

Article

# Analytical Properties of Degenerate Genocchi Polynomials of the Second Kind and Some of Their Applications

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**Abstract:** The main aim of this study is to define degenerate Genocchi polynomials and numbers of the second kind by using logarithmic functions, and to investigate some of their analytical properties and some applications. For this purpose, many formulas and relations for these polynomials, including some implicit summation formulas, differentiation rules and correlations with the earlier polynomials by utilizing some series manipulation methods, are derived. Additionally, as an application, the zero values of degenerate Genocchi polynomials and numbers of the second kind are presented in tables and multifarious graphical representations for these zero values are shown.

**Keywords:** degenerate Genocchi polynomials; degenerate Genocchi polynomials of the second kind; summation formulae; Stirling numbers

**MSC:** 05A10; 05A15; 11B68



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## 1. Introduction

Recently, many mathematicians, particularly Carlitz [1,2], Kim et al. [3], Sharma et al. [4,5], and Khan et al. [6–8], have studied and delivered diverse degenerate variations of many unique polynomials and numbers (such as degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Daehee polynomials, degenerate Fubini polynomials, degenerate Stirling numbers of the first and second kind, and so forth). It is noteworthy that studying degenerate variations is not always most effective when limited to polynomials, but also prolonged to transcendental features, like gamma functions. It is likewise terrific that the degenerate umbral calculus is delivered as a degenerate version of the classical umbral calculus. Degenerate versions of special numbers and polynomials have been explored by way of various techniques, such as combinatorial strategies, producing functions, umbral calculus techniques,  $p$ -adic analysis, differential equations, unique capabilities, probability principles, and analytic variety ideas. In this paper, we focus on degenerate Genocchi polynomials and numbers of the second kind. The intention of this paper is to introduce a degenerate version of the Genocchi polynomials and numbers of the second type, the so-called degenerate Genocchi polynomials and numbers of the second type, made from the degenerate exponential characteristic. We derive a few express expressions and identities for those numbers and polynomials. Further, we introduce degenerate Genocchi polynomials of the second kind attached to Dirichlet character  $\chi$  and establish some properties of these polynomials.

Let  $p$  be a fixed odd prime number. Throughout this paper,  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will denote the the ring of  $p$ -adic integers, the field of  $p$ -adic rational numbers, and the completion of algebraic closure of  $\mathbb{Q}_p$ , respectively. The  $p$ -adic norm  $|\cdot|_p$  is normalized as  $|\cdot|_p = p^{-1} = \frac{1}{p}$ . Let  $UD(\mathbb{Z}_p)$  be the space of  $\mathbb{C}_p$ -valued uniformly differentiable functions on  $\mathbb{Z}_p$ .

For  $f \in \cup D(\mathbb{Z}_p)$ , the  $p$ -adic invariant integral on  $\mathbb{Z}_p$  is defined as (see [8–11])

$$\begin{aligned}
 I_0(f) &= \int_{\mathbb{Z}_p} f(\xi) d\mu_0(\xi) = \lim_{N \rightarrow \infty} \sum_{\xi=0}^{p^N-1} f(\xi) \mu_0(\xi + p^N \mathbb{Z}_p) \\
 &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{\xi=0}^{p^N-1} f(\xi).
 \end{aligned}
 \tag{1}$$

For  $f \in \cup D(\mathbb{Z}_p)$ , the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  is defined by Kim as follows (see [11])

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-1}(\xi) = \lim_{N \rightarrow \infty} \sum_{\xi=0}^{p^N-1} f(\xi) (-1)^\xi.
 \tag{2}$$

From (1) and (2), we have

$$I_0(f_\omega) - I_0(f) = \sum_{l=0}^{\omega-1} f'(l) \quad (\omega \in \mathbb{N}),
 \tag{3}$$

and

$$I_{-1}(f_\omega) - (-1)^\omega I_{-1}(f) = 2 \sum_{a=0}^{\omega-1} (-1)^{\omega-1-a} f(a) \quad (\omega \in \mathbb{N}).
 \tag{4}$$

For any non-zero  $\lambda \in \mathbb{R}$  (or  $\mathbb{C}$ ), the degenerate exponential function is defined by (see [4–7])

$$e_\lambda^\xi(z) = (1 + \lambda z)^{\frac{\xi}{\lambda}}, \quad e_\lambda^1(z) = (1 + \lambda z)^{\frac{1}{\lambda}}.
 \tag{5}$$

By binomial expansion, we get

$$e_\lambda^\xi(z) = \sum_{\omega=0}^{\infty} (\xi)_{\omega, \lambda} \frac{z^\omega}{\omega!},
 \tag{6}$$

where  $(\xi)_{0, \lambda} = 1, (\xi)_{\omega, \lambda} = (\xi - \lambda)(\xi - 2\lambda) \cdots (\xi - (\omega - 1)\lambda) \quad (\omega \geq 1)$ .

Note that

$$\lim_{\lambda \rightarrow 0} e_\lambda^\xi(z) = \sum_{\omega=0}^{\infty} \xi^\omega \frac{z^\omega}{\omega!} = e^{\xi z}.$$

In [1], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{z}{(1 + \lambda z)^{\frac{1}{\lambda}} - 1} (1 + \lambda z)^{\frac{\xi}{\lambda}} = \sum_{\omega=0}^{\infty} \beta_{\omega, \lambda}(\xi) \frac{z^\omega}{\omega!} \quad (\lambda \in \mathbb{R}).
 \tag{7}$$

Here,  $\xi = 0, \beta_{\omega, \lambda} = \beta_{\omega, \lambda}(0)$  are called the degenerate Bernoulli numbers. The degenerate Genocchi polynomials  $\mathbb{G}_\omega(\xi; \lambda)$  are defined by (see [12])

$$\frac{2z}{e_\lambda(z) + 1} e_\lambda^\xi(z) = \sum_{\omega=0}^{\infty} \mathbb{G}_\omega(\xi; \lambda) \frac{z^\omega}{\omega!}.
 \tag{8}$$

In the case when  $\xi = 0, \mathbb{G}_\omega(\lambda) = \mathbb{G}_\omega(0; \lambda)$  are called the degenerate Genocchi numbers. From (8), we note that

$$\begin{aligned}
 \lim_{\lambda \rightarrow 0} \sum_{\omega=0}^{\infty} \mathbb{G}_\omega(\xi; \lambda) \frac{z^\omega}{\omega!} &= \lim_{\lambda \rightarrow 0} \frac{2z}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\
 &= \frac{2z}{e^z + 1} e^{\xi z} = \sum_{\omega=0}^{\infty} \mathbb{G}_\omega(\xi) \frac{z^\omega}{\omega!},
 \end{aligned}
 \tag{9}$$

where  $\mathbb{G}_\omega(\xi)$  are called the Genocchi numbers.

The partially degenerate Genocchi polynomials are defined by the generating function as follows (see [13])

$$\frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e^z + 1} e^{\xi z} = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \lambda}(\xi) \frac{z^\omega}{\omega!}. \tag{10}$$

At the point  $\xi = 0$ ,  $\mathbb{G}_{\omega, \lambda} = \mathbb{G}_{\omega, \lambda}(0)$  are called the partially degenerate Genocchi numbers. The new type of degenerate Changhee–Genocchi polynomials are defined by (see [14])

$$\frac{2 \log_\lambda(1 + z)}{2 + z} (1 + z)^\xi = \sum_{\omega=0}^{\infty} CG_{\omega, \lambda}(\xi) \frac{z^\omega}{\omega!}. \tag{11}$$

In the case when  $\xi = 0$ ,  $CG_{\omega, \lambda} = CG_{\omega, \lambda}(0)$  are called the new type of degenerate Changhee-Genocchi numbers.

For  $\omega \geq 0$ , the Stirling numbers of the first kind are defined by

$$(\xi)_\omega = \sum_{l=0}^{\omega} S_1(\omega, l) \xi^l, \tag{12}$$

where  $(\xi)_0 = 1$ , and  $(\xi)_\omega = \xi(\xi - 1) \cdots (\xi - \omega + 1)$  ( $\omega \geq 1$ ). From (12), it is easy to see that

$$\frac{1}{r!} (\log(1 + z))^r = \sum_{\omega=r}^{\infty} S_1(\omega, r) \frac{z^\omega}{\omega!} \quad (r \geq 0). \tag{13}$$

For  $\omega \geq 0$ , the Stirling numbers of the second kind are defined by

$$\xi^\omega = \sum_{l=0}^{\omega} S_2(\omega, l) (\xi)_l. \tag{14}$$

From (14), we attain that

$$\frac{1}{r!} (e^z - 1)^r = \sum_{\omega=r}^{\infty} S_2(\omega, r) \frac{z^\omega}{\omega!}. \tag{15}$$

This article is structured as follows. In Section 2, we consider degenerate Genocchi polynomials of the second kind and derive some basic properties of these polynomials by using different analytical means of their respective generating functions. In Section 3, we introduce degenerate Genocchi polynomials of the second kind attached to Dirichlet character  $\chi$  and derive some properties of these polynomials.

### 2. Degenerate Genocchi Polynomials of the Second Kind

Let  $\lambda, z \in \mathbb{C}_p$  be  $|\lambda z|_p < p^{-\frac{1}{p-1}}$ . Now, we consider the degenerate Genocchi polynomials of the second kind defined by

$$\frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} e_\lambda^\xi(z) = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \lambda}(\xi) \frac{z^\omega}{\omega!}. \tag{16}$$

In the case when  $\xi = 0$ ,  $\mathbb{G}_{\omega, \lambda}(0) = \mathbb{G}_{\omega, \lambda}$  are called the degenerate Genocchi numbers of the second kind.

Note that

$$\lim_{\lambda \rightarrow 0} \mathbb{G}_{\omega, \lambda}(\xi) = \mathbb{G}_\omega(\xi) \quad (\omega \geq 0).$$

**Theorem 1.** For  $\omega \geq 0$ , we have

$$\mathbb{G}_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \frac{(-1)^\nu}{\nu+1} \lambda^\nu \nu! \mathbb{G}_{\omega-\nu}(\xi; \lambda).$$

**Proof.** Using (8) and (16), we have

$$\begin{aligned} \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}(\xi) \frac{z^\omega}{\omega!} &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \frac{\log(1 + \lambda z)^{\frac{1}{\lambda}}}{\lambda z} \frac{2z}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \left( \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu+1} (\lambda z)^\nu \right) \left( \sum_{\omega=0}^{\infty} \mathbb{G}_\omega(\xi; \lambda) \frac{z^\omega}{\omega!} \right) \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \frac{(-1)^\nu}{\nu+1} \lambda^\nu \nu! \mathbb{G}_{\omega-\nu}(\xi; \lambda) \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{17}$$

Therefore, by (16) and (17), we obtain the result.  $\square$

**Theorem 2.** For  $\omega \geq 0$ , we have

$$\mathbb{G}_{\omega,\lambda}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} D_\nu \lambda^\nu \mathbb{G}_{\omega-\nu}(\xi; \lambda).$$

**Proof.** Following Equation (8) and (16), we find

$$\begin{aligned} \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}(\xi) \frac{z^\omega}{\omega!} &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \frac{\log(1 + \lambda z)^{\frac{1}{\lambda}}}{\lambda z} \frac{2z}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \left( \sum_{\nu=0}^{\infty} D_\nu \lambda^\nu \frac{z^\nu}{\nu!} \right) \left( \sum_{\omega=0}^{\infty} \mathbb{G}_\omega(\xi; \lambda) \frac{z^\omega}{\omega!} \right) \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} D_\nu \lambda^\nu \mathbb{G}_{\omega-\nu}(\xi; \lambda) \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{18}$$

Therefore, by (16) and (18), we get the result.  $\square$

**Theorem 3.** For  $\omega \geq 0$ , we have

$$\mathbb{G}_{\omega,\lambda}(\xi) = \omega \sum_{\nu=0}^{\omega-1} \binom{\omega-1}{\nu} D_\nu \lambda^\nu \mathbb{E}_{\omega-1-\nu,\lambda}(\xi).$$

**Proof.** Using the definition (8) and (16), we have

$$\begin{aligned} \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}(\xi) \frac{z^\omega}{\omega!} &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \frac{\log(1 + \lambda z)^{\frac{1}{\lambda}}}{\lambda z} \frac{2}{e_\lambda(z) + 1} e_\lambda^\xi(z) \end{aligned}$$

$$\begin{aligned}
 &= z \left( \sum_{\nu=0}^{\infty} D_{\nu}(\lambda)^{\nu} \frac{z^{\nu}}{\nu!} \right) \left( \sum_{\omega=0}^{\infty} \mathbb{E}_{\omega,\lambda}(\xi) \frac{z^{\omega}}{\omega!} \right) \\
 &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega-1} \binom{\omega-1}{\nu} D_{\nu} \lambda^{\nu} \mathbb{E}_{\omega-1-\nu,\lambda}(\xi) \right) \frac{z^{\omega}}{(\omega-1)!}.
 \end{aligned} \tag{19}$$

Comparing the coefficients of  $z$  on both sides, we obtain the result.  $\square$

**Theorem 4.** For  $\omega \geq 0$ , we have

$$\mathbb{G}_{\omega}(\xi) = \sum_{\nu=0}^{\omega} \mathbb{G}_{\nu,\lambda}(\xi) \lambda^{\omega-\nu} S_2(\omega, \nu).$$

**Proof.** By replacing  $z$  with  $\frac{1}{\lambda}(e^{\lambda z} - 1)$  in (16), we get

$$\begin{aligned}
 \frac{2z}{e^z + 1} e^{\xi z} &= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu,\lambda}(\xi) \frac{[\frac{1}{\lambda}(e^{\lambda z} - 1)]^{\nu}}{\nu!} \\
 &= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu,\lambda}(\xi) \lambda^{-\nu} \sum_{\omega=\nu}^{\infty} S_2(\omega, \nu) \frac{\lambda^{\omega} z^{\omega}}{\omega!} \\
 &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \mathbb{G}_{\nu,\lambda}(\xi) \lambda^{\omega-\nu} S_2(\omega, \nu) \right) \frac{z^{\omega}}{\omega!}.
 \end{aligned} \tag{20}$$

On the other hand, we have

$$\frac{2z}{e^z + 1} e^{\xi z} = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega}(\xi) \frac{z^{\omega}}{\omega!}. \tag{21}$$

In view of (20) and (21), we obtain the result.  $\square$

**Theorem 5.** For  $\omega \geq 0$ , we have

$$\frac{1}{2} [\mathbb{G}_{\omega,\lambda}(1) + \mathbb{G}_{\omega,\lambda}] = \lambda^{\omega} (-1)^{\omega} \omega!. \tag{22}$$

**Proof.** From (16), it is shown that

$$\begin{aligned}
 &\sum_{\omega=1}^{\infty} [\mathbb{G}_{\omega,\lambda}(1) + \mathbb{G}_{\omega,\lambda}] \frac{z^{\omega}}{\omega!} \\
 &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_{\lambda}(z) + 1} e_{\lambda}(z) + \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_{\lambda}(z) + 1} = 2 \log(1 + \lambda z)^{\frac{1}{\lambda}} \\
 &\sum_{\omega=0}^{\infty} \left[ \frac{\mathbb{G}_{\omega,\lambda}(1) + \mathbb{G}_{\omega,\lambda}}{\omega + 1} \right] \frac{z^{\omega}}{\omega!} \\
 &= 2 \sum_{\omega=0}^{\infty} \frac{\lambda^{\omega} (-1)^{\omega} \omega! z^{\omega}}{\omega + 1 \omega!}.
 \end{aligned}$$

Comparing the coefficients of  $z$ , we obtain the result (22).  $\square$

**Theorem 6.** For  $\omega \geq 0$  with  $d \in \mathbb{N}$ , we have

$$\mathbb{G}_{\omega,\lambda}(\xi) = d^{\omega-1} \sum_{a=0}^{d-1} \mathbb{G}_{\omega} \left( \frac{a + \xi}{d} \mid \frac{\lambda}{d} \right). \tag{23}$$

**Proof.** From (16), we find

$$\begin{aligned} \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}(\xi) \frac{z^\omega}{\omega!} &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} e_\lambda^\xi(z) \\ &= \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{1}{d}} + 1} (1 + \lambda z)^{\frac{a+\xi}{\lambda}} \\ &= \frac{1}{d} \left( \sum_{a=0}^{d-1} \frac{2 \log(1 + \lambda z)^{\frac{d}{\lambda}}}{(1 + \lambda z)^{\frac{a}{d}} + 1} (1 + \lambda z)^{\frac{d}{\lambda} \frac{a+\xi}{\lambda}} \right) \\ &= \sum_{\omega=0}^{\infty} \left( d^{\omega-1} \sum_{a=0}^{\omega-1} \mathbb{G}_\omega \left( \frac{a + \xi}{d} \middle| \frac{\lambda}{d} \right) \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{24}$$

Equating the coefficients of  $z$ , we get (23).  $\square$

**Theorem 7.** For  $\omega \geq 1$ , we have

$$D_{\omega-1} \lambda^{\omega-1} = \frac{1}{2\omega} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} (1)_{\nu,\lambda} \mathbb{G}_{\omega-\nu,\lambda} + \mathbb{G}_{\omega,\lambda} \right). \tag{25}$$

**Proof.** Using (16), we see

$$\begin{aligned} 2 \log(1 + \lambda z)^{\frac{1}{\lambda}} &= (e_\lambda(z) + 1) \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda} \frac{z^\omega}{\omega!} \\ \frac{2z \log(1 + \lambda z)}{\lambda z} &= \sum_{\nu=0}^{\infty} (1)_{\nu,\lambda} \frac{z^\nu}{\nu!} \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda} \frac{z^\omega}{\omega!} + \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}(\xi) \frac{z^\omega}{\omega!} \\ &= 2z \sum_{\omega=0}^{\infty} D_\omega \lambda^\omega \frac{z^\omega}{\omega!} \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} (1)_{\nu,\lambda} \mathbb{G}_{\omega-\nu,\lambda} + \mathbb{G}_{\omega,\lambda} \right) \frac{z^\omega}{\omega!} \\ &= 2 \sum_{\omega=1}^{\infty} D_{\omega-1} \lambda^{\omega-1} \frac{z^\omega}{(\omega-1)!} \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} (1)_{\nu,\lambda} \mathbb{G}_{\omega-\nu,\lambda} + \mathbb{G}_{\omega,\lambda} \right) \frac{z^\omega}{\omega!}. \end{aligned}$$

On comparing the coefficients of  $\frac{z^\omega}{\omega!}$ , we get the result (25).  $\square$

**Theorem 8.** Let  $\omega \geq 0$ . Then

$$C \mathbb{G}_{\omega,\lambda} = \sum_{\nu=0}^{\omega} \lambda^{\omega-\nu} \mathbb{G}_\nu(\lambda) S_1(\omega, \nu). \tag{26}$$

**Proof.** Replacing  $z$  by  $\log(1 + \lambda z)$  in (8), we find

$$\begin{aligned} \left( \frac{2 \log(1 + \lambda z)}{(1 + \lambda \log(1 + \lambda z))^{\frac{1}{\lambda}} + 1} \right) &= \sum_{\nu=0}^{\infty} \mathbb{G}_\nu(\lambda) \frac{(\log(1 + \lambda z))^\nu}{\nu!} \\ &= \sum_{\nu=0}^{\infty} \mathbb{G}_\nu(\lambda) \sum_{\omega=\nu}^{\infty} S_1(\omega, \nu) \lambda^{\omega-\nu} \frac{z^\omega}{\omega!} \end{aligned}$$

$$= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \lambda^{\omega-\nu} \mathbb{G}_{\nu}(\lambda) S_{1,\lambda}(\omega, \nu) \right) \frac{z^{\omega}}{\omega!}. \tag{27}$$

On the other hand,

$$\left( \frac{2 \log(1 + \lambda z)}{(1 + \lambda \log(1 + \lambda z))^{\frac{1}{\lambda}} + 1} \right) = \sum_{\omega=0}^{\infty} CG_{\omega,\lambda} \frac{z^{\omega}}{\omega!}. \tag{28}$$

By (27) and (28), we obtain the result (26).  $\square$

**Theorem 9.** Let  $\omega \geq 0$ . Then

$$\sum_{\nu=0}^{\omega} \mathbb{G}_{\nu,\lambda}(\xi) S_{1,\lambda}(\omega, \nu) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\omega-\nu} D_{\nu,\lambda}(\xi). \tag{29}$$

**Proof.** On changing  $z$  with  $\log_{\lambda}(1 + z)$  in (8), we get

$$\begin{aligned} \frac{2 \log_{\lambda}(1 + z)}{2 + z} (1 + z)^{\xi} &= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu,\lambda}(\xi) \frac{(\log_{\lambda}(1 + z))^{\nu}}{\nu!} \\ &= \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu,\lambda}(\xi) \sum_{\omega=\nu}^{\infty} S_{1,\lambda}(\omega, \nu) \frac{z^{\omega}}{\omega!} \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \mathbb{G}_{\nu,\lambda}(\xi) S_{1,\lambda}(\omega, \nu) \right) \frac{z^{\omega}}{\omega!}. \end{aligned} \tag{30}$$

On the other hand, we have

$$\begin{aligned} \frac{2 \log_{\lambda}(1 + z)}{2 + z} (1 + z)^{\xi} &= \frac{2z}{2 + z} \frac{\log_{\lambda}(1 + z)}{z} (1 + z)^{\xi} \\ &= \sum_{\omega=0}^{\infty} CG_{\omega} \frac{z^{\omega}}{\omega!} \sum_{\nu=0}^{\infty} D_{\nu,\lambda}(\xi) \frac{z^{\nu}}{\nu!} \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} CG_{\omega-\nu} D_{\nu,\lambda}(\xi) \right) \frac{z^{\omega}}{\omega!}. \end{aligned} \tag{31}$$

In view of (30) and (31), we obtain (29).  $\square$

For  $r \in \mathbb{N}$ , we define the higher-order degenerate Genocchi polynomials of the second kind given by the generating function

$$\left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_{\lambda}(z) + 1} \right)^r e_{\lambda}^{\xi}(z) = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^{\omega}}{\omega!}. \tag{32}$$

When  $\xi = 0$ ,  $\mathbb{G}_{\omega,\lambda}^{(r)} = \mathbb{G}_{\omega,\lambda}^{(r)}(0)$  are called the higher-order degenerate Genocchi numbers of the second kind.

It is worth noting that

$$\lim_{\lambda \rightarrow 0} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) = \mathbb{G}_{\omega}^{(r)}(\xi) \quad (\omega \geq 0).$$

**Theorem 10.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega,\lambda}^{(r)}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r-k)}(\xi) \mathbb{G}_{\nu,\lambda}^{(k)}. \tag{33}$$

**Proof.** Equation (32), we see

$$\begin{aligned}
 \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r e_\lambda^\xi(z) \\
 &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^{r-k} \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^k e_\lambda^\xi(z) \\
 &= \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r-k)}(\xi) \frac{z^\omega}{\omega!} \right) \left( \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu,\lambda}^{(k)} \frac{z^\nu}{\nu!} \right) \\
 &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r-k)}(\xi) \mathbb{G}_{\nu,\lambda}^{(k)} \right) \frac{z^\omega}{\omega!}, \tag{34}
 \end{aligned}$$

which gives the result (33).  $\square$

**Theorem 11.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega,\lambda}^{(r)}(\xi + \eta) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r)}(\xi)(\eta)_{\nu,\lambda}. \tag{35}$$

**Proof.** By (35), we note that

$$\begin{aligned}
 \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi + \eta) \frac{z^\omega}{\omega!} &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r e_\lambda^{(\xi+\eta)}(z) \\
 &= \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} \sum_{\nu=0}^{\infty} (\eta)_{\nu,\lambda} \frac{z^\nu}{\nu!} \\
 &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r)}(\xi)(\eta)_{\nu,\lambda} \right) \frac{z^\omega}{\omega!}. \tag{36}
 \end{aligned}$$

Comparing the coefficients of  $z$ , we get (35).  $\square$

**Theorem 12.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega,\lambda}^{(r)}(\xi) = \sum_{l=0}^{\omega} \sum_{m=0}^l \binom{\omega}{l} \mathbb{G}_{\omega-l,\lambda}^{(r)}(\xi)_m S_{2,\lambda}(l, m). \tag{37}$$

**Proof.** From (32), we observe that

$$\begin{aligned}
 \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r [e_\lambda(z) - 1 + 1]^\xi \\
 &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r \sum_{m=0}^{\infty} \binom{\xi}{m} (e_\lambda(z) - 1)^m \\
 &= \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)} \frac{z^\omega}{\omega!} \right) \left( \sum_{m=0}^{\infty} (\xi)_m \sum_{l=m}^{\infty} S_{2,\lambda}(l, m) \frac{z^l}{l!} \right) \\
 &= \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)} \frac{z^\omega}{\omega!} \right) \left( \sum_{l=0}^{\infty} \sum_{m=0}^l (\xi)_m S_{2,\lambda}(l, m) \frac{z^l}{l!} \right)
 \end{aligned}$$

$$= \sum_{\omega=0}^{\infty} \left( \sum_{l=0}^{\omega} \sum_{m=0}^l \binom{\omega}{l} \mathbb{G}_{\omega-l,\lambda}^{(r)}(\xi)_m S_{2,\lambda}(l, m) \right) \frac{z^\omega}{\omega!}, \tag{38}$$

which complete the proof.  $\square$

**Theorem 13.** Let  $\omega \geq 1$ . Then

$$\Delta_\lambda \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) = \omega \mathbb{G}_{\omega-1,\lambda}^{(r)}(\xi). \tag{39}$$

**Proof.** By applying the difference operator  $\Delta_\lambda$  to both sides of Equation (32), we get

$$\Delta_\lambda \left( \sum_{\omega=1}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} \right) = \Delta_\lambda \left( \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r (1 + \lambda z)^{\frac{\xi}{\lambda}} \right)$$

and then we have

$$\begin{aligned} \sum_{\omega=1}^{\infty} \Delta_\lambda \mathbb{G}_{\omega,\lambda}^{(r)} \frac{z^\omega}{\omega!} &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r \Delta_\lambda e_\lambda^\xi(z) \\ &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r e_\lambda^\xi(z) z \\ &= \sum_{\omega=1}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^{\omega+1}}{\omega!}. \end{aligned} \tag{40}$$

Therefore, by (40), we obtain (39).  $\square$

**Theorem 14.** Let  $\omega \geq 0$ . Then

$$\frac{\partial}{\partial \xi} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r)}(\xi) (1)_{\nu,\lambda}. \tag{41}$$

**Proof.** By applying the derivative operator  $\frac{\partial}{\partial \xi}$  with respect to  $\xi$  to both sides of Equation (32), we have

$$\begin{aligned} \frac{\partial}{\partial \xi} \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} \right) &= \frac{\partial}{\partial \xi} \left( \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r (1 + \lambda z)^{\frac{\xi}{\lambda}} \right) \\ &= \sum_{\omega=0}^{\infty} \frac{\partial}{\partial \xi} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} \\ &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r \frac{\partial}{\partial \xi} (1 + \lambda z)^{\frac{\xi}{\lambda}} \\ &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{e_\lambda(z) + 1} \right)^r (1 + \lambda z)^{\frac{\xi}{\lambda}} (1 + \lambda z)^{\frac{1}{\lambda}} \\ &= \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega,\lambda}^{(r)}(\xi) \frac{z^\omega}{\omega!} \right) \left( \sum_{\nu=0}^{\infty} (1)_{\nu,\lambda} \frac{z^\nu}{\nu!} \right) \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\omega-\nu,\lambda}^{(r)}(\xi) (1)_{\nu,\lambda} \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{42}$$

By, comparing the coefficients of  $z^\omega$  on both sides, we get the following theorem.  $\square$

### 3. Degenerate Genocchi Polynomials of the Second Kind Attached with Dirichlet Character $\chi$

Here, we introduce degenerate Genocchi polynomials of the second kind attached with Dirichlet character  $\chi$  and establish some properties of these polynomials by applying the generating function. First, we present the following definition.

Let  $d \in \mathbb{N}$  with  $d \equiv 1(mod 2)$  and  $\chi$  be a Dirichlet character with conductor  $d$ . We define generalized degenerate Genocchi polynomials of the second kind attached to  $\chi$  given by the following generating function

$$\begin{aligned} & \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}} \\ &= \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi, \lambda}(\xi) \frac{z^\omega}{\omega!}. \end{aligned} \tag{43}$$

When  $\xi = 0$ ,  $\mathbb{G}_{\omega, \chi, \lambda} = \mathbb{G}_{\omega, \chi, \lambda}(0)$  are called the generalized degenerate Genocchi numbers of the second kind attached to  $\chi$ .

We note that

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}} \\ &= \sum_{\omega=0}^{\infty} \lim_{\lambda \rightarrow 0} \mathbb{G}_{\omega, \chi, \lambda}(\xi) \frac{z^\omega}{\omega!} \\ &= \frac{2z}{e^{dz} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{(a+\xi)z} = \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi}(\xi) \frac{z^\omega}{\omega!}. \end{aligned} \tag{44}$$

Thus, by (43) and (44), we have

$$\lim_{\lambda \rightarrow 0} \mathbb{G}_{\omega, \chi, \lambda}(\xi) = \mathbb{G}_{\omega, \chi}(\xi) \quad (\omega \geq 0).$$

**Theorem 15.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega, \chi, \lambda}(\xi) = \sum_{l=0}^{\omega} \binom{\omega}{l} \lambda^l D_l \mathbb{G}_{\omega-l, \chi, \lambda}(\xi). \tag{45}$$

**Proof.** From (43), we have

$$\begin{aligned} & \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi, \lambda}(\xi) \frac{z^\omega}{\omega!} = \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}} \\ &= \left( \frac{\log(1 + \lambda z)}{\lambda z} \right) \left( \frac{2z}{(1 + \lambda z)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}} \right) \\ &= \left( \sum_{l=0}^{\infty} D_l \frac{\lambda^l z^l}{l!} \right) \left( \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi, \lambda}(\xi) \frac{z^\omega}{\omega!} \right) \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{l=0}^{\omega} \binom{\omega}{l} \lambda^l D_l \mathbb{G}_{\omega-l, \chi, \lambda}(\xi) \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{46}$$

which proves the identity (45).  $\square$

**Theorem 16.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega, \chi, \lambda}(\xi) = d^{\omega-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathbb{G}_{\omega, \frac{\lambda}{d}}\left(\frac{a+\xi}{d}\right). \tag{47}$$

**Proof.** From (43), we observe that

$$\begin{aligned} 2 \log(1 + \lambda z)^{\frac{1}{\lambda}} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}} &= \frac{1}{d} \sum_{a=0}^{d-1} (-1)^a \chi(a) \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda} + 1}} (1 + \lambda z)^{\frac{(a+\xi)}{d}} dz \\ &= \frac{1}{d} \sum_{a=0}^{d-1} \sum_{\omega=0}^{\infty} (-1)^a \chi(a) \mathbb{G}_{\omega, \frac{\lambda}{d}}\left(\frac{a+\xi}{d}\right) \frac{(dz)^\omega}{\omega!} \\ &= \sum_{\omega=0}^{\infty} \left( d^{\omega-1} \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathbb{G}_{\omega, \frac{\lambda}{d}}\left(\frac{a+\xi}{d}\right) \right) \frac{z^\omega}{\omega!}, \end{aligned} \tag{48}$$

which complete the proof.  $\square$

**Theorem 17.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega, \chi, \lambda}\left(\frac{a+\xi}{\lambda}\right) = \sum_{l=0}^{\omega} \binom{\omega}{l} \lambda^l \int_X (\gamma)_l d\mu_0(\gamma) \int_X \chi(\eta) (\xi + \eta)_{\omega-l, \lambda} d\mu_{-1}(\eta). \tag{49}$$

**Proof.** From (4) and (43), we can derive

$$\begin{aligned} &z \int_X \int_X (1 + \lambda z)^\gamma \chi(\eta) (1 + \lambda z)^{(\xi+\eta)z} d\mu_0(\gamma) d\mu_{-1}(\eta) \\ &= \left( \frac{\log(1 + \lambda z)}{\lambda z} \right) \left( \frac{2z \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{a}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda} + 1}} (1 + \lambda z)^{\frac{\xi}{\lambda}} \right) \\ &= \frac{\log(1 + \lambda z)^{\frac{1}{\lambda}} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{a+\xi}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda} + 1}} \\ &= \sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi, \lambda}\left(\frac{a+\xi}{\lambda}\right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{50}$$

On the other hand, we have

$$\begin{aligned} &z \int_X \int_X (1 + \lambda z)^\gamma \chi(\eta) (1 + \lambda z)^{(\xi+\eta)z} d\mu_0(\gamma) d\mu_{-1}(\eta) \\ &= \sum_{\omega=0}^{\infty} \left( \sum_{l=0}^{\omega} \binom{\omega}{l} \lambda^l \int_X (\gamma)_l d\mu_0(\gamma) \int_X \chi(\eta) (\xi + \eta)_{\omega-l, \lambda} d\mu_{-1}(\eta) \right) \frac{z^\omega}{\omega!}. \end{aligned} \tag{51}$$

Therefore, by (50) and (51), we obtain (49).  $\square$

**Theorem 18.** Let  $\omega \geq 0$ . Then

$$\mathbb{G}_{\omega, \chi, \lambda}(\xi) = \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\nu, \chi, \lambda}(\xi)_{\omega-\nu, \lambda}. \tag{52}$$

**Proof.** From (43), we see that

$$\sum_{\omega=0}^{\infty} \mathbb{G}_{\omega, \chi, \lambda}(\xi) \frac{z^\omega}{\omega!} = \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda} + 1}} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{(a+\xi)}{\lambda}}$$

$$\begin{aligned}
 &= \left( \frac{2 \log(1 + \lambda z)^{\frac{1}{\lambda}}}{(1 + \lambda z)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1 + \lambda z)^{\frac{a}{\lambda}} \right) (1 + \lambda z)^{\frac{\xi}{\lambda}} \\
 &= \left( \sum_{\nu=0}^{\infty} \mathbb{G}_{\nu, \chi, \lambda} \frac{z^\nu}{\nu!} \right) \left( \sum_{\omega=0}^{\infty} (\xi)_{\omega, \lambda} \frac{z^\omega}{\omega!} \right) \\
 &= \sum_{\omega=0}^{\infty} \left( \sum_{\nu=0}^{\omega} \binom{\omega}{\nu} \mathbb{G}_{\nu, \chi, \lambda} (\xi)_{\omega-\nu, \lambda} \right) \frac{z^\omega}{\omega!}. \tag{53}
 \end{aligned}$$

Equating the coefficients of  $z$ , we get (51).  $\square$

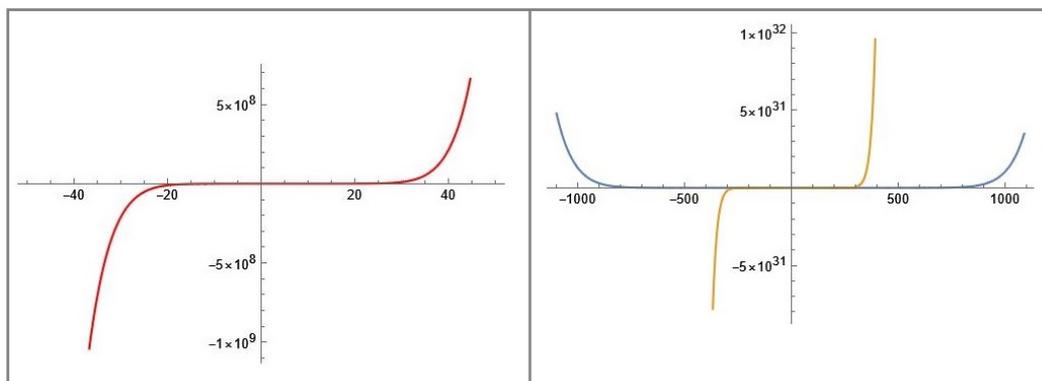
#### 4. Computational Values and Graphical Representations of Degenerate Genocchi Polynomials of the Second Kind

In this section, certain zeros of the degenerate Genocchi polynomials of the second kind  $\mathbb{G}_{\omega, \lambda}(\xi)$  and beautiful graphical representations are shown.

For  $\lambda \neq 0$ , the first five degenerate Genocchi polynomials of the second kind are:

$$\begin{aligned}
 \mathbb{G}_{0, \lambda}(\xi) &= 0, & \mathbb{G}_{1, \lambda}(\xi) &= 1, & \mathbb{G}_{2, \lambda}(\xi) &= \frac{1}{2}(2\xi - \lambda - 1), \\
 \mathbb{G}_{3, \lambda}(\xi) &= \frac{1}{6} (3\xi^2 - 3\xi - 6\lambda\xi + 2\lambda^2 + 3\lambda), \\
 \mathbb{G}_{4, \lambda}(\xi) &= \frac{1}{24} (1 - 6\lambda^3 - 18\lambda(\xi - 1)\xi - 6\xi^2 + 4\xi^3 + 11\lambda^2(2\xi - 1)).
 \end{aligned}$$

For instance, Figure 1 shows the plots of some degenerate Genocchi polynomials of the second kind.



**Figure 1.** Graphs of degenerate Genocchi polynomials of the second kind for  $\lambda = 1$ ,  $\omega = 10$  (red),  $\omega = 15$  (blue), and  $\omega = 20$  (orange).

Further, we calculate an approximate solution satisfying the degenerate Genocchi polynomials of the second kind  $\mathbb{G}_{\omega, \lambda}(\xi) = 0$ , for  $\lambda = \pm 1$ . The results are displayed in Tables 1 and 2.

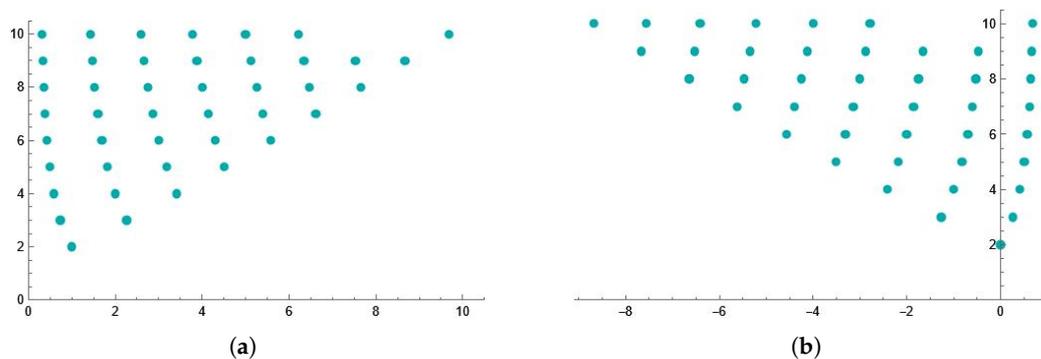
**Table 1.** Approximate solutions of  $\mathbb{G}_{\omega,1}(\zeta) = 0$ .

$\omega$	$\zeta$
2	1
3	0.736237, 2.26376
4	0.585786, 2, 3.41421
5	0.49031, 1.82092, 3.17908, 4.50969
6	0.42671, 1.69175, 3, 4.30825, 5.57329
7	0.383053, 1.59643, 2.85886, 4.14114, 5.40357, 6.61695
8	0.352346, 1.5254, 2.74661, 4, 5.25339, 6.4746, 7.64765
9	0.330213, 1.47227, 2.65719, 3.88103, 5.11897, 6.34281, 7.52773, 8.66979
10	0.313827, 1.43242, 2.58614, 3.7812, 5, 6.2188, 7.41386, 8.56758, 9.68617

**Table 2.** Approximate solutions of  $\mathbb{G}_{\omega,-1}(\zeta) = 0$ .

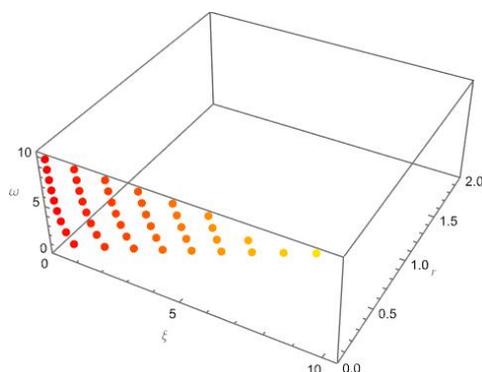
$\omega$	$\zeta$
2	0
3	-1.26376, 0.263763
4	2.41421, -1, 0.414214
5	-3.50969, -2.17908, -0.820923, 0.50969
6	-4.57329, -3.30825, -2, -0.691752, 0.57329
7	-5.61695, -4.40357, -3.14114, -1.85886, -0.596427, 0.616947
8	-6.64765, -5.4746, -4.25339, -3, -1.74661, -0.525404, 0.647654
9	-7.66979, -6.52773, -5.34281, -4.11897, -2.88103, -1.65719, -0.472272, 0.669787
10	-8.68617, -7.56758, -6.41386, -5.2188, -4, -2.7812, -1.58614, -0.432424, 0.686173

The plots of real zeros of  $\mathbb{G}_{\omega,\lambda}(\zeta)$ , for  $\lambda = \pm 1$  and  $\omega = 2, \dots, 10$  are presented in Figure 2.

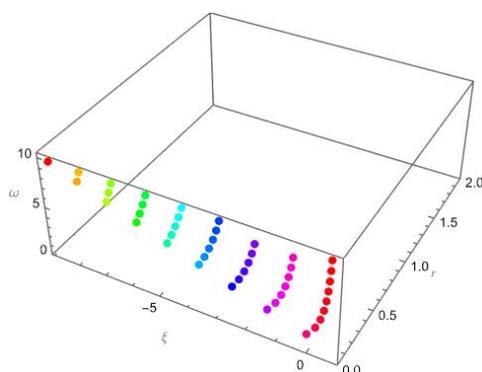


**Figure 2.** Plots of real zeros of  $\mathbb{G}_{\omega,\pm 1}(\zeta)$ , for  $\omega = 2, \dots, 10$ . (a) Plots of real zeros of  $\mathbb{G}_{\omega,1}(\zeta)$ , for  $\omega = 2, \dots, 10$ . (b) Plots of real zeros of  $\mathbb{G}_{\omega,-1}(\zeta)$ , for  $\omega = 2, \dots, 10$ .

The stacks of real zeros of  $\mathbb{G}_{\omega,\lambda}(\xi)$ , for  $\lambda = \pm 1$  and  $\omega = 2, \dots, 10$  are presented in Figures 3 and 4, respectively.



**Figure 3.** Stack of real zeros of degenerate Genocchi polynomials of the second kind  $\mathbb{G}_{\omega,1}(\xi)$ , for  $\omega = 2, \dots, 10$ .



**Figure 4.** Stack of real zeros of degenerate Genocchi polynomials of the second kind  $\mathbb{G}_{\omega,-1}(\xi)$ , for  $\omega = 2, \dots, 10$ .

## 5. Conclusions

Motivated by [5,13], in this paper, we defined degenerate Genocchi polynomials of the second kind, which turn out to be classical ones in exceptional cases. We have also derived their explicit expressions and some identities involving them. Later, we introduced the higher-order degenerate Genocchi polynomials of the second kind and deduced their explicit expressions and some identities by using the generating functions method, analytical means, and power series expansions. Additionally, we introduced degenerate Genocchi polynomials of the second kind attached to Dirichlet character  $\chi$  and obtained some properties of these polynomials.

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