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Generalized Inequalities of Hilbert-Type on Time Scales Nabla Calculus

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Abstract: In this paper, we prove some new generalized inequalities of Hilbert-type on time scales nabla calculus by applying Hölder's inequality, Young's inequality, and Jensen's inequality. Symmetrical properties play an essential role in determining the correct methods to solve inequalities.

Keywords: Hilbert-type inequalities; time scales nabla calculus; Hölder's inequality; Young's inequality

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1. Introduction

In the time (1862–1943), David Hilbert proved Hilbert's double series inequality without an exact determination of the constant in his lectures on integral equations. If $\{\beta_m\}$ and $\{d_n\}$ are two real sequences such that $0 < \sum_{m=1}^{\infty} \beta_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} d_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\beta_m d_n}{m+n} \le \pi \left(\sum_{m=1}^{\infty} \beta_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} d_n^2\right)^{\frac{1}{2}}. \tag{1}$$

In 1911, Schur [1] proved that π in (1) is sharp and also discovered the integral analogue of (1), which became known as the Hilbert integral inequality in the form

$$\int_0^{\infty}\int_0^{\infty} \frac{\Xi(\tau)\Upsilon(y)}{\tau+y} d\tau dy \le \pi \left(\int_0^{\infty} \Xi^2(\tau)d\tau\right)^{\frac{1}{2}} \left(\int_0^{\infty} \Upsilon^2(y)dy\right)^{\frac{1}{2}}, \tag{2}$$

where Ξ and Υ are measurable functions such that, $0 < \int_0^{\infty} \Xi^2(\tau)d\tau < \infty$ and $0 < \int_0^{\infty} \Upsilon^2(y)dy < \infty$.

In 1925, by introducing one pair of conjugate exponents (p, q) with $1/p + 1/q = 1$, Hardy [2] gave an extension of (1) as follows. If $p, q > 1$, $\beta_m, d_n \ge 0$ such that $0 < \sum_{m=1}^{\infty} \beta_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} d_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\beta_m d_n}{m+n} \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\sum_{m=1}^{\infty} \beta_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} d_n^q\right)^{\frac{1}{q}}, \tag{3}$$

where the constant $\pi/\sin(\pi/p)$ in (3) is sharp. In 1934, Hardy et al. [3] proved the equivalent integral analog of (3) in the form

$$\int_0^\infty \int_0^\infty \frac{\Xi(\tau)\Upsilon(y)}{\tau+y} d\tau dy \le \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_0^\infty \Xi^p(\tau) d\tau\right)^{\frac{1}{p}} \left(\int_0^\infty \Upsilon^q(y) dy\right)^{\frac{1}{q}}, \tag{4}$$

where Ξ and Υ are measurable non-negative functions such that $0 < \int_0^\infty \Xi^p(\tau)d\tau < \infty$ and $0 < \int_0^\infty \Upsilon^q(y)dy < \infty$.

In 1998, Pachpatte [4] gave a new inequality closed to that of Hilbert as follows: Let $\beta(\iota) : \{0,1,2,\ldots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $d(\vartheta) : \{0,1,2,\ldots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $\beta(0) = d(0) = 0$. Then

$$\begin{aligned} \sum_{\iota=1}^{p} \sum_{\vartheta=1}^{q} \frac{|\beta_\iota||d_\vartheta|}{\iota+\vartheta} \le C(p,q) \left(\sum_{\iota=1}^{p} (p-\iota+1)|\nabla\beta_\iota|^2\right)^{\frac{1}{2}} \\ \times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla d_\vartheta|^2\right)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where $\nabla\beta_\iota = \beta_\iota - \beta_{\iota-1}$, $\nabla d_\vartheta = d_\vartheta - d_{\vartheta-1}$ and

$$C(p,q) = \frac{1}{2}\sqrt{pq}.$$

In 2000, Pachpatte [5] generalized (5) by introducing one pair of conjugate exponents (λ,μ), such that $\lambda,\mu > 1$ with $1/\lambda + 1/\mu = 1$. Then, it is established that if $\beta(\iota) : \{0,1,2,\ldots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $d(\vartheta) : \{0,1,2,\ldots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $\beta(0) = d(0) = 0$, then

$$\begin{aligned} \sum_{\iota=1}^{p} \sum_{\vartheta=1}^{q} \frac{|\beta_\iota||d_\vartheta|}{\mu\iota^{\lambda-1}+\lambda\vartheta^{\mu-1}} \le D(\lambda,\mu,p,q) \left(\sum_{\iota=1}^{p} (p-\iota+1)|\nabla\beta_\iota|^\lambda\right)^{\frac{1}{\lambda}} \\ \times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla d_\vartheta|^\mu\right)^{\frac{1}{\mu}}, \end{aligned} \tag{6}$$

where $\nabla\beta_\iota = \beta_\iota - \beta_{\iota-1}$, $\nabla d_\vartheta = d_\vartheta - d_{\vartheta-1}$ and

$$D(\lambda,\mu,p,q) = \frac{1}{\lambda\mu} p^{\frac{\lambda-1}{\lambda}} q^{\frac{\mu-1}{\mu}}.$$

In 2002, Kim et al. [6] generalized (6) and proved that if $\lambda,\mu > 1$, $\beta(\iota) : \{0,1,2,\ldots,p\} \subset \mathbb{N} \to \mathbb{R}$ and $d(\vartheta) : \{0,1,2,\ldots,q\} \subset \mathbb{N} \to \mathbb{R}$ with $\beta(0) = d(0) = 0$, then

$$\begin{aligned} \sum_{\iota=1}^{p} \sum_{\vartheta=1}^{q} \frac{|\beta_\iota||d_\vartheta|}{\mu\iota^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \le D^*(\lambda,\mu,p,q) \left(\sum_{\iota=1}^{p} (p-\iota+1)|\nabla\beta_\iota|^\lambda\right)^{\frac{1}{\lambda}} \\ \times \left(\sum_{\vartheta=1}^{q} (q-\vartheta+1)|\nabla d_\vartheta|^\mu\right)^{\frac{1}{\mu}}, \end{aligned} \tag{7}$$

where $\nabla\beta_\iota = \beta_\iota - \beta_{\iota-1}$, $\nabla d_\vartheta = d_\vartheta - d_{\vartheta-1}$ and

$$D^*(\lambda,\mu,p,q) = \frac{1}{\lambda+\mu} p^{\frac{\lambda-1}{\lambda}} q^{\frac{\mu-1}{\mu}}.$$

Furthermore, the researchers [6] proved the continuous analog of (7) and proved that if $\lambda,\mu > 1$, and $\Xi(\iota)$, $\Upsilon(t)$ are real continuous functions on the intervals $(0,\tau)$, $(0,y)$, respectively, and let $\Xi(0) = \Upsilon(0) = 0$. Then

$$\begin{aligned} \int_0^\tau \int_0^y \frac{|\Xi(\iota)||\Upsilon(t)|}{\mu\iota^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} d\iota dt \\ \le M(\lambda,\mu,\tau,y)\left(\int_0^\tau (\tau-\iota)|\Xi'(\iota)|^\lambda d\iota\right)^{\frac{1}{\lambda}} \left(\int_0^y (y-t)|\Upsilon'(t)|^\mu dt\right)^{\frac{1}{\mu}}, \end{aligned} \tag{8}$$

for $\tau, y \in (0,\infty)$, where

$$M(\lambda,\mu,\tau,y) = \frac{1}{\lambda+\mu}\tau^{\frac{\lambda-1}{\lambda}} y^{\frac{\mu-1}{\mu}}.$$

In recent decades, a new theory has been discovered to unify the continuous calculus and discrete calculus. Many authors proved some dynamic inequalities of Hilbert type, its generalizations and also the reversed forms, see the papers [7–9].

The goal of this article is to use time scales nabla calculus to prove various dynamic inequalities for Hilbert-type. Furthermore, we establish some generalized inequalities of Hilbert type by using submuliplicative and bounded functions.

The organization of the paper as follows. In Section 2, we show some basics of the time scale theory and some lemmas needed in Section 3 where we prove our results. Our results as special cases give the inequalities (7) and (8) proved by Kim et al. [6].

2. Preliminaries

For a time scale $\mathbb{T}$, we define the backward jump operator as following $\rho(\tau) := \sup\{\iota \in \mathbb{T} : \iota < \tau\}$. Let $\Xi : \mathbb{T} \to \mathbb{R}$ be a function, we say that Ξ is ld-continuous if it is continuous at each left dense point in $\mathbb{T}$ and the right limit exists as a finite number for all right dense points $t \in \mathbb{T}$. The set of all such ld–continuous functions is ushered by $C_{ld}(\mathbb{T},\mathbb{R})$ and for any function $\Xi : \mathbb{T} \to \mathbb{R}$, the notation $\Xi^{\rho}(\tau)$ denotes $\Xi(\rho(\tau))$. For more information about the time scale calculus, see [10,11].

The nabla derivative of uv and u/v (where $v(\tau)v^{\rho}(\tau) \neq 0$) are

$$\begin{aligned}(uv)^{\nabla}(\tau) &= u^{\nabla}(\tau)v(\tau) + u^{\rho}(\tau)v^{\nabla}(\tau) \\ &= u(\tau)v^{\nabla}(\tau) + u^{\nabla}(\tau)v^{\rho}(\tau),\end{aligned}$$

and

$$\left(\frac{u}{v}\right)^{\nabla}(\tau) = \frac{u^{\nabla}(\tau)v(\tau) - u(\tau)v^{\nabla}(\tau)}{v(\tau)v^{\rho}(\tau)}.$$

Definition 1 ([10])**.** *A function $\Xi : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $\Xi : \mathbb{T} \to \mathbb{R}$ provided $\Xi^{\nabla}(t) = \Xi(t)$ holds $\forall t \in \mathbb{T}$. We then define the integral of Ξ by*

$$\int_{\beta}^{t} \Xi(\tau)\nabla\tau = \Xi(t) - \Xi(\beta) \quad \forall t \in \mathbb{T}.$$

Theorem 1 ([10])**.** *If $\beta, d \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $\Xi, \Upsilon \in C_{ld}(\mathbb{T},\mathbb{R})$, then*

(1) $\int_{\beta}^{d}[\Xi(\tau) + \Upsilon(\tau)]\nabla\tau = \int_{\beta}^{d}\Xi(\tau)\nabla\tau + \int_{\beta}^{d}\Upsilon(\tau)\nabla\tau$;

(2) $\int_{\beta}^{d}\alpha\Xi(\tau)\nabla\tau = \alpha\int_{\beta}^{d}\Xi(\tau)\nabla\tau$;

(3) $\int_{\beta}^{\beta}\Xi(\tau)\nabla\tau = 0$.

The integration by parts formula on $\mathbb{T}$ is

$$\int_{\beta}^{d} u(\tau)v^{\nabla}(\tau)\nabla\tau = [u(\tau)v(\tau)]_{\beta}^{d} - \int_{\beta}^{d} u^{\nabla}(\tau)v^{\rho}(\tau)\nabla\tau. \tag{9}$$

The Hölder inequality on $\mathbb{T}$ is

$$\int_{\beta}^{d} |\Xi(\tau)\Upsilon(\tau)|\nabla\tau \le \left[\int_{\beta}^{d} |\Xi(\tau)|^{\gamma}\nabla\tau\right]^{\frac{1}{\gamma}} \left[\int_{\beta}^{d} |\Upsilon(\tau)|^{\nu}\nabla\tau\right]^{\frac{1}{\nu}}, \tag{10}$$

where $\beta, d \in \mathbb{T}$, $\Xi, \Upsilon \in C_{ld}(\mathbb{I},\mathbb{R})$, $\gamma > 1$ and $1/\gamma + 1/\nu = 1$.

Definition 2 ([12]). *A function $G : J \to \mathbb{R}^+$ is sub-multiplicative if*

$$G(\tau\eta) \le G(\tau)G(\eta), \qquad \forall \tau, \eta \in J \subset \mathbb{R}. \tag{11}$$

The inequality (11) holds with equality when G is the identity map (i.e., $G(\tau) = \tau$).

Lemma 1 (Young's inequality [13]). *Let $\beta, d \ge 0$ be real numbers. Then we have for $p, q > 1$ and $1/p + 1/q = 1$, that*

$$\beta d \le \frac{\beta^p}{p} + \frac{d^q}{q}.$$

Lemma 2. *Let $\omega_1,\ \omega_2,\ \beta_1,\ \beta_2 > 0$. Then*

$$\beta_1^{\omega_1}\beta_2^{\omega_2} \le \frac{1}{\omega_1+\omega_2}\Big(\omega_1\beta_1^{\omega_1+\omega_2} + \omega_2\beta_2^{\omega_1+\omega_2}\Big). \tag{12}$$

Proof. Applying Lemma 1 with $p = (\omega_1+\omega_2)/\omega_1 > 1$, $q = (\omega_1+\omega_2)/\omega_2 > 1$, $\beta = \beta_1^{\omega_1}$ and $d = \beta_2^{\omega_2}$, we obtain that

$$\begin{aligned}\beta_1^{\omega_1}\beta_2^{\omega_2} &\le \frac{\omega_1}{\omega_1+\omega_2}\beta_1^{\omega_1+\omega_2} + \frac{\omega_2}{\omega_1+\omega_2}\beta_2^{\omega_1+\omega_2} \\ &= \frac{1}{\omega_1+\omega_2}\Big(\omega_1\beta_1^{\omega_1+\omega_2} + \omega_2\beta_2^{\omega_1+\omega_2}\Big),\end{aligned}$$

which is (12). □

In the following, we present Jensen's inequality in the time scale nabla calculus which is a special case of ([14] Theorem 3.3) by taking $\alpha = 0$.

Lemma 3. *Let $\zeta_0, \zeta \in \mathbb{T}$ and $r_0, \delta \in \mathbb{R}$. If $\lambda \in C_{ld}([\zeta_0,\zeta]_{\mathbb{T}}, \mathbb{R})$, $\varphi : [\zeta_0,\zeta]_{\mathbb{T}} \to (r_0,\delta)$ is ld-continuous and $\Psi : (r_0, r) \to \mathbb{R}$ is continuous and convex, then*

$$\Psi\left(\frac{1}{\int_{\zeta_0}^{\zeta}\lambda(\tau)\nabla\tau}\int_{\zeta_0}^{\zeta}\lambda(\tau)\varphi(\tau)\nabla\tau\right) \le \frac{1}{\int_{\zeta_0}^{\zeta}\lambda(\tau)\nabla\tau}\int_{\zeta_0}^{\zeta}\lambda(\tau)\Psi(\varphi(\tau))\nabla\tau. \tag{13}$$

3. Main Results

Throughout the article, we will assume that the functions are ld-continuous functions on $[\beta, d]_{\mathbb{T}} := [\beta, d] \cap \mathbb{T}$ and the integrals considered are assumed to exist.

Theorem 2. *Let $\iota, \vartheta, r_0 \in \mathbb{T}$, $\lambda, \mu > 1$, $\Xi \in C^1_{ld}([r_0,\tau)_{\mathbb{T}}, \mathbb{R})$, $\Upsilon \in C^1_{ld}([r_0,y)_{\mathbb{T}}, \mathbb{R})$ and $\Xi(r_0) = \Upsilon(r_0) = 0$. Assume that $h \in C^1_{ld}([r_0,\tau)_{\mathbb{T}}, \mathbb{R})$, $l \in C^1_{ld}([r_0,y)_{\mathbb{T}}, \mathbb{R})$ and $\Omega_1,\ \Omega_2,\ \Phi,\ \Psi$ are non-negative functions such that Φ, Ψ are increasing, convex and submultiplicative functions with*

$$\beta\Phi \le \Omega_1 \le d\Phi \quad \textit{and} \quad \beta\Psi \le \Omega_2 \le d\Psi, \tag{14}$$

where β, d are positive constants such that $\beta \le d$. If

$$H(\iota) = \int_{r_0}^{\iota}|h(\tau)|\nabla\tau \quad \textit{and} \quad L(\vartheta) = \int_{r_0}^{\vartheta}|l(\zeta)|\nabla\zeta,$$

then we have for $\iota \in [r_0, \tau)_{\mathbb{T}}$, $\vartheta \in [r_0, y)_{\mathbb{T}}$ *that*

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau} \frac{\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|)}{\mu(\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \nabla\iota\nabla\vartheta \\
&\leq \frac{d^2}{\beta^2}\Upsilon(\lambda,\mu,\tau,y)\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Omega_1\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau - \rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times \left[\int_{r_0}^{y}\left[|l(\vartheta)|\Omega_2\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y - \rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}},
\end{aligned} \tag{15}$$

where $\tau, y \in (r_0, \infty)_{\mathbb{T}}$ *and*

$$\begin{aligned}
\Upsilon(\lambda,\mu,\tau,y) &= \frac{1}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\
&\quad \times \left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}}.
\end{aligned} \tag{16}$$

Proof. From (14) and (13) and using the fact that Φ is a non-negative, increasing, submultiplicative and convex function, we have

$$\begin{aligned}
\Omega_1(|\Xi(\iota)|) &\leq d\Phi(|\Xi(\iota)|) \\
&\leq d\Phi\left(\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|\nabla\tau\right) \\
&= d\Phi\left(H(\iota)\frac{\int_{r_0}^{\iota}|h(\tau)|\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\nabla\tau}{\int_{r_0}^{\iota}|h(\tau)|\nabla\tau}\right) \\
&\leq d\Phi(H(\iota))\Phi\left(\frac{\int_{r_0}^{\iota}|h(\tau)|\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\nabla\tau}{\int_{r_0}^{\iota}|h(\tau)|\nabla\tau}\right) \\
&\leq d\frac{\Phi(H(\iota))}{H(\iota)}\left(\int_{r_0}^{\iota}|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau\right).
\end{aligned} \tag{17}$$

Applying (10) on the term

$$\int_{r_0}^{\iota}|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau,$$

with indices $\lambda > 1$ and $\lambda/(\lambda-1)$, we observe that

$$\begin{aligned}
&\int_{r_0}^{\iota}|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau \\
&\leq \left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}\left[\int_{r_0}^{\iota}1\nabla\tau\right]^{\frac{\lambda-1}{\lambda}} \\
&= (\iota - r_0)^{\frac{\lambda-1}{\lambda}}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}.
\end{aligned} \tag{18}$$

From (17) and (18), we obtain

$$\begin{aligned}\Omega_1(|\Xi(\iota)|) \quad &\le \quad d(\iota - r_0)^{\frac{\lambda-1}{\lambda}} \frac{\Phi(H(\iota))}{H(\iota)} \\ &\quad \times \left[\int_{r_0}^{\iota} \left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda} \nabla\tau\right]^{\frac{1}{\lambda}}. \end{aligned} \tag{19}$$

Similarly, we have that

$$\begin{aligned}\Omega_2(|\Upsilon(\vartheta)|) \quad &\le \quad d(\vartheta - r_0)^{\frac{\mu-1}{\mu}} \frac{\Psi(L(\vartheta))}{L(\vartheta)} \\ &\quad \times \left[\int_{r_0}^{\vartheta} \left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu} \nabla\tau\right]^{\frac{1}{\mu}}. \end{aligned} \tag{20}$$

From (19) and (20), we observe that

$$\begin{aligned}&\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|) \\ &\le d^2(\iota - r_0)^{\frac{\lambda-1}{\lambda}}(\vartheta - r_0)^{\frac{\mu-1}{\mu}} \\ &\times \frac{\Phi(H(\iota))}{H(\iota)} \left[\int_{r_0}^{\iota} \left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda} \nabla\tau\right]^{\frac{1}{\lambda}} \\ &\times \frac{\Psi(L(\vartheta))}{L(\vartheta)} \left[\int_{r_0}^{\vartheta} \left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu} \nabla\tau\right]^{\frac{1}{\mu}}. \end{aligned} \tag{21}$$

From Lemma 2, the inequality (21) becomes

$$\begin{aligned}&\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|) \\ &\le d^2 \frac{\lambda\mu}{\lambda+\mu}\left(\frac{1}{\lambda}(\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \frac{1}{\mu}(\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right) \\ &\times \frac{\Phi(H(\iota))}{H(\iota)} \left[\int_{r_0}^{\iota} \left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda} \nabla\tau\right]^{\frac{1}{\lambda}} \\ &\times \frac{\Psi(L(\vartheta))}{L(\vartheta)} \left[\int_{r_0}^{\vartheta} \left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu} \nabla\tau\right]^{\frac{1}{\mu}} \\ &= d^2 \frac{1}{\lambda+\mu}\left(\mu(\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right) \\ &\times \frac{\Phi(H(\iota))}{H(\iota)} \left[\int_{r_0}^{\iota} \left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda} \nabla\tau\right]^{\frac{1}{\lambda}} \\ &\times \frac{\Psi(L(\vartheta))}{L(\vartheta)} \left[\int_{r_0}^{\vartheta} \left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu} \nabla\tau\right]^{\frac{1}{\mu}}. \end{aligned} \tag{22}$$

Dividing the two sides of (22) on the term

$$\left(\mu(\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right),$$

and then integrating with respect to ι from r_0 to τ and for ϑ from r_0 to y, to obtain

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq \frac{d^2}{\lambda+\mu}\int_{r_0}^{\tau}\frac{\Phi(H(\iota))}{H(\iota)}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}\nabla\iota \\
&\times\int_{r_0}^{y}\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}}\nabla\vartheta.
\end{aligned} \tag{23}$$

Applying (10) on the R.H.S of (23), we see that

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq \frac{d^2}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\
&\times\left[\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}} \\
&\times\left[\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\nabla\vartheta\right]^{\frac{1}{\mu}}.
\end{aligned} \tag{24}$$

By applying the Formula (9), we have that

$$\begin{aligned}
&\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\nabla\iota \\
= &\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota,
\end{aligned}$$

and also,

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\nabla\vartheta \\
= &\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta
\end{aligned}$$

and then substituting into (24), we obtain

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq \frac{d^2}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\
&\times\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}} \\
&\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}}.
\end{aligned}$$

From (14) and (16), the last inequality becomes

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Omega_1(|\Xi(\iota)|)\Omega_2(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq d^2\Upsilon(\lambda,\mu,\tau,y)\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}} \\
&\leq \frac{d^2}{\beta^2}\Upsilon(\lambda,\mu,\tau,y)\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Omega_1\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Omega_2\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}},
\end{aligned}$$

which is (15). □

Corollary 1. *Let ι, ϑ, $r_0\in\mathbb{T}$, λ, $\mu>1$, $\Xi\in C^1_{ld}([r_0,\tau)_{\mathbb{T}},\mathbb{R})$, $\Upsilon\in C^1_{ld}([r_0,y)_{\mathbb{T}},\mathbb{R})$ and $\Xi(r_0)=\Upsilon(r_0)=0$. Then we have for $\iota\in[r_0,\tau)_{\mathbb{T}}$, $\vartheta\in[r_0,y)_{\mathbb{T}}$ that*

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{|\Xi(\iota)||\Upsilon(\vartheta)|}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
\leq\;&\frac{1}{\lambda+\mu}(\tau-r_0)^{\frac{\lambda-1}{\lambda}}(y-r_0)^{\frac{\mu-1}{\mu}} \\
&\times\left[\int_{r_0}^{y}\left|\Upsilon^{\nabla}(\vartheta)\right|^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}}\left[\int_{r_0}^{\tau}\left|\Xi^{\nabla}(\iota)\right|^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}}.
\end{aligned}\tag{25}$$

for $\tau,y\in(r_0,\infty)_{\mathbb{T}}$.

Proof. Since $\Xi(r_0)=\Upsilon(r_0)=0$, we have

$$\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|\nabla\tau\geq\left|\int_{r_0}^{\iota}\Xi^{\nabla}(\tau)\nabla\tau\right|=|\Xi(\iota)-\Xi(r_0)|=|\Xi(\iota)|,\tag{26}$$

and

$$\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right| \nabla\tau \geq \left| \int_{r_0}^{\vartheta} \Upsilon^{\nabla}(\tau) \nabla\tau \right| = |\Upsilon(\vartheta) - \Upsilon(r_0)| = |\Upsilon(\vartheta)| \tag{27}$$

By applying (10) on L.H.S of (26) and (27), we obtain

$$\begin{aligned} |\Xi(\iota)| &\leq \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}} \left(\int_{r_0}^{\iota} (1)^{\frac{\lambda}{\lambda-1}} \nabla\tau \right)^{\frac{\lambda-1}{\lambda}} \\ &= \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}} (\iota - r_0)^{\frac{\lambda-1}{\lambda}}, \end{aligned} \tag{28}$$

and also,

$$|\Upsilon(\vartheta)| \leq \left(\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right|^{\mu} \nabla\tau \right)^{\frac{1}{\mu}} (\vartheta - r_0)^{\frac{\mu-1}{\mu}}. \tag{29}$$

From (28) and (29), we see that

$$|\Xi(\iota)||\Upsilon(\vartheta)| \leq (\iota - r_0)^{\frac{\lambda-1}{\lambda}} (\vartheta - r_0)^{\frac{\mu-1}{\mu}} \left(\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right|^{\mu} \nabla\tau \right)^{\frac{1}{\mu}} \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}}. \tag{30}$$

Applying (12) with $\beta_1 = (\iota - r_0)^{\lambda-1}$, $\beta_2 = (\vartheta - r_0)^{\mu-1}$, $\omega_1 = 1/\lambda$, $\omega_2 = 1/\mu$, we observe that

$$\begin{aligned} &(\iota - r_0)^{\frac{\lambda-1}{\lambda}} (\vartheta - r_0)^{\frac{\mu-1}{\mu}} \\ \leq\ & \frac{1}{\frac{1}{\lambda} + \frac{1}{\mu}} \left(\frac{1}{\lambda} (\iota - r_0)^{(\lambda-1)\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)} + \frac{1}{\mu} (\vartheta - r_0)^{(\mu-1)\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)} \right) \\ =\ & \frac{\lambda\mu}{\lambda + \mu} \left(\frac{1}{\lambda} (\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \frac{1}{\mu} (\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}} \right). \end{aligned} \tag{31}$$

Substituting (31) into (30), we obtain

$$\begin{aligned} |\Xi(\iota)||\Upsilon(\vartheta)| &\leq \frac{\lambda\mu}{\lambda + \mu} \left(\frac{1}{\lambda} (\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \frac{1}{\mu} (\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}} \right) \\ &\quad \times \left(\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right|^{\mu} \nabla\tau \right)^{\frac{1}{\mu}} \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}} \\ &= \frac{1}{\lambda + \mu} \left(\mu (\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda (\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}} \right) \\ &\quad \times \left(\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right|^{\mu} \nabla\tau \right)^{\frac{1}{\mu}} \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}}. \end{aligned} \tag{32}$$

By dividing (32) on the term

$$\mu (\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda (\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}},$$

and then integrating with respect to ι from r_0 to τ and for ϑ from r_0 to y, to obtain

$$\begin{aligned} &\int_{r_0}^{y} \int_{r_0}^{\tau} \frac{|\Xi(\iota)||\Upsilon(\vartheta)|}{\mu (\iota - r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda (\vartheta - r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} \nabla\iota \nabla\vartheta \\ \leq\ & \frac{1}{\lambda + \mu} \left[\int_{r_0}^{y} \left(\int_{r_0}^{\vartheta} \left| \Upsilon^{\nabla}(\tau) \right|^{\mu} \nabla\tau \right)^{\frac{1}{\mu}} \nabla\vartheta \right] \left[\int_{r_0}^{\tau} \left(\int_{r_0}^{\iota} \left| \Xi^{\nabla}(\tau) \right|^{\lambda} \nabla\tau \right)^{\frac{1}{\lambda}} \nabla\iota \right]. \end{aligned} \tag{33}$$

Applying (10) on the two parts of the right term of (33), we see

$$
\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{|\Xi(\iota)||\Upsilon(\vartheta)|}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
\leq\;& \frac{1}{\lambda+\mu}\left[\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|^{\lambda}\nabla\tau\nabla\iota\right]^{\frac{1}{\lambda}}(\tau-r_0)^{\frac{\lambda-1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\nabla\vartheta\right]^{\frac{1}{\mu}}(y-r_0)^{\frac{\mu-1}{\mu}}.
\end{aligned}
\tag{34}
$$

Applying (9) on the term

$$
\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\nabla\vartheta,
$$

with $u(\vartheta)=\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau$ and $v^{\nabla}(\vartheta)=1$, we obtain

$$
\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\nabla\vartheta \\
=\;& v(\vartheta)\left(\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\right)\Big|_{r_0}^{y}-\int_{r_0}^{y}\left|\Upsilon^{\nabla}(\vartheta)\right|^{\mu}v^{\rho}(\vartheta)\nabla\vartheta,
\end{aligned}
$$

where $v(\vartheta)=\vartheta-y$, and then

$$
\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\nabla\vartheta \\
=\;& (\vartheta-y)\left(\int_{r_0}^{\vartheta}\left|\Upsilon^{\nabla}(\tau)\right|^{\mu}\nabla\tau\right)\Big|_{r_0}^{y}+\int_{r_0}^{y}\left|\Upsilon^{\nabla}(\vartheta)\right|^{\mu}(y-\rho(\vartheta))\nabla\vartheta \\
=\;& \int_{r_0}^{y}\left|\Upsilon^{\nabla}(\vartheta)\right|^{\mu}(y-\rho(\vartheta))\nabla\vartheta.
\end{aligned}
\tag{35}
$$

Similarly, by applying (9) on the term

$$
\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|^{\lambda}\nabla\tau\nabla\iota,
$$

we observe that

$$
\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|^{\lambda}\nabla\tau\nabla\iota=\int_{r_0}^{\tau}\left|\Xi^{\nabla}(\iota)\right|^{\lambda}(\tau-\rho(\iota))\nabla\iota. \tag{36}
$$

Substituting (35) and (36) into (34), to obtain

$$
\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{|\Xi(\iota)||\Upsilon(\vartheta)|}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
\leq\;& \frac{1}{\lambda+\mu}(\tau-r_0)^{\frac{\lambda-1}{\lambda}}(y-r_0)^{\frac{\mu-1}{\mu}} \\
&\times\left[\int_{r_0}^{y}\left|\Upsilon^{\nabla}(\vartheta)\right|^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}}\left[\int_{r_0}^{\tau}\left|\Xi^{\nabla}(\iota)\right|^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}},
\end{aligned}
$$

which is (25). □

Remark 1. *As particular cases of Corollary 1, (when $\mathbb{T}=\mathbb{N}$, $r_0=0$), we obtain the inequality (7) and (when $\mathbb{T}=\mathbb{R}$, $r_0=0$), we obtain the inequality (8).*

In what follows, we generalize Corollary 1 by using a submultiplicative function.

Corollary 2. *Let ι, ϑ, $r_0 \in \mathbb{T}$, λ, $\mu > 1$, $\Xi \in C^1_{ld}([r_0,\tau)_{\mathbb{T}},\mathbb{R})$, $\Upsilon \in C^1_{ld}([r_0,y)_{\mathbb{T}},\mathbb{R})$ with $\Xi(r_0) = \Upsilon(r_0) = 0$. Assume that $h \in C^1_{ld}([r_0,\tau)_{\mathbb{T}},\mathbb{R})$, $l \in C^1_{ld}([r_0,y)_{\mathbb{T}},\mathbb{R})$ and $\Phi,\Psi \geq 0$ are increasing, convex and submultiplicative functions. If*

$$H(\iota) = \int_{r_0}^{\iota} |h(\tau)|\nabla\tau \quad \text{and} \quad L(\vartheta) = \int_{r_0}^{\vartheta} |l(\zeta)|\nabla\zeta,$$

for $\iota \in [r_0,\tau)_{\mathbb{T}}$, $\vartheta \in [r_0,y)_{\mathbb{T}}$, then

$$\begin{aligned} &\int_{r_0}^{y}\int_{r_0}^{\tau} \frac{\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\ &\leq \Upsilon(\tau,y,\lambda,\mu)\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\ &\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}}, \end{aligned} \tag{37}$$

where $\tau, y \in (r_0,\infty)_{\mathbb{T}}$ and

$$\begin{aligned} \Upsilon(\tau,y,\lambda,\mu) &= \frac{1}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\ &\quad\times\left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}}. \end{aligned}$$

Proof. Using the fact that Φ is a non-negative, increasing and submultiplicative function, we have

$$\begin{aligned} \Phi(|\Xi(\iota)|) &\leq \Phi\left(\int_{r_0}^{\iota}\left|\Xi^{\nabla}(\tau)\right|\nabla\tau\right) \\ &= \Phi\left(H(\iota)\frac{\int_{r_0}^{\iota}|h(\tau)|\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\nabla\tau}{\int_{r_0}^{\iota}|h(\tau)|\nabla\tau}\right) \\ &\leq \Phi(H(\iota))\Phi\left(\frac{\int_{r_0}^{\iota}|h(\tau)|\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\nabla\tau}{\int_{r_0}^{\iota}|h(\tau)|\nabla\tau}\right). \end{aligned} \tag{38}$$

Applying (13) on the R.H.S of (38), we observe that

$$\Phi(|\Xi(\iota)|) \leq \frac{\Phi(H(\iota))}{H(\iota)}\left(\int_{r_0}^{\iota}|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau\right). \tag{39}$$

Applying (10) on the term

$$\int_{r_0}^{\iota}|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau,$$

with indices $\lambda > 1$ and $\lambda/(\lambda - 1)$, we obtain

$$\begin{aligned}
&\int_{r_0}^{\iota} |h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\nabla\tau \\
&\leq \left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}\left[\int_{r_0}^{\iota} 1\nabla\tau\right]^{\frac{\lambda-1}{\lambda}} \\
&= (\iota - r_0)^{\frac{\lambda-1}{\lambda}}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}.
\end{aligned} \tag{40}$$

From (39) and (40), we obtain

$$\begin{aligned}
\Phi(|\Xi(\iota)|) \quad \leq \quad & \frac{\Phi(H(\iota))}{H(\iota)}(\iota - r_0)^{\frac{\lambda-1}{\lambda}} \\
& \times\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}.
\end{aligned} \tag{41}$$

Similarly, we have that

$$\begin{aligned}
\Psi(|\Upsilon(\vartheta)|) \quad \leq \quad & \frac{\Psi(L(\vartheta))}{L(\vartheta)}(\vartheta - r_0)^{\frac{\mu-1}{\mu}} \\
& \times\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}}.
\end{aligned} \tag{42}$$

From (41) and (42), we observe that

$$\begin{aligned}
&\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|) \\
&\leq (\iota - r_0)^{\frac{\lambda-1}{\lambda}}(\vartheta - r_0)^{\frac{\mu-1}{\mu}} \\
&\times \frac{\Phi(H(\iota))}{H(\iota)}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}} \\
&\times \frac{\Psi(L(\vartheta))}{L(\vartheta)}\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}}.
\end{aligned} \tag{43}$$

From Lemma 2, the inequality (43) becomes

$$\begin{aligned}
&\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|) \\
&\leq \frac{\lambda\mu}{\lambda+\mu}\left(\frac{1}{\lambda}(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\frac{1}{\mu}(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right) \\
&\times\frac{\Phi(H(\iota))}{H(\iota)}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}} \\
&\times\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}} \\
&=\frac{1}{\lambda+\mu}\left(\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right) \\
&\times\frac{\Phi(H(\iota))}{H(\iota)}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}} \\
&\times\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}}.
\end{aligned} \tag{44}$$

Dividing the two sides of (44) on the term

$$\left(\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}\right),$$

and then integrating with respect to ι from r_0 to τ and for ϑ from r_0 to y, to obtain

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq\frac{1}{\lambda+\mu}\int_{r_0}^{\tau}\frac{\Phi(H(\iota))}{H(\iota)}\left[\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\right]^{\frac{1}{\lambda}}\nabla\iota \\
&\times\int_{r_0}^{y}\frac{\Psi(L(\vartheta))}{L(\vartheta)}\left[\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\right]^{\frac{1}{\mu}}\nabla\vartheta.
\end{aligned} \tag{45}$$

Applying (10) on the R.H.S of (45), we see that

$$\begin{aligned}
&\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\
&\leq\frac{1}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\
&\times\left[\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\nabla\iota\right]^{\frac{1}{\lambda}} \\
&\times\left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}} \\
&\times\left[\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\nabla\vartheta\right]^{\frac{1}{\mu}}.
\end{aligned} \tag{46}$$

By applying (9), we have that

$$\begin{aligned} &\int_{r_0}^{\tau}\int_{r_0}^{\iota}\left[|h(\tau)|\Phi\left(\left|\frac{\Xi^{\nabla}(\tau)}{h(\tau)}\right|\right)\right]^{\lambda}\nabla\tau\nabla\iota \\ = &\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota, \end{aligned}$$

and also,

$$\begin{aligned} &\int_{r_0}^{y}\int_{r_0}^{\vartheta}\left[|l(\tau)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\tau)}{l(\tau)}\right|\right)\right]^{\mu}\nabla\tau\nabla\vartheta \\ = &\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta \end{aligned}$$

and then substituting into (46), we obtain

$$\begin{aligned} &\int_{r_0}^{y}\int_{r_0}^{\tau}\frac{\Phi(|\Xi(\iota)|)\Psi(|\Upsilon(\vartheta)|)}{\mu(\iota-r_0)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(\vartheta-r_0)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla\iota\nabla\vartheta \\ &\leq \frac{1}{\lambda+\mu}\left[\int_{r_0}^{\tau}\left[\frac{\Phi(H(\iota))}{H(\iota)}\right]^{\frac{\lambda}{\lambda-1}}\nabla\iota\right]^{\frac{\lambda-1}{\lambda}} \\ &\times\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\ &\times\left[\int_{r_0}^{y}\left[\frac{\Psi(L(\vartheta))}{L(\vartheta)}\right]^{\frac{\mu}{\mu-1}}\nabla\vartheta\right]^{\frac{\mu-1}{\mu}} \\ &\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}} \\ &= \Upsilon(\tau,y,\lambda,\mu)\left[\int_{r_0}^{\tau}\left[|h(\iota)|\Phi\left(\left|\frac{\Xi^{\nabla}(\iota)}{h(\iota)}\right|\right)\right]^{\lambda}(\tau-\rho(\iota))\nabla\iota\right]^{\frac{1}{\lambda}} \\ &\times\left[\int_{r_0}^{y}\left[|l(\vartheta)|\Psi\left(\left|\frac{\Upsilon^{\nabla}(\vartheta)}{l(\vartheta)}\right|\right)\right]^{\mu}(y-\rho(\vartheta))\nabla\vartheta\right]^{\frac{1}{\mu}}, \end{aligned}$$

which is (37). □

Remark 2. *As a specific case of Corollary 2, when $\beta = d = 1$, we obtain Corollary 1.*

4. Conclusions and Future Work

In this work, we explored some new generalization inequalities of Hilbert type on time scales by using nabla calculus, which are used in various problems involving symmetry. Further, we also applied our inequalities to discrete and continuous calculus to obtain some new Hilbert type inequalities as special cases. Moreover, some new inequalities as special cases are discussed. In future work, we will continue to generalize more dynamic inequalities by conformable fractional calculus on time scales by using Specht's ratio, Kantorovich's ratio and n-tuple fractional integral. It will also be very enjoyable to introduce such inequalities on quantum calculus.

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