathbb{I}_3)$ $h(t,0) = 0, \forall t\in[0,T]$;
- $(\mathbb{I}_4)$ There exists a positive constant C and q with $0\le q\le\infty$ if $n=2$ and $0\le q\le \dfrac{n}{n-2}$ if $n\ge 2$ such that $|\ h(t,u)\ | \le C(1+ |\ v\ |^q)$.

Remark 1. *With the assumption made on the nonlinear boundary function h, one could easily see that:*

- $h(v(x,t)).v(x,t) \ge 0$ *for all functions* $v : D\times(0,t)\mapsto\mathbb{R}$;
- *For any functions* $v,\psi\in L_2(0,T;H^1(D\backslash S)$ *we have:*

$$\int_0^T\int_{\partial S}|h(v(x,t))\psi(x,t)|dtd\sigma < \infty.$$

With the preceding context, we state the main result of this paper:

Theorem 1. *Assume that the hypotheses (A_1)–(A_4) hold and the function h having the properties $\mathbb{I}_1$–$\mathbb{I}_4$. Then there exists a unique strong probabilistic solution (v,B) of the problem (I).*

The proof of this result will be carried out in the sections that follow.

2. Tightness Property

This section establishes the tightness of probability measures. We refer to [19,24,25]. for more information on relatively compactness, tightness of probability measures, and Prokhorov and Skorokhod compactness results. Let $\mathcal{J}$ be a separable Banach space and consider its Borel σ-field to be $\mathcal{B}(\mathcal{J})$.

We also consider $\{\mu_m\}$ and $\{\nu_m\}$, $\mu_m \geq 0$ and $\nu_m \geq 0$ for all $m \in \mathbb{N}$ such that $\mu_m \to 0$ and $\nu_m \to 0$ as $m \to \infty$ and $\sum_m \dfrac{\sqrt{\mu_m}}{\nu_m} < \infty$, and define the set $\mathcal{N}_*$ as follows:

$$\mathcal{N}_* = \Bigg\{ v : \sup_{0\leq t\leq T} \| v \|^2_{L_2(D)} \leq C_1, \quad \int_0^T \| v \|^2_{H^1_0(D)} \, dt \leq C_2,$$
$$\sup_{0\leq t\leq T} \frac{1}{\nu_m} \sup_{|\theta|\leq\mu_m} \int_0^{T-\theta} \| v(t+\theta) - v(t) \|^2_{H^{-1}(D)} \, dt \leq C3 \Bigg\}.$$

The norm of $\mathcal{N}_*$ is given by:

$$\| v, \|_{\mathcal{N}_*} = \sup_{0\leq t\leq T} \| v \|_{L_2(D)} + \left(\int_0^T \| v \|^2_{H^1_0(D)} \, dt \right)^{\frac{1}{2}}$$
$$+ \sup_{0\leq t\leq T} \frac{1}{\nu_m} \sup_{|\theta|\leq\mu_m} \left(\int_0^{T-\theta} \| v(t+\theta) - v(t) \|^2_{H^{-1}(D)} \, dt \right)^{\frac{1}{2}}.$$

For more details on the following lemma, see [19,26]:

Lemma 1. *The set $\mathcal{N}_*$ is a compact subset of $L_2(0,T;L_2(D))$.*

Define $\mathcal{J} = C(0,T;\mathbb{R}^m) \times L_2(0,T;L_2(D))$ and $\mathcal{B}(\mathcal{J})$ the σ-algebra of its Borel sets. Let Ψ_n be the map:

$$\Psi_n : D \to \mathcal{J}, \quad w \to (B(w,.), v^n(w,.)).$$

For each n we introduce a measure Π_n on $(\mathcal{J}, \mathcal{B}(\mathcal{J}))$ by:

$$\Pi_n(A) = \mathbb{P}(\Psi_n^{-1}(A)), \quad \forall A \in \mathcal{B}(\mathcal{J}).$$

With this setting, we have the following result:

Theorem 2. *The family of measures $\{\Pi_n\}$ is tight.*

Proof. For arbitrary $\propto > 0$, we look for compact subsets $H_\propto \subset C(0,T;\mathbb{R}^m)$ and $K_\propto \subset L_2(0,T;L_2(D))$, such that:

$$\mathbb{P}\{w : B(w,.) \notin H_\propto\} \leq \frac{\propto}{2}. \tag{1}$$

$$\mathbb{P}\{w : v^n(w,.) \notin K_\propto\} \leq \frac{\propto}{2}. \tag{2}$$

Define $H_\propto$ consisting of all $B(.) \in C(0,T;\mathbb{R}^m)$ such that

$$\sup\left\{ |B(t_2) - B(t_1)| \leq \frac{L_\propto}{m} : t_1, t_2 \in [0,T], | t_2 - t_1 | < m^{-6} \right\}.$$

It is clear that $H_\propto$ is compactly included in $C(0,T;\mathbb{R}^m)$ this is due to the well known result of Arzela and Ascoli. Using Markov's inequality as in [26], see also [19]. We get:

$$\mathbb{P}\{w : B(w,.) \notin H_{\propto}\} \leq \mathbb{P}\left\{\bigcup_{n}\left\{w : \sup_{t_1,t_2\in[0,T],|t_2-t_1|<m^{-6}} |B(t_2) - B(t_1)| > \frac{L_{\propto}}{m}\right\}\right\}$$

$$\leq \sum_{m=1}^{\infty}\sum_{i=0}^{m^{-6}}\left(\frac{m}{L_{\propto}}\right)^4 \mathbb{E} \sup_{iTm^{-6}\leq t\leq(i+1)Tm^{-6}}\left|B(t) - B(iTm^{-6})\right|^4$$

$$\leq C\sum_{m=1}^{\infty}\left(\frac{m}{L_{\propto}}\right)^4 (Tm^{-6})^2 m^6 = \frac{C}{L_{\propto}^4}\sum_{m=1}^{\infty}\frac{1}{m^2} \leq \frac{\propto}{2}.$$

We choose $L_{\propto} = \frac{1}{2C\propto}\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^{-1}$.

Now,

$$\mathbb{P}\{v(w,.) \notin K_{\propto}\} \leq \mathbb{P}\left\{\sup_{0\leq t\leq T}\|v\|^2_{L_2(D)} > M_{\propto}\right\} + \mathbb{P}\left\{\int_0^T \|v\|^2_{H_0^1(D)}dt > N_{\propto}\right\}$$

$$+\mathbb{P}\left\{\sup_{0\leq t\leq T}\frac{1}{\nu_m}\sup_{|\theta|\leq\mu_m}\int_0^{T-\mu_{\propto}} \| v(t+\theta) - v(t) \|^2_{H^{-1}(D)}\, dt > Q_{\propto}\right\}$$

$$\leq \frac{1}{M_{\propto}}\sup_{0\leq t\leq T}\|v\|^2_{L_2(D)} + \frac{1}{N_{\propto}}\mathbb{E}\int_0^T \|v\|^2_{H_0^1(D)}dt$$

$$+\sum_m \frac{1}{\nu_m Q_{\propto}}\mathbb{E}\left\{\sup_{|\theta|\leq\mu_m}\int_0^{T-\mu_{\propto}} \| v(t+\theta) - v \|^2_{H^{-1}(D)}\, dt\right\}$$

$$\leq \frac{C}{M_{\propto}} + \frac{C}{N_{\propto}} + \frac{C}{Q_{\propto}}\sum_m \frac{\sqrt{\mu_m}}{\nu_m} = \frac{\propto}{2}.$$

Choose $M_{\propto} = N_{\propto} = \frac{6C}{\propto}$ and $Q_{\propto} = \frac{6C\sum_m \frac{\sqrt{\mu_m}}{\nu_m}}{\propto}$. From (1)and (2), we get:

$$\mathbb{P}\{w : B(w,.) \in H_{\propto}; v(w,.) \in K_{\propto}\} \geq 1- \propto .$$

This proves that:

$$\Pi_n(H_{\propto} \times K_{\propto}) \geq 1- \propto .$$

As a result, the family Π_n is tight, and the proof is complete. □

3. Galerkin Approximation

In this section, we will build a weak solution using Galerkin approximation. For this, we let $H_n = span(w_1, w_2, \cdots, w_n) \subset H_0^1(D)$. We seek a finite-dimensional approximation of a solution to our problem in the form of a vector $v^n \in H_n$, which can be written as the Fourier series:

$$v^n = \sum_{k=1}^{n}\zeta_n^k(t)w_k(x).$$

We require that v^n satisfies:

$$\begin{aligned}&(dv^n, w_k)_{L_2(D)} + (A\nabla v^n, \nabla w_k)_{L_2(D)}dt + (h(v^n), w_k)_{L_2(\partial S)}\\ &= (g, w_k)_{L_2(\partial S)} + (F_1(v^n), w_k)_{L_2(D)}dt + (F_2(v^n), w_k)dB,\end{aligned} \tag{3}$$

$k = 1, \cdots, n$.

A Priori Bounds

In this subsection, we derive energy and finite difference bounds that will be used to prove the existence result of problem (I).

Lemma 2. *Suppose that the assumptions A_1–A_4 are satisfied, then:*
(i) $\mathbb{E} \quad ess\ sup_{[0,T]} \parallel v^n \parallel^2_{L_2(D)} \leq C$,
(ii) $\mathbb{E} \parallel v^n \parallel^2_{L_2(0,T;[H^1(D)]^n)} \leq C$.

Proof. Let us introduce the following stopping time:

$$\tau_M = \begin{cases} \inf\{0 \leq t : \|v^n\|_{H^1_0(D)} + \|v^n\|_{L_2(\partial S)} > M\} \\ T \quad if \quad \{\|v^n\|_{H^1_0(D)} + \|v^n\|_{L_2(\partial S)} > M\} = \emptyset, \end{cases} \tag{4}$$

for any integer $M > 0$.

We multiply Equation (3) by (v^n, w_k) and sum over $k = 1, \cdots, n$, to obtain:

$$\begin{aligned} &(dv^n, \sum_{k=1}^{n} (v^n, w_k) w_k)_{L_2(D)} + (A\nabla v^n, \sum_{k=1}^{n} (v^n, w_k) \nabla w_k)_{L_2(D)} dt \\ &\quad + (h(v^n), \sum_{k=1}^{n} (v^n, w_k) w_k)_{L_2(\partial S)} dt = (g, \sum_{k=1}^{n} (v^n, w_k) w_k)_{L_2(\partial S)} dt \\ &\quad + (F_1(v^n), \sum_{k=1}^{n} (v^n, w_k) w_k)_{L_2(D)} dt + (F_2(t, x, v^n), \sum_{k=1}^{n} (v^n, w_k) w_k)_{L_2(D)} dB, \end{aligned}$$

by definition $\zeta^k_n(t) = (v^n, w_k)$, then we get:

$$\begin{aligned} &(dv^n, v^n)_{L_2(D)} + (A\nabla v^n, \nabla v^n)_{L_2(D)} dt + (h(v^n), v^n)_{L_2(\partial S)} dt \\ &= (g, v^n)_{L_2(\partial S)} dt + (F_1(v^n), v^n)_{L_2(D)} dt + (F_2(t, x, v^n), v^n)_{L_2(D)} dB, \end{aligned} \tag{5}$$

applying Itô formula:

$$\begin{aligned} &d \parallel v^n \parallel^2_{L_2(D)} + 2(A\nabla v^n, \nabla v^n)_{L_2(D)} dt + 2(h(v^n), v^n)_{L_2(\partial S)} = 2(g, v^n)_{L_2(\partial S)} \\ &+ 2(F_1(v^n), v^n)_{L_2(D)} dt + (F_2(t, x, v^n), v^n)_{L_2(D)} dB + \parallel F_2(t, x, v^n) \parallel^2_{L_2(D)} dt, \end{aligned}$$

integrating over $0 \leq s \leq t \wedge \tau_m$, and $t \leq T$, we have:

$$\begin{aligned} &\parallel v^n(t) \parallel^2_{L_2(D)} + 2\int_0^s (A\nabla v^n(\tau), \nabla v^n(\tau))_{L_2(D)} d\tau + 2\int_0^s \int_{\partial S} h(v^n(\tau)).v^n(\tau) d\tau d\sigma(\tau) \\ &\quad \parallel v^n_0 \parallel^2_{L_2(D)} + 2\int_0^s \int_{\partial S} (g(\tau), v^n(\tau)) d\tau d\sigma + 2\int_0^s (F_1(v^n(\tau)), v^n(\tau))_{L_2(D)} d\tau \\ &\quad + \int_0^s (F_2(t, x, v^n(\tau)), v^n(\tau))_{L_2(D)} dB(\tau) + \int_0^s \parallel F_2(t, x, v^n(\tau)) \parallel^2 d\tau, \end{aligned}$$

using the assumption on the matrix A, taking the supremum over $0 \leq s \leq t \wedge \tau_m$ and the expectation, we have:

$$\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_m}\Bigg[\parallel v^n(s)\parallel^2_{L_2(D)}+\int_0^s\parallel\nabla v^n(\tau)\parallel^2_{L_2(D)}d\tau$$
$$+2\Bigg|\int_0^s\int_{\partial S}h(v^n(\tau)).v^n(\tau)d\tau d\sigma(\tau)\Bigg|\Bigg]$$
$$\leq\alpha\parallel v_0^n\parallel^2_{L_2(D)}+2\alpha\mathbb{E}\int_0^{t\wedge\tau_m}\int_{\partial S}|(g(s),v^n(s))d\tau d\sigma|$$
$$+2\alpha\mathbb{E}\int_0^{t\wedge\tau_m}|(F_1(v^n(s)),v^n(s))_{L_2(D)}ds|$$
$$+\alpha\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_m}\int_0^{t\wedge\tau_m}|(F_2(t,x,v^n(s)),v^n(s))|_{L_2(D)}dB(s)$$
$$+\alpha\mathbb{E}\int_0^{t\wedge\tau_m}\parallel F_2(t,x,v^n(s))\parallel^2 ds. \quad (6)$$

From Remark 1, we have:

$$\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_m}\Bigg[\parallel v^n(s)\parallel^2_{L_2(D)}+\int_0^s\parallel\nabla v^n(\tau)\parallel^2_{L_2(D)}d\tau\Bigg]$$
$$\leq\alpha\parallel v_0^n\parallel^2_{L_2(D)}+2\alpha\mathbb{E}\int_0^{t\wedge\tau_m}\int_{\partial S}|(g(s),v^n(s))d\tau d\sigma|$$
$$+2\alpha\mathbb{E}\int_0^{t\wedge\tau_m}|(F_1(v^n(s)),v^n(s))_{L_2(D)}ds|$$
$$+\alpha\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_m}\int_0^{t\wedge\tau_m}|(F_2(t,x,v^n(s)),v^n(s))|_{L_2(D)}dB(s)$$
$$+\alpha\mathbb{E}\int_0^{t\wedge\tau_m}\parallel F_2(t,x,v^n(s))\parallel^2 ds. \quad (7)$$

Let us find bound for the right hand side of Equation (7). Using Cauchy–Schwarz's, Burkhölder–Davis–Gundy's and Young's inequalities and the hypotheses on F_2, we obtain:

$$\mathbb{E}\sup\Bigg|\int_0^s(F_2(\tau,x,v^n(\tau)),v^n(\tau))_{L_2(D)}dB(\tau)\Bigg|$$
$$\leq C\mathbb{E}\Bigg(\int_0^s(F_2(\tau,x,v^n(\tau)),v^n(\tau))^2\Bigg)^{\frac{1}{2}}dv$$
$$\leq C\mathbb{E}\Bigg(\int_0^t\parallel v^n(v)\parallel^2_{L_2(D)}\parallel F_2(\tau,x,v^n(\tau)\parallel^2_{L_2(D)}d\tau\Bigg)^{\frac{1}{2}}$$
$$\leq\rho\mathbb{E}\sup\parallel v^n(\tau)\parallel^2_{L_2(D)}+C(\rho)\int_0^T\parallel F_2(\tau,v^n(\tau))\parallel^2_{L_2(D)}d\tau$$
$$\leq\rho\mathbb{E}\sup\parallel v^n(v)\parallel^2_{L_2(D)}+C(\rho)T+C(\rho)\int_0^T\parallel v^n(\tau)\parallel_{L_2(D)}d\tau, \quad (8)$$

for $\rho>0$.

Now, from ([21], Proposition 3) we get:

$$\mathbb{E}\Bigg|\int_0^T(g,v^n)_{L_2(\partial S)}dt\Bigg|$$
$$\leq C\Bigg[\Big|\mathcal{M}_{L_2(\partial S)}(g)\Big|\mathbb{E}\int_0^T\parallel v^n\parallel_{L_2(D)}dt$$
$$+\mathbb{E}\int_0^T\parallel\nabla v^n\parallel_{[L_2(D)]^n}dt\Bigg]. \quad (9)$$

Using Poincar'e inequality, we obtain:

$$\| (F_1(v^n), v^n) \| \leq \varrho^2 \| \nabla v^n \| \| \nabla F_1(v^n) \|,$$

where ϱ is Poincare constant. We also mention that:

$$(\varrho^2 + \beta)^{-1} \| \psi \|^2_{L_2(D)} \leq \| \nabla \psi \|^2 \leq (\beta)^{-1} \| \psi \|^2_{L_2(D)},$$

for all $\psi \in L_2(D)$. Then, from the assumption on F_1 and this, we get:

$$2 \| (F_1(v^n), v^n) \|_{L_2(D)} \leq 2C \frac{\varrho^2}{\alpha} \Big(1 + \| v^n(s) \|^2_{L_2(D)} \Big). \tag{10}$$

Using Equations (8)–(10) into (6), together with Grönwall's inequality, gives the desired estimates. Thus, the proof is complete. □

Lemma 3. *For* $4 \leq p < \infty$ *we have:*

$$\mathbb{E} \quad ess\, sup_{[0,T]} \| v^n \|^p_{L_2(D)} \leq C, \tag{11}$$

$$\mathbb{E} \| v^n \|^p_{L_2(0,T;[H^1(D)]^n)} \leq C, \tag{12}$$

for some positive constant C.

Proof. For $4 \leq p < \infty$, we apply Itô's lemma to $\| v^n \|^{\frac{p}{2}}_{L_2(D)}$, we have:

$$\begin{aligned} \| v^n(t) \|^{\frac{p}{2}}_{L_2(D)} + \frac{p}{2} \int_0^t \| v^n(s) \|^{(\frac{p}{2}-2)}_{L_2(D)} & \Big[(A \nabla v^n(s), \nabla v^n(s))_{L_2(D)} \\ & + (h(v^n(s)), v^n(s))_{L_2(\partial S)} \Big] ds \\ & = \| u_0^n \|^{\frac{p}{2}} + \frac{p}{2} \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big[(g, v^n(s))_{L_2(\partial S)} \\ & + (F_1(v^n(s)), v^n(s))_{L_2(D)} \Big] ds \\ & + \frac{p}{2} \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} + (F_2(t, x, v^n(s)), v^n(s)) dB \\ & + \frac{p}{4} (\frac{p}{2} - 1) \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \| F_2(t, x, v^n(s)) \|^2 \, ds. \end{aligned}$$

From Remark 1, we have:

$$\begin{aligned} \| v^n(t) \|^{\frac{p}{2}}_{L_2(D)} + \frac{p}{2} \int_0^t \| v^n(s) \|^{(\frac{p}{2}-2)}_{L_2(D)} & \Big[(A \nabla v^n(s), \nabla v^n(s))_{L_2(D)} \Big] ds \\ & = \| u_0^n \|^{\frac{p}{2}} + \frac{p}{2} \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big[(g, v^n(s))_{L_2(\partial S)} \\ & + (F_1(v^n(s)), v^n(s))_{L_2(D)} \Big] ds \\ & + \frac{p}{2} \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} + (F_2(t, x, v^n(s)), v^n(s)) dB \\ & + \frac{p}{4} (\frac{p}{2} - 1) \int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \| F_2(t, x, v^n(s)) \|^2 \, ds. \end{aligned}$$

Taking the Square on both sides, using the assumption on the matrix A, and some elementary inequalities, we obtain:

$$\| v^n(t) \|^p_{L_2(D)} + C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big[\| \nabla v^n(s) \|^2_{L_2(D)}\Big] ds\Big)^2$$
$$\leq C \| u_0^n \|^p_{L_2(D)} + C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big[(g, v^n(s))_{L_2(\partial S)}$$
$$+ (F_1(v^n(s)), v^n(s))_{L_2(D)}\Big] ds\Big)^2$$
$$C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} (F_2(t,x,v^n(s)), v^n(s)) dB\Big)^2$$
$$+ C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \| F_2(t,x,v^n(s)) \|^2 ds\Big)^2.$$

We conclude from Poincar′e inequality, the hypothesis on h and Equations (8)–(10) that:

$$\| v^n(t) \|^p_{L_2(D)} + \int_0^t \| \nabla v^n(s) \|^p_{L_2(D)} ds \leq C \| v_0^n \|^p_{L_2(D)}$$
$$+ C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big(\| v^n(s) \|_{L_2(D)} + \| \nabla v^n \|_{[L_2(D)]^n}\Big)\Big)^2$$
$$+ C\Big(\int_0^t \| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \Big(1 + \| v^n(s) \|^2_{L_2(D)}\Big)\Big)^2 ds$$
$$+ C\Big(\| v^n(s) \|^{\frac{p}{2}-2}_{L_2(D)} \| v^n(s) \|^2_{L_2(D)}\Big)^2,$$

we know that:

$$\| v^n \|^{p-4} \leq (1 + \| v^n(t) \|_{L_2(D)})^{p-4}$$

taking the expectation and the supremum on both sides to the above equation and using Grönwall's inequality, we arrive at the estimates (11) and (12). Thus, the proof is complete. □

Remark 2. *Lemmas 2 and 3 imply the following:*

(i) $\mathbb{E}$ *ess* $sup_{[0,T]} \| v^n \|^p_{L_2(D)} \leq C$,

(ii) $\mathbb{E} \| v^n \|^p_{L_2(0,T;[H^1(D)]^n)} \leq C$,

for any $1 \leq p \leq \infty$.

Lemma 4. *For every* $\delta \in (0,1)$ *there exists* $C > 0$ *such that for all natural number* n*, we get:*

$$\mathbb{E} \sup_{|\theta| < \delta} \int_0^{T-\delta} \|v^n(s+\theta) - v^n(s)\|^2_{H^{-1}(D)} ds \leq C\delta^{\frac{1}{2}}$$

Proof. Assume that $\tilde{v^n} = 0$ in $\mathbb{R} \backslash [0,T]$. We write:

$$v^n(s+\theta) - v^n(s) = \int_s^{s+\theta} \operatorname{div}(A\nabla v^n(t)) dt + \int_s^{s+\theta} F_1(v^n(t)) dt$$
$$+ \int_s^{s+\theta} F_2(t,x,v^n(t)) dB_t$$

Then,

$$\|v^n(s+\theta) - v^n(s)\|_{H^{-1}(D)} \leq \Big\| \int_s^{s+\theta} \operatorname{div}(A\nabla v^n(t)) dt \Big\|_{H^{-1}(D)}$$
$$+ \Big\| \int_s^{s+\theta} F_1(v^n(t)) dt \Big\|_{H^{-1}(D)} + \Big\| \int_s^{s+\theta} F_2(t,x,v^n(t)) dB_t \Big\|_{H^{-1}(D)}. \tag{13}$$

We will argue as in ([19], Lemma 2). By definition of the norm in $H^{-1}(D)$, and assumption A_1, we have:

$$\begin{aligned}
&\left\|\int_s^{s+\theta} div(A\nabla v^n(t))dt\right\|_{H^{-1}(D)} \\
&\leq \sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\left|\left\langle\int_s^{s+\theta}\mathrm{div}(A\nabla v^n(t))dt,\Psi\right\rangle_{H^{-1}(D),H_0^1(D)}\right| \\
&= \sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\int_D\int_s^{s+\theta}A\nabla v^n\nabla\Psi dxdt \\
&\leq C\sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\int_s^{s+\theta}\|v^n\|_{H_0^1(D)}\|\Psi\|_{H_0^1(D)}dt\leq C\theta.
\end{aligned} \tag{14}$$

From assumption A_2, we deduce:

$$\begin{aligned}
&\left\|\int_s^{s+\theta}F_1(v^n(t))dt\right\|_{H^{-1}(D)} \\
&\leq \sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\left|\left\langle\int_s^{s+\theta}F_1(v^n(t))dt,\Psi\right\rangle_{H^{-1}(D),H_0^1(D)}\right| \\
&= \sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\int_D\int_s^{s+\theta}F_1(v^n(t))\Psi dxdt \\
&\leq C\sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\int_s^{s+\theta}(1+\|v^n\|_{L_2(D)})\|\Psi\|_{L_2(D)}dt\leq C\theta.
\end{aligned} \tag{15}$$

Finally, we shall bound the stochastic term, for that we use Fubini's theorem, Burkholder–Davis–Gundy's inequality, Cauchy–Schwarz's inequality, and hypothesis A_3, we get:

$$\begin{aligned}
&\mathbb{E}\sup_{|\theta|<\delta}\left\|\int_s^{s+\theta}F_2(t,v^n)dB_t\right\|^2_{H^{-1}(D)}\leq \\
&\leq\mathbb{E}\sup_{|\theta|<\delta}\left(\int_s^{s+\theta}\left(\int_D F_2(t,v^n)\Psi dx\right)dB_t\right)^2 \\
&\leq C\sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\mathbb{E}\left(\int_s^{s+\delta}\left(\int_D F_2(t,v^n)\Psi dx\right)^2dt\right) \\
&\leq C\sup_{\Psi\in H_0^1(D):\|\Psi\|=1}\mathbb{E}\left(\int_s^{s+\delta}\|F_2(t,v^n)\|^2_{L_2(D)}\|\Psi\|^2_{L_2(D)}dt\right) \\
&\leq C\mathbb{E}\left[\delta+\int_s^{s+\delta}\|v^n\|^2_{L_2(D)}ds\right]\leq C\delta.
\end{aligned} \tag{16}$$

Substituting (14)–(16) into (13), integrating over $(0,T-\delta$, taking the expectation and the $\sup_{|\theta|<\delta}$ we have:

$$\mathbb{E}\sup_{|\theta|<\delta}\int_0^{T-\delta}\|v^n(s+\theta)-v^n(s)\|^2_{H^{-1}(D)}ds\leq C\delta^{\frac{1}{2}}$$

□

4. Implementation of the Tightness Property

From the estimates obtained in Lemmas 2 and 4, one can easily see that $v^n \in \mathcal{N}_*$. Then from the tightness Theorem 2 and Prokhorov's Lemma, there exists a subsequence Π_{n_j} of Π_{n_j} and Π such that Π_{n_j} weakly convergent to Π. With this in hand, we apply Skorokhod's Lemma, to have the existence of a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ and $\mathcal{B}$-valued random variables (B_{n_j}, v^{n_j}) and $(\widehat{B}, v)$ with the property that the probability law of (B_{n_j}, v^{n_j}) is Π_{n_j} and the probability law of $(\widehat{B}, v)$ is Π. Then, we have:

$$(B_{n_j}, v^{n_j}) \to (\widehat{B}, v) \quad in \quad \mathcal{B}, \quad \widehat{\mathbb{P}}\text{-}a.s. \tag{17}$$

We now define:

$$\widehat{\mathcal{F}}_t = \sigma\Big\{\widehat{B}(s), v(s)\Big\}_{s\in[0,t]},$$

and prove $\widehat{B}(t)$ is an $\widehat{\mathcal{F}}_t$-Wiener process. To do this, we will follow the same steps as in [27]. Define the sub intervals $(0, t_1), (t_1, t_2), \cdots, (t_{m-1}, t_m = T)$, it is enough to prove that:

- $(B(t_j) - B(t_{j-1}))$ are independent for all $i = 1, 2, \cdots m$;
- $(B(t_j) - B(t_{j-1}))$ are normally distributed with $\mu = 0$ and $\sigma = t_j - t_{j-1}$ for all $i = 1, 2, \cdots m$.

To prove the above assertions, we show,

$$\widehat{\mathbb{E}} \exp\left\{\sum_{j=1}^{m} i\lambda_j [B(t_j) - B(t_{j-1})]\right\} = \exp\left\{-\frac{1}{2}\sum_{j=1}^{m} \lambda_j^2 (t_j - t_{j-1})\right\}, \tag{18}$$

where $\lambda_j \in \mathbb{R}^m$ and $\widehat{\mathbb{E}}$ denotes the mathematical expectation on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$.
Note that (18) is similar to:

$$\widehat{\mathbb{E}} \exp\{i\lambda[B(t+k) - B(t)]/\mathcal{F}\} = \exp\left\{-\frac{\lambda^2 k}{2}\right\}, \tag{19}$$

for any $k > 0$ and λ in $\mathbb{R}^m$. Now, let us consider an arbitrary bounded continuous functional $\hbar(\widehat{B}, v)$ depending only on the values of $\widehat{B}$ and v on $(0, T)$, from classical probability theory, we get:

$$\begin{aligned} \widehat{\mathbb{E}}\{\exp i\lambda[B(t+k) - B(t)]\hbar(\hat{B}, v)\} &= \widehat{\mathbb{E}} \exp\{i\lambda[B(t+k) - B(t)]\}\widehat{\mathbb{E}}\hbar(\widehat{B}, v) \\ &= \exp\left\{-\frac{\lambda^2 k}{2}\right\}\widehat{\mathbb{E}}\hbar(\widehat{B}, v). \end{aligned} \tag{20}$$

Since $[B_{n_j}(t+k) - B_{n_j}(t)]$ are independent of $\hbar(B_{n_j}, v^{n_j})$ and B_{n_j} is a Wiener process, it follows that:

$$\begin{aligned} &\widehat{\mathbb{E}}\Big\{\exp i\lambda[B_{n_j}(t+k) - B_{n_j}(t)]\hbar(B_{n_j}, v^{n_j})\Big\} \\ &= \widehat{\mathbb{E}} \exp\Big\{i\lambda[B_{n_j}(t+k) - B_{n_j}(t)]\Big\}\widehat{\mathbb{E}}\hbar(B_{n_j}, v^{n_j}) \\ &= \exp\left\{-\frac{\lambda^2 k}{2}\right\}\widehat{\mathbb{E}}\hbar(B_{n_j}, v^{n_j}), \end{aligned} \tag{21}$$

Using (17) and the continuity of $\hbar$ we can pass to the limit in this equality and get (20), which, in view of (18), implies (19) and therefore $\widehat{B}(t)$ is an $\widehat{\mathcal{F}}_t$-Wiener process.

Theorem 3. *The pair* (B_{n_j}, v^{n_j}) *satisfies* $\mathbb{P}$*-a.s.*

$$
\begin{aligned}
(v^{n_j}(s), w_k)_{L_2(D)} - (v_0, w_k) + \int_0^t (A\nabla v^{n_j}(s), \nabla w_k)_{L_2(D)} ds \\
+ \int_0^t (h(v^{n_j}(s)), w_k)_{L_2(\partial S)} ds = \int_0^t (g, w_k)_{L_2(\partial S)} ds \\
+ \int_0^t (F_1(v^{n_j}(s)), w_k)_{L_2(D)} ds + \int_0^t (F_2(v^{n_j}(s)), w_k) dB_{n_j}(s)
\end{aligned}
\tag{22}
$$

for any $k \geq 0$.

Proof. Let us set:

$$
\begin{aligned}
\aleph^n = \int_0^T \Big\| (v^n(s), w_k)_{L_2(D)} - (v_0, w_k) + \int_0^t (A\nabla v^n(s), \nabla w_k)_{L_2(D)} ds \\
+ \int_0^t (h(v^n(s)), w_k)_{L_2(\partial S)} ds - \int_0^t (g, w_k)_{L_2(\partial S)} ds \\
- \int_0^t (F_1(v^n(s)), w_k)_{L_2(D)} ds - \int_0^t (F_2(v^n(s)), w_k) dB_n \Big\|^2_{H^{-1}(D)} dt
\end{aligned}
\tag{23}
$$

Clearly,

$$\aleph^n = 0 \quad \mathbb{P}\text{-}a.s.,$$

and,

$$\hat{\mathbb{E}} \frac{\aleph^n}{1 + \aleph^n} = 0.$$

If we define:

$$
\begin{aligned}
\Upsilon^{n_j} = \int_0^T \Big| (v^{n_j}(s), w_k)_{L_2(D)} - (v_0, w_k) + \int_0^t (A\nabla v^{n_j}(s), \nabla w_k)_{L_2(D)} ds \\
+ \int_0^t (h(v^{n_j}(s)), w_k)_{L_2(\partial S)} ds - \int_0^t (g, w_k)_{L_2(\partial S)} ds \\
- \int_0^t (F_1(v^{n_j}(s)), w_k)_{L_2(D)} ds - \int_0^t (F_2(v^{n_j}(s)), w_k) dB_{n_j} \Big|^2 dt,
\end{aligned}
\tag{24}
$$

we will show that,

$$\hat{\mathbb{E}} \frac{\Upsilon^{n_j}}{1 + \Upsilon^{n_j}} = 0. \tag{25}$$

The main difficulty in the achievement of this is the that $\aleph^n$ is a stochastic functional of v^n and B_n. To overcome this, we define a regularization of F_2 by:

$$F_2^{\varepsilon}(v(t), t) = \frac{1}{\varepsilon} \int_0^T \gamma(-\frac{t-s}{\varepsilon}) F_2(v(s), s) ds, \tag{26}$$

such that γ is the standard mollifier, from which we have:

$$\mathbb{E} \int_0^T |F_2^{\varepsilon}(v(t), t)|^2 dt \leq \mathbb{E} \int_0^T |F_2(v(t), t)|^2 dt \tag{27}$$

and,

$$F_2^{\varepsilon}(v(.), .) \to F_2(v(.), .) \quad in \quad L_2(\widehat{\Omega}, \widehat{\mathbb{P}}, L_2(0, T; L_2(D))),$$

from this, we have for all $k \geq 1$,

$$(F_2^{\varepsilon}(v(.), .), w_k) \to (F_2(v(.), .), w_k) \; in \; L_2(\widehat{\Omega}, \widehat{\mathbb{P}}, L_2(0, T)). \tag{28}$$

By replacing F_2 by its regularization in the stochastic term, we introduce the corresponding functionals $\aleph^{n,\varepsilon}$ and $\Upsilon^{n_j,\varepsilon}$. We now define the application:

$$\psi^{n,\varepsilon} : C(0,T;\mathbb{R}^m) \times L_2(0,T;L_2((D))) \to (\widehat{\Omega}, \widehat{\mathbb{P}}, \widehat{\mathcal{F}})$$

$$\psi^{n,\varepsilon}(B_t, v^n) = \frac{\aleph^{n,\varepsilon}}{1+\aleph^{n,\varepsilon}}$$

This is a continuous bounded functional on B. Now, introduce,

$$\psi^{n_j,\varepsilon}(B_t, v^{n_j}) = \frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}},$$

we have:

$$\widehat{\mathbb{E}}\psi^{n_j,\varepsilon}(B_{n_j}, v^n) = \widehat{\mathbb{E}}\frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}}. \tag{29}$$

Since $\psi^{n_j,\varepsilon}(B_{n_j}, v^{n_j})$ is bounded in $\mathcal{B}$ and the law of $\psi^{n_j}(B_{n_j}, v^{n_j})$ is Π_{n_j}, then:

$$\widehat{\mathbb{E}}\frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}} = \int_{\mathcal{B}} \psi(w,v)d\Pi_{n_j}. \tag{30}$$

Further, the law of (B_t, v^{n_j}) is Π_{n_j}, therefore,

$$\int_{\mathcal{B}} \psi(w,v)d\Pi_{n_j} = \widehat{\mathbb{E}}\psi^{n_j}(B_t, v^{n_j}) = \widehat{\mathbb{E}}\frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}}. \tag{31}$$

From (24)and (31), we have:

$$\widehat{\mathbb{E}}\frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}} = \widehat{\mathbb{E}}\frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}}.$$

Although,

$$\widehat{\mathbb{E}}\frac{\Upsilon^{n_j}}{1+\Upsilon^{n_j}} - \mathbb{E}\frac{\aleph^{n_j}}{1+\aleph^{n_j}} = \widehat{\mathbb{E}}\left(\frac{\Upsilon^{n_j}}{1+\Upsilon^{n_j}} - \frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}}\right) + \widehat{\mathbb{E}}\frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}}$$
$$-\mathbb{E}\frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}} + \mathbb{E}\left(\frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}} - \frac{\aleph^{n_j}}{1+\aleph^{n_j}}\right),$$

furthermore,

$$\begin{aligned}\widehat{\mathbb{E}}\left|\frac{\Upsilon^{n_j}}{1+\Upsilon^{n_j}} - \frac{\Upsilon^{n_j,\varepsilon}}{1+\Upsilon^{n_j,\varepsilon}}\right| &= \widehat{\mathbb{E}}\left|\frac{\Upsilon^{n_j} - \Upsilon^{n_j,\varepsilon}}{(1+\Upsilon^{n_j})(1+\Upsilon^{n_j,\varepsilon})}\right| \\ &\leq \widehat{\mathbb{E}}|\Upsilon^{n_j} - \Upsilon^{n_j,\varepsilon}| \\ &\leq C\left(\mathbb{E}\int_0^T |(F_2^\varepsilon(v^{n_j}(t),t) - F_2(v^{n_j}(t),t), w_k|^2 dt\right)^{\frac{1}{2}},\end{aligned}$$

we also have,

$$\mathbb{E}\left(\frac{\aleph^{n_j}}{1+\aleph^{n_j}} - \frac{\aleph^{n_j,\varepsilon}}{1+\aleph^{n_j,\varepsilon}}\right) \leq C\left(\mathbb{E}\int_0^T |(F_2^\varepsilon(v^{n_j}(t),t) - F_2(v^{n_j}(t),t), w_k|^2 dt\right)^{\frac{1}{2}}.$$

The above estimates and (29), gives:

$$\begin{aligned}&\left|\left|\widehat{\mathbb{E}}\frac{\Upsilon^{n_j}}{1+\Upsilon^{n_j}}\right| - \left|\mathbb{E}\frac{\aleph^{n_j}}{1+\aleph^{n_j}}\right|\right| \\ &\qquad \leq C\left(\mathbb{E}\int_0^T |(F_2^\varepsilon(v^{n_j}(t),t) - F_2(v^{n_j}(t),t), w_k|^2 dt\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the limit as ε goes to zero, we get the relation (25). This proves the result. $\square$

5. Passage to the Limits

The following Theorem is given in [28]. We use this result in the prove of the existences of the strong probabilistic solutions.

Theorem 4. *Let O be an open set in $\mathbb{R}^n$, $h : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function and $N : v \longrightarrow N(v)$ its associated Nemytskii operator (i.e., class of nonlinear operators on L_P spaces with continuity and boundedness properties). The following three equivalent:*

(i) $h : L_p(O) \longrightarrow L_q(O), \quad \frac{1}{p} + \frac{1}{q} = 1.$
(ii) *h is bounded and continuous from $L_p(O)$ to $L_q(O)$.*
(iii) *There exist $a > 0$ and $b > 0$ such that:*

$$|h(s)| \leq a + b|s|^{\frac{p}{q}} \quad \forall s \in \mathbb{R}.$$

Now we are in the situation to pass to the limit in (3). From the Prokhorov and Skorokhod results, we have:

$$v^{n_j} \to v \quad in \quad L_2(0, T; H_0^1(D)) \quad \mathbb{P} - a.e. \tag{32}$$

Since v^{n_j} verifies the Equation (22), then it satisfies the same estimates as v^n.
Thanks to the uniform integrability and the estimate in Lemma 3, we can use Vitali's theorem to conclude that:

$$v^{n_j} \to v \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; H_0^1(D))), \tag{33}$$

which gives:

$$v^{n_j} \to v \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; L_2(D))) \tag{34}$$

Thanks to (33), there exists a subsequence of v^{n_j}, which, for simplicity, has the same notation as v^{n_j} such that for almost all (w, t):

$$v^{n_j} \to v \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; L_2(D))) \tag{35}$$

with respect to the measure $dt \times d\mathbb{P}$. It is easily seen that:

$$(v^{n_j}, w_k) \to (v, w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; L_2(D))). \tag{36}$$

Using Lemma 3 and assumption on A, we get:

$$(A\nabla v^{n_j}, \nabla w_k) \to (A\nabla v, \nabla w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; H^{-1}(D))). \tag{37}$$

Lemma 3, convergence (35), hypothesis on F_1 and Vitali's theorem imply:

$$F_1(v^{n_j}(.), .) \to F_1(v(.), .) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; H_0^1(D))), \tag{38}$$

from which, we have:

$$(F_1(v^{n_j}(.), .), w_k) \to (F_1(v(.), .), w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; L_2(D))), \tag{39}$$

for all k. Let us know show that:

$$(h(v^{n_j}(.), .), w_k) \to (h(v(.), .), w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T; L_2(\partial S))). \tag{40}$$

To do so, we define the nonlinear operator $N : u \in H_S \to N(v) \in H_S^{'}$ with:

$$N(v) : \phi \to \int_{\partial S} h(v)\phi d\sigma(x),$$

where $H_S = \{\phi \in H^1(D) | \phi = 0 \quad on \quad \partial D \backslash \partial S\}$ and proves that it is continuous. From Theorem 4 we have that $v \in H_S \to v \in L_2(\partial S)$ and $h : v \in L_2(\partial S) \to h(v) \in L_2(\partial S)$ are continuous. Hence, the mapping,

$$M : v \in H_S \to h(v) \in L_2(\partial S)$$

is also continuous. If we consider (36), we have:

$$\begin{aligned}
&| \langle N(v^{n_j}(.),.), w_k) - N(v(.),.), w_k), \phi \rangle_{H'_S, H_S} | \\
&\qquad = \left| \int_{\partial S} ((h(v^{n_j}(.),.), w_k) - (h(v(.),.), w_k) \phi d\sigma \right| \\
&\qquad \leq \| \phi \|_{L_2(\partial S)} \| M(v^{n_j}(.),.), w_k) - M(v(.),.), w_k) \|_{L_2(\partial S)}
\end{aligned}$$

Since M is continuous, we obtain:

$$\begin{aligned}
&\|N(v^{n_j}(.),.), w_k) - N(v(.),.), w_k) \| \\
&\qquad = \sup_{\phi \neq 0} \frac{\langle N(v^{n_j}(.),.), w_k) - N(v(.),.), w_k), \phi \rangle_{H'_S, H_S}}{\| \phi \|_{H_S}} \\
&\qquad \leq \| M(v^{n_j}(.),.), w_k) - M(v(.),.), w_k \|_{L_2(\partial S)} \longrightarrow 0.
\end{aligned}$$

Then Equation (40) is proved. Finally, we will show that:

$$\int_0^t (F_2(v^{n_j}, s), w_k) dB_{n_j} \rightharpoonup \int_0^t (F_2(v, s), w_k) d\widehat{B} \quad in \quad L_2(\Omega, \mathbb{P}; L_\infty(0, T)), \tag{41}$$

for any $t \in (0, T)$ and k as $j \to \infty$. Arguing as in [29], we will show that:

$$\int_0^T (F_2(v^{n_j}, s), w_k) dB_{n_j} \rightharpoonup \int_0^T (F_2(v, s), w_k) d\widehat{B} \quad in \quad L_2(\Omega, \mathbb{P}), \tag{42}$$

from (35) the Lemma 3, the assumption on F_1, and Vitali's theorem, we have:

$$(F_2(v^{n_j}(.),.), w_k) \to (F_2(v(.),.), w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T)), \tag{43}$$

as $j \to \infty$. Considering the regularization $F_2^\varepsilon(v(.),.)$ in (26). We can easily show that:

$$(F_2^\varepsilon(v(.),.), w_k) \to (F_2(v(.),.), w_k) \quad in \quad L_2(\Omega, \mathbb{P}, L_2(0, T)), \tag{44}$$

and:

$$\begin{aligned}
&\mathbb{E} \int_0^T |(F_2^\varepsilon(v^{n_j}, t) - F_2^\varepsilon(v, t), w_k)|^2 dt \\
&\qquad \leq \mathbb{E} \int_0^T |(F_2(v^{n_j}, t) - F_2(v, t), w_k)|^2 dt.
\end{aligned} \tag{45}$$

The difficulty is to show that:

$$\int_0^T (F_2^\varepsilon(v^{n_j}, s), w_k) dB_{n_j} \rightharpoonup \int_0^T (F_2^\varepsilon(v, s), B_k) d\widehat{B} \quad in \quad L_2(\Omega, \mathbb{P}). \tag{46}$$

Since:

$$\mathbb{E} \left| \int_0^T (F_2^\varepsilon(v^{n_j}, t), w_k) dB_{n_j} \right|^2 = \mathbb{E} \left| \int_0^T (F_2^\varepsilon(v^{n_j}, t), w_k) dt \right|^2 < \infty, \tag{47}$$

then, the left-hand-side weakly converges to some α in $L_2(\Omega,\mathbb{P})$. An integration by parts gives:

$$\int_0^T (F_2^\varepsilon(v^{n_j},t),w_k)dB_{n_j} \\ = (F_2^\varepsilon(v^{n_j}(T),T),w_k) - \int_0^T B_{n_j}\frac{d}{dt}F_2^\varepsilon(v^{n_j}(t),t),w_k)dt, \tag{48}$$

where:

$$\frac{d}{dt}F_2^\varepsilon(v^{n_j}(t),t),w_k) = \frac{1}{\varepsilon}\int_0^T \frac{d}{dt}\gamma(-\frac{t-s}{\varepsilon})F_2(v(s),s)ds. \tag{49}$$

From (17), we have:

$$\int_0^T (F_2^\varepsilon(v^{n_j},t),w_k)dB_{n_j} \\ \to (F_2^\varepsilon(v(T),T),w_k) - \int_0^T B(t)\frac{d}{dt}F_2^\varepsilon(v(t),t),w_k)dt \tag{50}$$

for almost all $w \in \Omega$. The term in the left-hand-side of (50) is equal to:

$$\int_0^T (F_2^\varepsilon(v(t),t),w_k)d\widehat{B} \tag{51}$$

Now, let us choose an element $\xi \in L_\infty(\Omega,\mathbb{P})$, we have:

$$\int_0^T (F_2^\varepsilon(v^{n_j},t),\xi w_k)dB_{n_j} \to \int_0^T (F_2^\varepsilon(v(t),t),\xi w_k)d\widehat{B}. \tag{52}$$

Therefore,

$$\alpha = \int_0^T (F_2^\varepsilon(v(t),t),w_k)d\widehat{B} \tag{53}$$

Thanks to the estimate (27), Lemma 2 the sequence of random variables $\int_0^T (F_2^\varepsilon(v^{n_j},t),\xi w_k)dB_{n_j}$ is uniformly integrable. Using convergence (50), and Vitali's Theorem, we get (52). We also have (46) since $L_\infty(\Omega,\mathbb{F},\mathbb{P})$ is dense in $L_2(\Omega,\mathbb{P})$.

Let $\xi \in L_\infty(\Omega,\mathbb{P})$, we have:

$$\left|\mathbb{E}\int_0^T (F_2(v^{n_j},t),\xi w_k)dB_{n_j} - \mathbb{E}\int_0^T (F_2(v(t),t),\xi w_k)d\widehat{B}\right| \le I_1 + I_2 + I_3. \tag{54}$$

where:

$$I_1 = \left|\mathbb{E}\int_0^T (F_2^\varepsilon(v^{n_j},t),\xi w_k)dB_{n_j} - \mathbb{E}\int_0^T (F_2(v^{n_j}(t),t),\xi w_k)dB_{n_j}\right| \tag{55}$$

$$I_2 = \left|\mathbb{E}\int_0^T (F_2^\varepsilon(v^{n_j},t),\xi w_k)dB_{n_j} - \mathbb{E}\int_0^T (F_2^\varepsilon(v(t),t),\xi w_k)dB\right| \tag{56}$$

$$I_3 = \left|\mathbb{E}\int_0^T (F_2^\varepsilon(v,t),\xi w_k)dB - \mathbb{E}\int_0^T (F_2(v(t),t),\xi w_k)dB\right| \tag{57}$$

By the Cauchy−Schwarz's inequality, using estimate (45) and convergence (43) and (44), it easily seen that I_1 tends to 0 when $\varepsilon \to 0$ and $j \to \infty$. Using (52) we get that I_2 does converge to zero as $j \to \infty$. Again, from the Cauchy−Schwarz's inequality, and convergence (44), I_3 tends to zero as $j \to \infty$. From these convergences, one can pass to the limit in (54) as $\varepsilon \to 0$ and $j \to \infty$; thus, we obtain (46). Collecting all those results and passing to the limit in (22), we show that v satisfies Problem I in the weak sense.

6. Proof of Uniqueness

Theorem 5. *Suppose that the assumption of Theorem 1 can be fulfilled; furthermore, $F_1(v,t)$ is Lipschitz continuous. Then, v is a unique solution of problem (I).*

Proof. Let v^1 and v^2 be solutions of Problem (I), we have:

$$\Big\langle d(v^1-v^2), v\Big\rangle_{H^{-1}(D),H_0^1(D)} + \int_D A\nabla(v^1-v^2)\nabla v dx \\ + \int_D \Big(h(v^1,t)-h(v^2,t)\Big) v d\sigma(x) = \int_D \Big(F_1(v^1,t)-F_1(v^2,t)\Big) v dx \\ + \int_D \Big(F_2(v^1,t)-F_2(v^2,t)v\Big)\widehat{B}. \tag{58}$$

Let $v = v^1 - v^2$,

$$\Big\langle d(v^1-v^2), v^1-v^2\Big\rangle_{H^{-1}(D),H_0^1(D)} + \int_D A\nabla(v^1-v^2)\nabla(v^1-v^2) dx \\ + \int_D \Big(h(v^1,t)-h(v^2,t)\Big)(v^1-v^2) d\sigma(x) \\ = \int_D \Big(F_1(v^1,t)-F_1(v^2,t)\Big)(v^1-v^2) dx \\ + \int_D \Big(F_2(v^1,t)-F_2(v^2,t)\Big)(v^1-v^2) d\widehat{B},$$

so, we have,

$$\frac{1}{2}\frac{d}{dt}\|v^1-v^2\|^2_{L_2(D)} + \int_D A\nabla(v^1-v^2)\nabla(v^1-v^2) dx \\ + \int_D \Big(h(v^1,t)-h(v^2,t)\Big)(v^1-v^2) d\sigma(x) \\ = \int_D \Big(F_1(v^1,t)-F_1(v^2,t)\Big)(v^1-v^2) dx \\ + \int_D \Big(F_2(v^1,t)-F_2(v^2,t)\Big)(v^1-v^2) d\widehat{B}.$$

Now, integrating over $(0,t)$, using the assumption on A and taking the expectation we have:

$$\mathbb{E}\|(v^1-v^2)(t)-(v^1-v^2)(0)\|^2_{L_2(D)} \\ + \alpha\mathbb{E}\int_0^t \Big[\|(v^1-v^2)(s)\|^2_{H_0^1(D)} + I_1\Big] ds leq \mathbb{E}\int_0^t |I_2+I_3| ds. \tag{59}$$

where:

$$I_1 = \int_{\partial S} \Big(h(v^1,t)-h(v^2,t)\Big)(v^1-v^2) d\sigma(x)$$
$$I_2 = \int_D \Big(F_1(v^1,t)-F_1(v^2,t)\Big)(v^1-v^2) dx$$
$$I_3 = \int_D \Big(F_2(v^1,t)-F_2(v^2,t)(v^1-v^2)\Big) d\widehat{B}.$$

First, since h is a monotonously non-decreasing function, then the product $\big(h(v^1,t)-h(v^2,t)\big)\,(v^1-v^2) \geq 0$, so (59) becomes:

$$\mathbb{E}\|(v^1-v^2)(t)-(v^1-v^2)(0)\|^2_{L_2(D)} \\ + \alpha\mathbb{E}\int_0^t \|(v^1-v^2)(s)\|^2_{H_0^1(D)} ds \leq \mathbb{E}\int_0^t |I_2+I_3| ds. \tag{60}$$

Now, we estimate I_3. Thanks to Burkhölder–Davis–Gundy's inequality, followed by the Cauchy–Schwarz's inequality, we can estimate:

$$\mathbb{E}\left|\int_0^t \left(F_2(v^1,t) - F_2(v^2,t)(v^1-v^2)\right)dB(s)\right|$$
$$\leq C\mathbb{E}\left(\int_0^t \left(F_2(v^1,t) - F_2(v^2,t)(v^1-v^2)\right)^2\right)^{\frac{1}{2}} ds$$
$$\leq C\mathbb{E}\left(\int_0^t \parallel (v^1-v^2)(s) \parallel^2_{L_2(D)} \parallel (F_2(v^1,t) - F_2(v^2,t))(s) \parallel^2_{L_2(D)} ds\right)^{\frac{1}{2}},$$

using Young's inequality and the assumption on F_2, we get:

$$\begin{aligned}\mathbb{E}\left(\int_0^t \parallel (v^1-v^2)(s) \parallel^2_{L_2(D)} \parallel (F_2(v^1,t) - F_2(v^2,t))(s) \parallel^2_{L_2(D)} ds\right)^{\frac{1}{2}}\\ \leq \rho\mathbb{E} \parallel (v^1-v^2)(s) \parallel^2_{L_2(D)}\\ +C(\rho)\int_0^t \parallel (F_2(v^1,t) - F_2(v^2,t))(s) \parallel^2_{L_2(D)} ds\\ \leq \rho\mathbb{E}\sup \parallel (v^1-v^2)(s) \parallel^2_{L_2(D)}\\ +C(\rho)T + C(\rho)\int_0^T \parallel (v^1-v^2)(s) \parallel_{L_2(D)} ds,\end{aligned} \tag{61}$$

for $\rho > 0$.

Finally, let us estimate I_2 by Cauchy–Schwarz's and the assumption on F_1, we get:

$$\mathbb{E}\left|\int_0^t \int_D \left(F_1(v^1,t) - F_1(v^2,t)(v^1-v^2)\right)dxds\right|$$
$$\leq C\mathbb{E}\sup \parallel (v^1-v^2)(s) \parallel^2_{L_2(D)} + CT + C\int_0^T \parallel (v^1-v^2)(s) \parallel_{L_2(D)} ds \tag{62}$$

considering (62) and (61) we have:

$$\mathbb{E}\|(v^1-v^2)(t) - (v^1-v^2)(0)\|^2_{L_2(D)} + \alpha\mathbb{E}\int_0^t \|(v^1-v^2)(s)\|^2_{H^1_0(D)} ds \leq C.$$

where C independent of s.

$$\mathbb{E}\|(v^1-v^2)(t) - (v^1-v^2)(0)\|^2_{L_2(D)} + \alpha\mathbb{E}\int_0^t \|(v^1-v^2)(s)\|^2_{H^1_0(D)} ds \leq 0.$$

This implies that $v^1 - v^2 \equiv 0$. □

Thanks to these pathwise uniqueness results, one can easily apply the celebrated result of Yamada–Watanabe, see [30], to obtain the existence of a strong probabilistic solution.

7. Conclusions

In this paper, we have investigated existence and uniqueness of strong probabilistic solutions for nonlinear parabolic stochastic partial differential equations with nonlinear Robin boundary condition in a domain with holes. The tools used are: Galerkin's approximation method, probabilistic compactness results, and some results from stochastic calculus. The problem considered in this paper describe interesting physical models such as the effect of external random forces on climatization or some chemical reaction, which is more realistic than the obtained deterministic models. As we mentioned in the Introduction, the results obtained in this paper are mainly keyed towards the study of homogenization and asymptotic analysis. With this results in hand, one can derive homogenization results for linear and nonlinear stochastic PDES with nonlinear boundary conditions.

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