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Regularized Solution of the Cauchy Problem in an Unbounded Domain

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Abstract: In this paper, using the construction of the Carleman matrix, we explicitly find a regularized solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a three-dimensional unbounded domain.

Keywords: Carleman matrix; regularized solution; matrix factorizations; Helmholtz equation

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1. Introduction

A fundamental problem in the theory of differential equations (ordinary and partial) is the determination of a solution that verifies certain initial conditions.

Regarding Cauchy problems, certain questions arise: Does a solution exist (even locally only)? Is this unique? In this case, the solution continuously depends on the initial data, that is, is the problem well posed? The concept of a well-posed problem is connected with investigations by the famous French mathematician Hadamard [1]. The problems that are not well-posed are called ill-posed problems. The theory of ill-posed problems has been the subject of research by many mathematicians in the last years, with applicability in various fields: theoretical physics, optimization of control, astronomy, management and planning, automatic systems, etc., all of which have been influenced by the rapid development of computing technology.

Tikhonov [2] answered certain questions that are posed in the class of ill-posed problems, such as: what does an approximate solution mean, and what algorithm can be used to find such an approximate solution? This involves including additional assumptions. This process is known as regularization. Tikhonov regularization is one of the most commonly used for the regularization of linear ill-posed problems. Lavrent'ev [3,4] also established a regularization method. Based on this method, Yarmukhamedov [5,6] constructed the Carleman functions for the Laplace and Helmholtz, when the data is unknown on a conical surface or a hyper surface. Carleman-type formulas allow a solution to an elliptic equation to be found when the Cauchy data are known only on a part of the boundary of the domain. Carleman [7] obtained a formula for a solution to Cauchy–Riemann equations, on domains of certain forms. Based on [7], Goluzin and Krylov [8] gave a formula for establishing the values of analytic functions on arbitrary domains. The multidimensional case was treated in [9]. The Cauchy problem for elliptic equations was considered by Tarkhanov [10,11]. In [12], the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain was considered. Certain boundary value problems and the determination of numerical solutions was investigated in [13–25]. In [21] is studied the Cauchy problem of a modified Helmholtz equation. An efficient D-N alternating algorithm for solving an inverse problem for Helmholtz equation was investigated in [18]. The Cauchy problem for elliptic equations, was studied in [2–11] and then it was investigated in [12,26–37].

In this article, based on previous works [30–32,37], we find an explicit formula for an approximate solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a three-dimensional unbounded domain of $\mathbb{R}^3$. The approximate solution formula requires construction of a family of fundamental solutions of the Helmholtz operator in space. This family is parametrized by some entire function $K(z)$. Relying on the works [30–37], we obtain better results, due to a special selection of the function $K(z)$. This helped us to obtain good results when finding an approximate solution based on the Carleman matrix.

Let

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3, \quad \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3,$$
$$\zeta' = (\zeta_1, \zeta_2) \in \mathbb{R}^2, \quad \eta' = (\eta_1, \eta_2) \in \mathbb{R}^2,$$

and $\Omega \subset \mathbb{R}^3$ an unbounded, simply connected domain, having the boundary $\partial\Omega$ piecewise smooth, such that $\partial\Omega = \Sigma \bigcup D$, where D is the plane $\eta_3 = 0$ and Σ is a smooth surface lying in the half-space $\eta_3 > 0$.

The following notations are used in the paper:

$$r = |\eta - \zeta|, \quad \alpha = |\eta' - \zeta'|, \quad z = i\sqrt{a^2 + \alpha^2} + \eta_3, \quad a \geq 0,$$

$$\partial_\zeta = \left(\partial_{\zeta_1}, \partial_{\zeta_2}, \partial_{\zeta_3}\right)^T, \quad \partial_\zeta \to \chi^T, \quad \chi^T = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} \text{ transposed vector } \chi,$$

$$W(\zeta) = (W_1(\zeta), \ldots, W_n(\zeta))^T, \quad v^0 = (1, \ldots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m = 3,$$

$$E(w) = \begin{Vmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & w_n \end{Vmatrix} - \text{diagonal matrix}, w = (w_1, \ldots, w_n) \in \mathbb{R}^n.$$

$P(\chi^T)$ is an $n \times n$ matrix, having the elements linear functions with constant coefficients from $\mathbb{C}$, such that

$$P^*(\chi^T)P(\chi^T) = E((|\chi|^2 + \lambda^2)v^0),$$

where $P^*(\chi^T)$ is the Hermitian conjugate matrix of $P(\chi^T)$ and $\lambda \in \mathbb{R}$.

Next, we consider the system

$$P(\partial_\zeta)W(\zeta) = 0, \quad \eta \in \Omega, \tag{1}$$

where $P(\partial_\zeta)$ is the matrix differential operator of order one.

Additionally, consider the set

$$S(\Omega) = \{W : \overline{\Omega} \longrightarrow \mathbb{R}^n\},$$

where W is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$ and W satisfies (1).

2. Statement of the Cauchy Problem

We formulate now the following Cauchy problem for the system (1):

Let $f : \Sigma \longrightarrow \mathbb{R}^n$ be a continuous given function on Σ.

Suppose $W(\eta) \in S(\Omega)$ and

$$W(\eta)|_\Sigma = f(\eta), \quad \eta \in \Sigma. \tag{2}$$

Our purpose is to determine the function $W(\eta)$ in the domain Ω when its values are known on Σ.

If $W(\eta) \in S(\Omega)$, then

$$W(\zeta) = \int\limits_{\partial\Omega} L(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{3}$$

where

$$L(\eta, \zeta; \lambda) = \left(E\left(\Gamma_3(\lambda r) v^0\right) P^*(\partial_\zeta)\right) P(t^T),$$

$t = (t_1, t_2, t_3)$ is the unit exterior normal at a point η on the surface $\partial\Omega$ and $\Gamma_3(\lambda r)$ denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$(see, [38]), that is

$$\Gamma_3(\lambda r) = -\frac{e^{i\lambda r}}{4\pi r}. \tag{4}$$

Let $K(z)$ be an entire function taking real values for real z ($z = a + ib$, $a, b \in \mathbb{R}$), satisfying

$$K(a) \neq 0, \quad \sup_{b \geq 1} \left| b^p K^{(p)}(z) \right| = B(a, p) < \infty, \\ -\infty < a < \infty, \quad p = \overline{0, 3}. \tag{5}$$

Define

$$\Psi(\eta, \zeta; \lambda) = -\frac{1}{2\pi^2 K(\zeta_3)} \int\limits_0^\infty \mathrm{Im}\left[\frac{K(z)}{z - \zeta_3}\right] \frac{\cos(\lambda a)}{\sqrt{a^2 + \alpha^2}} da, \text{ for } \eta \neq \zeta. \tag{6}$$

Consider $\Psi(\eta, \zeta; \lambda)$ in (3) instead $\Gamma_3(\lambda r)$, where

$$\Psi(\eta, \zeta; \lambda) = \Gamma_3(\lambda r) + G(\eta, \zeta; \lambda), \tag{7}$$

$G(\eta, \zeta; \lambda)$ being the regular solution of Helmholtz's equation with respect to η, including the case $\eta = \zeta$.

We obtain

$$W(\zeta) = \int\limits_{\partial\Omega} L(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{8}$$

where

$$L(\eta, \zeta; \lambda) = \left(E\left(\Psi(\eta, \zeta; \lambda) v^0\right) P^*(\partial_\zeta)\right) P(t^T).$$

We generalize (8) for the case when the domain Ω is unbounded.

Hence, in what follows, we consider the domain $\Omega \subset \mathbb{R}^3$ be unbounded.

Suppose that Ω is situated inside the layer of smallest width defined by the inequality

$$0 < \eta_3 < h, \quad h = \frac{\pi}{\rho}, \quad \rho > 0,$$

and $\partial\Omega$ extends to infinity.

Let

$$\Omega_R = \{\eta : \eta \in \Omega, \quad |\eta| < R\}, \quad \Omega_R^\infty = \Omega \backslash \Omega_R, \quad R > 0.$$

Theorem 1. *Let $W(\eta) \in S(\Omega)$. If for each fixed $\zeta \in \Omega$ we have the equality*

$$\lim_{R \to \infty} \int\limits_{\Omega_R^\infty} L(\eta, \zeta; \lambda) W(\eta) ds_\eta = 0, \tag{9}$$

then (8) *is satisfied.*

Proof. Fix $\zeta \in \Omega$, $|\zeta| < R$. Using (8) we obtain

$$\int_{\partial\Omega} L(\eta,\zeta;\lambda)W(\eta)ds_\eta = \int_{\partial\Omega_R} L(\eta,\zeta;\lambda)W(\eta)ds_y$$

$$+ \int_{\partial\Omega_R^\infty} L(\eta,\zeta;\lambda)W(\eta)ds_\eta = W(\zeta) + \int_{\partial\Omega_R^\infty} L(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta \in \Omega_R.$$

Using (9), we obtain (8).

Also assume that the length $\partial\Omega$ satisfies the following growth condition

$$\int_{\partial\Omega} \exp\left[-d_0\rho_0|\eta'|\right] ds_\eta < \infty, \quad 0 < \rho_0 < \rho, \tag{10}$$

for some $d_0 > 0$. Suppose $W(\eta) \in S(\Omega)$ satisfies

$$|W(\eta)| \le \exp\left[\exp\rho_2|\eta'|\right], \quad \rho_2 < \rho, \quad \eta \in \Omega. \tag{11}$$

We consider in (6):

$$K(z) = \exp\left[-d\,i\rho_1\left(z - \frac{h}{2}\right) - d_1\,i\rho_0\left(z - \frac{h}{2}\right)\right],$$

$$K(\zeta_3) = \exp\left[d\,\cos\rho_1\left(\zeta_3 - \frac{h}{2}\right) + d_1\,\cos i\rho_0\left(\zeta_3 - \frac{h}{2}\right)\right], \tag{12}$$

$$0 < \rho_1 < \rho, \quad 0 < \zeta_3 < h,$$

where

$$d = 2c\exp\left(\rho_1|\zeta'|\right), \quad d_1 > \frac{d_0}{\cos\left(\rho_0\dfrac{h}{2}\right)}, \quad c \ge 0, \quad d > 0.$$

Then (8) is valid.

Let $\zeta \in \Omega$ be fixed and $\eta \to \infty$. We estimate $\Psi(\eta,\zeta;\lambda)$, $\dfrac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}$, $j = \overline{1,2}$ and $\dfrac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_3}$. To estimate $\dfrac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}$, we use the equality

$$\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j} = \frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial\eta_j} = 2(\eta_j - \zeta_j)\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial s}, \quad j = \overline{1,2}. \tag{13}$$

Really,

$$\left|\exp\left[-d\,i\rho_1\left(z - \frac{h}{2}\right) - d_1 i\rho_0\left(z - \frac{h}{2}\right)\right]\right|$$

$$= \exp\mathrm{Re}\left[-d\,i\rho_1\left(z - \frac{h}{2}\right) - d_1 i\rho_0\left(z - \frac{h}{2}\right)\right]$$

$$= \exp\left[-d\,\rho_1\sqrt{a^2+\alpha^2}\cos\rho_1\left(\eta_3 - \frac{h}{2}\right) - d_1\rho_0\sqrt{a^2+\alpha^2}\cos\rho_0\left(\eta_3 - \frac{h}{2}\right)\right].$$

As
$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \leq \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2},$$
$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho} \cdot \frac{\pi}{2} \leq \rho_0\left(y_3 - \frac{h}{2}\right) \leq \frac{\rho_1}{\rho} \cdot \frac{\pi}{2} < \frac{\pi}{2}.$$
Consequently,
$$\cos\rho\left(\eta_3 - \frac{h}{2}\right) > 0, \quad \cos\rho_0\left(\eta_3 - \frac{h}{2}\right) \geq \cos\frac{h\rho_0}{2} > \delta_0 > 0.$$
It does not vanish in the region Ω and
$$|\Psi(\eta,\zeta;\lambda)| = O[\exp(-\varepsilon\rho_1|\eta'|)], \quad \varepsilon > 0, \quad \eta \to \infty, \quad \eta \in \Omega \bigcup \partial\Omega,$$
$$\left|\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}\right| = O[\exp(-\varepsilon\rho_1|\eta'|)], \quad \varepsilon > 0, \quad \eta \to \infty, \quad \eta \in \Omega \bigcup \partial\Omega, \quad j = \overline{1,2}.$$
$$\left|\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_3}\right| = O[\exp(-\varepsilon\rho_1|\eta'|)], \quad \varepsilon > 0, \quad \eta \to \infty, \quad \eta \in \Omega \bigcup \partial\Omega.$$

We now choose ρ_1 with the condition $\rho_2 < \rho_1 < \rho$. Hence, (10) is satisfied and (8) is true. □

Condition (12) can be weakened.
Denote
$$S_\rho(\Omega) = \{W(\eta) : W(\eta) \in S(\Omega), |W(\eta)| \leq \exp[O(\exp\rho|\eta_1|)], \eta \to \infty, \eta \in \Omega\}. \tag{14}$$

Theorem 2. *If $W(\eta) \in S_\rho(\Omega)$ satisfies*
$$|W(\eta)| \leq C\exp\left[c\cos\rho_1\left(\eta_3 - \frac{h}{2}\right)\exp(\rho_1|\eta'|)\right],$$
$$C \text{ constant}, \quad c \geq 0, \quad 0 < \rho_1 < \rho, \quad \eta \in \partial\Omega, \tag{15}$$
then (8) *is true.*

Proof. Divide Ω by a line $\eta_3 = \dfrac{h}{2}$ into the domains
$$\Omega_1 = \left\{\eta : 0 < \eta_3 < \frac{h}{2}\right\} \text{ and } \Omega_2 = \left\{\eta : \frac{h}{2} < \eta_3 < h\right\}.$$
Consider the domain Ω_1. We put
$$K_1(z) = K(z)\exp\left[-\delta i\tau\left(z - \frac{h}{2}\right) - \delta_1 i\rho\left(z - \frac{h}{2}\right)\right],$$
$$\rho < \tau < 2\rho, \quad \delta > 0, \quad \delta_1 > 0, \tag{16}$$
in (6), $K(z)$ being defined in (12) and we obtain that (10) is valid.

Really,

$$\left|\exp\left[-i\tau\left(z-\frac{h}{4}\right)-\delta_1 i\rho\left(z-\frac{h}{4}\right)\right]\right|$$

$$=\exp\left[-\delta\tau\sqrt{a^2+\alpha^2}\cos\tau\left(\eta_3-\frac{h}{4}\right)\right]$$

$$=\exp\left[-\delta\tau\sqrt{a^2+\alpha^2}\right]\le\exp\left[-\delta\exp\tau|\eta'|\right],$$

$$-\frac{\pi}{2}\le-\tau\frac{\pi}{4}\le\tau\left(\eta_3-\frac{h}{4}\right)\le\tau\frac{\pi}{2}<\frac{h}{2}\text{ and }\cos\tau\left(\eta_3-\frac{h}{4}\right)\ge\cos\tau\frac{h}{4}\ge\delta_0>0.$$

We denote the corresponding $\Psi(\eta,\zeta;\lambda)$ by $\Psi^+(\eta,\zeta;\lambda)$.
Since

$$\cos\tau\left(\eta_3-\frac{h}{4}\right)\ge\delta_0,\quad \eta\in\Omega_1\bigcup\partial\Omega_1,$$

then for fixed $\zeta\in\Omega_1$, $\eta\in\Omega_1\bigcup\partial\Omega_1$ we have

$$|\Psi^+(\eta,\zeta;\lambda)|=O[\exp(-\delta_0\exp(\tau|\eta'|)],\quad \eta\to\infty,\quad \rho<\tau<2\rho,$$

$$\left|\frac{\partial\Psi^+(\eta,\zeta;\lambda)}{\partial\eta_j}\right|=O[\exp(-\delta_0\exp(\tau|\eta'|)],\quad \eta\to\infty,\quad \rho<\tau<2\rho,\quad j=\overline{1,2}.$$

$$\left|\frac{\partial\Psi^+(\eta,\zeta;\lambda)}{\partial\eta_3}\right|=O[\exp(-\delta_0\exp(\tau|\eta'|)],\quad \eta\to\infty,\quad \rho<\tau<2\rho.$$

Suppose $W(\eta)\in S_\rho(\Omega_1)$ satisfies

$$|W(\eta)|\le C\exp\left[\exp(2\rho-\varepsilon)|\eta'|\right],\quad \varepsilon>0,\quad \eta\in\Omega_1. \tag{17}$$

Consider τ in (16) satisfying $2\rho-\varepsilon<\tau<2\rho$.
We obtain that (16) is valid in Ω_1, and we have

$$W(\zeta)=\int\limits_{\partial\Omega_1}L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\quad \zeta\in\Omega_1. \tag{18}$$

where

$$L(\eta,\zeta;\lambda)=\left(E\left(\Psi^+(\eta,\zeta;\lambda)v^0\right)P^*(\partial_\zeta)\right)P(t^T).$$

If $W(\eta)\in S_\rho(\Omega_2)$ satisfies (15) in Ω_2, then for $2\rho-\varepsilon<\tau<2\rho$ analog we have

$$W(\zeta)=\int\limits_{\partial\Omega_2}L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\quad \zeta\in\Omega_2, \tag{19}$$

where

$$L(\eta,\zeta;\lambda)=\left(E\left(\Psi^-(\eta,\zeta;\lambda)v^0\right)P^*(\partial_\zeta)\right)P(t^T),$$

and $\Psi^-(\eta,\zeta;\lambda)$ it is given by (6), in which $K(z)$ it is replaced by the function $K_2(z)$:

$$K_2(z)=K(z)\exp\left[-\delta i\tau(z-h_1)-\delta_1 i\rho\left(z-\frac{h}{2}\right)\right], \tag{20}$$

where

$$h_1=\frac{h}{2}+\frac{h}{4},\quad \frac{h}{2}<\eta_3<h,\quad \frac{h}{2}<\zeta_3<h_1,\quad \delta>0,\quad \delta_1>0.$$

The integrals converge uniformly for $\delta \geq 0$, and $W(\eta) \in S_\rho(\Omega)$. We consider $\delta = 0$ and we find

$$W(\zeta) = \int\limits_{\partial\Omega} L(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \quad \zeta_3 \neq \frac{h}{2}, \tag{21}$$

where

$$L(\eta, \zeta; \lambda) = \Big(E\Big(\tilde{\Psi}(\eta, \zeta; \lambda) v^0 \Big) P^*(\partial_\zeta) \Big) P(t^T),$$

$$\tilde{\Psi}(\eta, \zeta; \lambda) = (\Psi^+(\eta, \zeta; \lambda))_{\delta=0} = (\Psi^-(\eta, \zeta; \lambda))_{\delta=0}.$$

Here, $\tilde{\Psi}(\eta, \zeta; \lambda)$ is given by (6), and $K(z)$ by (16), for $\delta = 0$. According to the continuation principle, Formula (21) is valid for every $\zeta \in \Omega$. Using (18) and (21) holds for every $\delta_1 \geq 0$. Supposing $\delta_1 = 0$, Theorem 2 is proved. □

We choose

$$K(z) = \frac{1}{(z - \zeta_3 + 2h)^2} \exp(\sigma z^2),$$
$$K(\zeta_3) = \frac{1}{(2h)^2} \exp(\sigma \zeta_3^2), \qquad 0 < \zeta_3 < h, \qquad h = \frac{\pi}{\rho}, \tag{22}$$

in (6) and we obtain

$$\Psi_\sigma(\eta, \zeta; \lambda) = -\frac{e^{-\sigma\zeta_3^2}}{\pi^2 (2h)^{-1}} \int\limits_0^\infty \mathrm{Im} \frac{\exp(\sigma z^2)}{(z - \zeta_3 + 2h)^2 (z - \zeta_3)} \frac{\cos(\lambda a)}{\sqrt{a^2 + \alpha^2}} da. \tag{23}$$

The Formula (8) becomes:

$$W(\zeta) = \int\limits_{\partial\Omega} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{24}$$

where

$$L_\sigma(\eta, \zeta; \lambda) = \Big(E\Big(\Psi_\sigma(\eta, \zeta; \lambda) v^0 \Big) P^*(\partial_\zeta) \Big) P(t^T).$$

3. Regularized Solution of the Problem

Theorem 3. *Let $W(\eta) \in S_\rho(\Omega)$ satisfying*

$$|W(\eta)| \leq M, \quad \eta \in D. \tag{25}$$

If

$$W_\sigma(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \eta \in \Omega, \tag{26}$$

then

$$|W(\zeta) - W_\sigma(\zeta)| \leq K_\rho(\lambda, \zeta) \sigma^2 M e^{-\sigma\zeta_3^2}, \quad \zeta \in \Omega, \tag{27}$$

$$\left| \frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} \right| \leq K_\rho(\lambda, \zeta) \sigma^2 M e^{-\sigma\zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,3}, \tag{28}$$

where $K_\rho(\lambda, \zeta)$ are bounded on compact subsets of Ω.

Proof. From (24) and (26), we obtain

$$W(\zeta) = \int_{\Sigma} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta + \int_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta$$

$$= W_\sigma(\zeta) + \int_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta \in \Omega.$$

Now using (25), we obtain

$$|W(\zeta) - W_\sigma(\zeta)| \le \left| \int_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta \right|$$
$$\le \int_{D} |L_\sigma(\eta,\zeta;\lambda)||W(\eta)|ds_\eta \le M \int_{D} |L_\sigma(\eta,\zeta;\lambda)|ds_\eta, \quad \zeta \in \Omega. \tag{29}$$

Next, we estimate the integrals $\int_{D} |\Psi_\sigma(\eta,\zeta;\lambda)|ds_\eta$, $\int_{D} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta$, $j = \overline{1,2}$ and $\int_{D} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_\eta$ on the part D of the plane $\eta_3 = 0$.

Separating the imaginary part of (23), we obtain

$$\Psi_\sigma(\eta,\zeta;\lambda) = \frac{e^{\sigma(\eta_3^2-\zeta_3^2)}}{\pi^2(2h^2)^{-1}} \left[\int_0^\infty \left(\frac{e^{-\sigma(u^2+\alpha^2)}(-\alpha_1^2+\beta_1^2+2\beta_1)\cos\gamma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)} \right.\right.$$
$$\left.\left. - \frac{e^{-\sigma(u^2+\alpha^2)}(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)} \frac{\sin\gamma\alpha_1}{\alpha_1} \right) \cos(\lambda a)\, da \right], \tag{30}$$

where

$$\gamma = 2\sigma\eta_3, \quad \alpha_1^2 = a^2+\alpha^2, \quad \beta = \eta_3-\zeta_3, \quad \beta_1 = \eta_3-\zeta_3+2h.$$

Given equality (30), we have

$$\int_{D} |\Psi_\sigma(\eta,\zeta;\lambda)|ds_\eta \le K_\rho(\lambda,\zeta)\sigma^2 e^{-\sigma\zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{31}$$

Now using the equality

$$\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j} = \frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial \eta_j} = 2(\eta_j-\zeta_j)\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s},$$
$$s = \alpha^2, \quad j = \overline{1,2}, \tag{32}$$

the equality (30) and (32), we have

$$\int_{D} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta \le K_\rho(\lambda,\zeta)\sigma^2 e^{-\sigma\zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,2}. \tag{33}$$

Now, we estimate the integral $\int_{D} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_3}\right| ds_\eta$.

Taking into account equality (30), we obtain

$$\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \eta_3} \right| ds_\eta \le K_\rho(\lambda, \zeta)\sigma^2 e^{-\sigma \zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \tag{34}$$

From inequalities (29), (31), (33), and (34), we obtain (27).

Now we prove the inequality (28). Taking the derivatives from equalities (24) and (26) with respect to ζ_j, $j = \overline{1,3}$, we obtain:

$$\begin{gathered} \frac{\partial W(\zeta)}{\partial \zeta_j} = \int_\Sigma \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_\eta + \int_D \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_\eta, \\ \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \int_\Sigma \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_\eta, \quad \zeta \in \Omega, \quad j = \overline{1,3} \end{gathered} \tag{35}$$

From (25) and (35), we have

$$\begin{gathered} \left| \frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial_\sigma W(\zeta)}{\partial \zeta_j} \right| \le \left| \int_D \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_\eta \right| \\ \le \int_D \left| \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} \right| |W(\eta)| ds_\eta \le M \int_D \left| \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} \right| ds_\eta, \\ \zeta \in \Omega, \quad j = \overline{1,3}. \end{gathered} \tag{36}$$

To prove (36), we estimate $\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} \right| ds_\eta$, $j = \overline{1,2}$, and $\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_3} \right| ds_\eta$, on the part D of the plane $\eta_3 = 0$.

For the first integrals, we use:

$$\begin{gathered} \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} = \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial s} \frac{\partial s}{\partial \zeta_j} = -2(\eta_j - \zeta_j) \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial s}, \\ s = \alpha^2, \quad j = \overline{1,2}. \end{gathered} \tag{37}$$

Applying equality (30) and equality (37), we obtain

$$\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} \right| ds_\eta \le K_\rho(\lambda, \zeta)\sigma^2 e^{-\sigma \zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,2}. \tag{38}$$

Now, we estimate the integral $\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_3} \right| ds_\eta$.

Taking into account equality (30), we obtain

$$\int_D \left| \frac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_3} \right| ds_\eta \le K_\rho(\lambda, \zeta)\sigma^2 e^{-\sigma \zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{39}$$

Using (36), (38) and (39), we obtain (27). □

Corollary 1. *For every* $\zeta \in \Omega$,

$$\lim_{\sigma\to\infty} W_\sigma(\zeta) = W(\zeta), \quad \lim_{\sigma\to\infty} \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = \overline{1,3}.$$

We define $\overline{\Omega}_\varepsilon$ as

$$\overline{\Omega}_\varepsilon = \left\{ (\zeta_1, \zeta_2, \zeta_3) \in \Omega, \quad q > \zeta_3 \geq \varepsilon, \quad q = \max_D \psi(\zeta'), \quad 0 < \varepsilon < q \right\}.$$

Here, $\psi(\zeta')-$ is a surface. We remark that the set $\overline{\Omega}_\varepsilon \subset \Omega$ is compact.

Corollary 2. *If* $\zeta \in \overline{\Omega}_\varepsilon$, *then the families of functions* $\{W_\sigma(\zeta)\}$ *and* $\left\{\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right\}$ *converge uniformly for* $\sigma \to \infty$, *that is:*

$$W_\sigma(\zeta) \rightrightarrows W(\zeta), \quad \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} \rightrightarrows \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = \overline{1,3}.$$

Remark that $E_\varepsilon = \Omega \backslash \overline{\Omega}_\varepsilon$ is a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

Suppose that the surface Σ is given by the equation

$$\eta_m = \psi(\eta'), \quad \eta' \in \mathbb{R}^2,$$

where $\psi(\eta')$ satisfies the condition

$$|\psi'(\eta')| \leq C < \infty, \quad C = const.$$

Consider

$$q = \max_D \psi(\eta'), \quad l = \max_D \sqrt{1 + \psi'^2(\eta')}.$$

Theorem 4. *If* $W(\eta) \in S_\rho(\Omega)$ *satisfies* (25), *and the inequality*

$$|W(\eta)| \leq \delta, \quad 0 < \delta < 1,\ \eta \in \Sigma,\ \Sigma \textit{ a smooth surface}, \tag{40}$$

then

$$|W(\zeta)| \leq K_\rho(\lambda, \zeta)\sigma^2 M^{1-\frac{\zeta_3^2}{q^2}} \delta^{\frac{\zeta_3^2}{q^2}}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{41}$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq K_\rho(\lambda, \zeta)\sigma^2 M^{1-\frac{\zeta_3^2}{q^2}} \delta^{\frac{\zeta_3^2}{q^2}}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \tag{42}$$

Proof. Using (24), we obtain

$$W(\zeta) = \int_\Sigma L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta + \int_D L_\sigma(\eta, \zeta; \lambda)) W(\eta) ds_\eta, \quad \zeta \in \Omega. \tag{43}$$

We estimate the following

$$|W(\zeta)| \leq \left| \int_\Sigma L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta \right| + \left| \int_D L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta \right|, \quad \zeta \in \Omega. \tag{44}$$

Given inequality (40), we estimate the first integral of inequality (44).

$$\left| \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} \right| \leq \int_{\Sigma} |L_{\sigma}(\eta, \zeta; \lambda)| |W(\eta)| ds_{\eta}$$
$$\leq \delta \int_{\Sigma} |L_{\sigma}(\eta, \zeta; \lambda)| ds_{\eta}, \quad \zeta \in \Omega. \tag{45}$$

We estimate now the integrals $\int\limits_{\Sigma} |\Psi_{\sigma}(\eta, \zeta; \lambda)| ds_{\eta}$, $\int\limits_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(\eta, \zeta; \lambda)}{\partial \eta_j} \right| ds_{\eta}, j = \overline{1,2}$ and $\int\limits_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(\eta, \zeta; \lambda)}{\partial \eta_3} \right| ds_{\eta}$ on Σ.

Using (30), we have

$$\int_{\Sigma} |\Psi_{\sigma}(\eta, \zeta; \lambda)| ds_{\eta} \leq K_{\rho}(\lambda, \zeta) \sigma^2 e^{\sigma(q^2 - \zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{46}$$

From (30) and (32), we have

$$\int_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(\eta, \zeta; \lambda)}{\partial y_j} \right| ds_{\eta} \leq K_{\rho}(\lambda, \zeta) \sigma^2 e^{\sigma(q^2 - \zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,2}. \tag{47}$$

Using (30), we obtain

$$\int_{\Sigma} \left| \frac{\partial \Psi_{\sigma}(\eta, \zeta; \lambda)}{\partial \eta_3} \right| ds_{\eta} \leq K_{\rho}(\lambda, \zeta) \sigma^2 e^{\sigma(q^2 - \zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{48}$$

From (46)–(48) and applying (45), we obtain

$$\left| \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} \right| \leq K_{\rho}(\lambda, \zeta) \sigma^2 \delta \, e^{\sigma(q^2 - \zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{49}$$

The following is known

$$\left| \int_{D} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} \right| \leq K_{\rho}(\lambda, \zeta) \sigma^2 M e^{-\sigma \zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{50}$$

Now using (44), (49) and (50), we have

$$|W(\zeta)| \leq \frac{K_{\rho}(\lambda, \zeta) \sigma^2}{2} (\delta \, e^{\sigma q^2} + M) e^{-\sigma \zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{51}$$

Choosing

$$\sigma = \frac{1}{q^2} \ln \frac{M}{\delta}, \tag{52}$$

we obtain (41).

We compute now the partial derivative from Formula (24) with respect to ζ_j, $j = \overline{1,3}$:

$$\begin{aligned} \frac{\partial W(\zeta)}{\partial \zeta_j} &= \int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta + \int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta \\ &= \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} + \int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \end{aligned} \tag{53}$$

Here

$$\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta. \tag{54}$$

Now we have

$$\begin{aligned} \left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| &\leq \left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \\ &+ \left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq \left|\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right| \\ &+ \left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right|, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \end{aligned} \tag{55}$$

Given inequality (40), we obtain:

$$\begin{aligned} \left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| &\leq \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |W(\eta)| ds_\eta \\ &\leq \delta \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \end{aligned} \tag{56}$$

To prove (56), we estimate now $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_y$, $j = \overline{1,2}$ and $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_\eta$ on a smooth surface Σ.

Given equality (30) and equality (35), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta \leq K_\rho(\lambda,\zeta)\sigma^2 e^{\sigma(q^2-\zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,2}. \tag{57}$$

Taking into account (30), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_y \leq K_\rho(\lambda,\zeta)\sigma^2 e^{\sigma(q^2-\zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega, \tag{58}$$

From (57) and (58), bearing in mind (56), we obtain

$$\begin{aligned} \left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq K_\rho(\lambda,\zeta)\sigma^2 \delta e^{\sigma(q^2-\zeta_3^2)}, \quad \sigma > 1, \quad \zeta \in \Omega, \\ j = \overline{1,3}. \end{aligned} \tag{59}$$

We have

$$\left|\int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \le K_\rho(\lambda,\zeta)\sigma^2 M e^{-\sigma\zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \tag{60}$$

From (55), (59) and (60),we obtain

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \le \frac{K_\rho(\lambda,\zeta)\sigma^2}{2}(\delta e^{\sigma q^2} + M)e^{-\sigma\zeta_3^2}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \tag{61}$$

Choosing σ as in (52) we get (42). □

Suppose now that $W(\eta) \in S_\rho(\Omega)$ is defined on Σ and $f_\delta(\eta)$ is its approximation with an error $0 < \delta < 1$. Then

$$\max_\Sigma |W(\eta) - f_\delta(\eta)| \le \delta. \tag{62}$$

We put

$$W_{\sigma(\delta)}(\zeta) = \int_\Sigma L_\sigma(\eta,\zeta;\lambda) f_\delta(\eta) ds_\eta, \quad \zeta \in \Omega. \tag{63}$$

Theorem 5. *Let $W(\eta) \in S_\rho(\Omega)$ satisfying the condition (25) on the part of the plane $\eta_3 = 0$. Then*

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| \le K_\rho(\lambda,\zeta)\sigma^2 M^{1-\frac{\zeta_3^2}{q^2}} \delta^{\frac{\zeta_3^2}{q^2}}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{64}$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| \le K_\rho(\lambda,\zeta)\sigma^2 M^{1-\frac{\zeta_3^2}{q^2}} \delta^{\frac{\zeta_3^2}{q^2}}, \quad \sigma > 1, \quad \zeta \in \Omega, \quad j = \overline{1,3}. \tag{65}$$

Proof. From (24) and (63), we obtain

$$W(\zeta) - W_{\sigma(\delta)}(\zeta) = \int_{\partial\Omega} L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta$$

$$- \int_\Sigma L_\sigma(\eta,\zeta;\lambda) f_\delta(\eta) ds_\eta = \int_\Sigma L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta$$

$$+ \int_D L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta - \int_\Sigma L_\sigma(\eta,\zeta;\lambda) f_\delta(\eta) ds_\eta$$

$$= \int_\Sigma L_\sigma(\eta,\zeta;\lambda)\{W(\eta) - f_\delta(\eta)\} ds_\eta + \int_D L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta.$$

and

$$\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} = \int\limits_{\partial\Omega} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta$$

$$-\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} f_\delta(y) ds_y = \int\limits_{S} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y$$

$$+\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta - \int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} f_\delta(\eta) ds_\eta$$

$$= \int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} \{W(\eta) - f_\delta(\eta)\} ds_\eta + \int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta,$$

$$j = \overline{1,3}.$$

Using (25) and (62), we obtain:

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| = \left|\int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)\{W(\eta) - f_\delta(\eta)\} ds_\eta\right|$$

$$+\left|\int\limits_{D} L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta\right| \le \int\limits_{\Sigma} |L_\sigma(\eta,\zeta;\lambda)||\{W(\eta) - f_\delta(\eta)\}| ds_\eta$$

$$+\int\limits_{D} |L_\sigma(\eta,\zeta;\lambda)||W(\eta)| ds_\eta \le \delta \int\limits_{\Sigma} |L_\sigma(\eta,\zeta;\lambda)| ds_\eta$$

$$+M \int\limits_{D} |L_\sigma(\eta,\zeta;\lambda)| ds_\eta.$$

and

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| = \left|\int\limits_{\sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} \{W(\eta) - f_\delta(\eta)\} ds_\eta\right|$$

$$+\left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \le \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |\{W(\eta) - f_\delta(\eta)\}| ds_\eta$$

$$+\int\limits_{D} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |W(\eta)| ds_\eta \le \delta \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta$$

$$+M \int\limits_{D} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \quad j = \overline{1,3}.$$

We obtain, similarly repeating the proof of Theorems 3 and 4, that

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| \le \frac{K_\rho(\lambda,\zeta)\sigma^2}{2} (\delta e^{\sigma q^2} + M) e^{-\sigma \zeta_3^2}.$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| \le \frac{K_\rho(\lambda,\zeta)\sigma^2}{2} (\delta e^{\sigma q^2} + M) e^{-\sigma \zeta_3^2}, \quad j = \overline{1,3}.$$

Considering σ from (52), we obtain (64) and (65). $\square$

Corollary 3. *For every $\zeta \in \Omega$,*

$$\lim_{\delta\to 0} W_{\sigma(\delta)}(\zeta) = W(\zeta), \quad \lim_{\delta\to 0} \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} = \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = \overline{1,3}.$$

Corollary 4. *If $\zeta \in \overline{\Omega}_\varepsilon$, then the families of functions $\left\{W_{\sigma(\delta)}(\zeta)\right\}$ and $\left\{\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right\}$ are convergent uniformly for $\delta \to 0$, that is:*

$$W_{\sigma(\delta)}(\zeta) \rightrightarrows W(\zeta), \quad \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} \rightrightarrows \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = \overline{1,3}.$$

4. Conclusions

In this paper, as a continuation of some previous papers, we explicitly found a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation in an unbounded domain from $\mathbb{R}^3$. When applied problems are solved, the approximate values of $W(\zeta)$ and $\frac{\partial W(\zeta)}{\partial \zeta_j}$, $\zeta \in \Omega$, $j = \overline{1,3}$ must be found.

We have built, in this paper, a family of vector-functions $W(\zeta, f_\delta) = W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial W(\zeta, f_\delta)}{\partial \zeta_j} = \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$, $j = \overline{1,3}$, depending on σ. Moreover, we have proved that for $\sigma = \sigma(\delta)$, at $\delta \to 0$, specially chosen, $W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ are convergent to a solution $W(\zeta)$ and its derivative $\frac{\partial W(\zeta)}{\partial \zeta_j}$, $\zeta \in \Omega$. Such a family of vector functions $W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ are called a regularized solution of the problem. A regularized solution determines a stable method to find the approximate solution of the problem.

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