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On Some Important Dynamic Inequalities of Hardy–Hilbert-Type on Timescales

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Abstract: In this article, by using some algebraic inequalities, nabla Hölder inequalities, and nabla Jensen's inequalities on timescales, we proved some new nabla Hilbert-type dynamic inequalities on timescales. These inequalities extend some known dynamic inequalities on timescales and unify some continuous inequalities and their corresponding discrete analogues. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Hilbert's inequality; dynamic inequality; timescale; nabla calculus

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1. Introduction

The form of the established classical discrete Hardy–Hilbert double series inequality [1] is given as follows: If $\{a_m\} \geqslant 0$, $\{b_n\} \geqslant 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} b_m^q\right)^{\frac{1}{q}}, \tag{1}$$

where $p > 1$, $q = p/p-1$.

The continuous versions of inequality (1) is given by:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^{p'}(x)dx\right)^{\frac{1}{q}}, \tag{2}$$

unless $f \equiv 0$ or $g \equiv 0$, where f and g are measurable non-negative functions such that $\int_0^{\infty} f^p(x)dx < \infty$ and $\int_0^{\infty} g^p(x)dx < \infty$. The constant $\frac{\pi}{\sin\frac{\pi}{p}}$, in (1) and (2), is the best possible.

In [2], Pachpatte proved that if $f \in C^1[[0,x],\mathbb{R}^+]$, $g \in C^1[[0,y],\mathbb{R}^+]$ with $f(0) = g(0) = 0$ and p, q are two positive functions defined for $t \in [0,x)$ and $\tau \in [0,y)$, with $P(t) = \int_0^t p(\tau)d\tau$ and $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in [0,x)$ and $t \in [0,y)$ where x, y are positive real numbers. Let Φ and Ψ be two real-valued non-negative, convex, and sub-multiplicative functions defined on $[0,\infty)$. Then

$$\int_0^x\int_0^y \frac{\Phi(f(s))\Psi(g(t))}{s+t}dsdt \leqslant L(x,y)\left(\int_0^x (x-s)\left(p(s)\Phi\left(\frac{f'(s)}{p(s)}\right)^2 ds\right)^{\frac{1}{2}} \times \left(\int_0^y (y-t)\left(q(t)\Psi\left(\frac{g'(t)}{q(t)}\right)^2 dt\right)^{\frac{1}{2}}, \tag{3}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^y\left(\frac{\Psi(Q(t))}{Q(t)}\right)^2 dt\right)^{\frac{1}{2}}.$$

In [3], Handley et al. proved that if $\{a_{\ell,m_\ell}\}$ $(\ell=1,2,\ldots,n)$ are n sequences of non-negative real numbers defined for $m_\ell=1,\ldots,k_\ell$ with $a_{1,0}=a_{2,0}\ldots a_{n,0}=0$. Let $\{p_{\ell,m_\ell}\}$ be n sequences of positive real numbers defined for $m_\ell=1,\ldots,k_\ell$, where k_ℓ are natural numbers. Set $P_{\ell,m_\ell}=\sum_{s_\ell}^{m_\ell} p_{\ell,s_\ell}$. Let Φ_ℓ $(\ell=1,2,\ldots,n)$ be n real valued non-negative convex and sub-multiplicative functions defined on $(0,\infty)$. Let $\alpha_\ell\in(0.1)$, and set $\alpha'_\ell=1-\alpha$, $\alpha=\sum_{\ell=1}^n\alpha_\ell$, and $\alpha'_\ell=n-\alpha$. Then

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{\ell=1}^n\Phi_\ell(a_{\ell,m_\ell})}{\left(\sum_{\ell=1}^n\alpha'_\ell m_\ell\right)^{\alpha'}} \leqslant M(k_1,\ldots,k_n)\prod_{\ell=1}^n\left(\sum_{m_\ell=1}^{k_\ell}(k_\ell-m_\ell+1)\left(p_{\ell,m_\ell}\Phi_\ell\left(\frac{\nabla a_{\ell,m_\ell}}{p_{\ell,m_\ell}}\right)^{\frac{1}{\alpha_\ell}}\right)^{\alpha_\ell} \tag{4}$$

where

$$M(k_1,\ldots,k_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{\ell=1}^n\left(\sum_{m_\ell=1}^{k_\ell}\left(\frac{\Phi_\ell(P_{\ell,m_\ell})}{P_{\ell,m_\ell}}\right)^{\frac{1}{\alpha'_\ell}}\right)^{\alpha'_\ell}.$$

Moreover, in the same paper [3], the authors proved that if $f_\ell\in C^1([0,k_\ell],\mathbb{R}_+])$ $\ell=1,\ldots,n$, with $f_\ell(0)=0$. Let $p_\ell(\xi_\ell)$ be n positive functions defined for $\xi_\ell\in[0,x_\ell]$ $(\ell=1,\ldots,n)$. Set $P_\ell(s_\ell)=\int_0^{s_\ell}p_\ell(\xi_\ell)d\xi_\ell$ for $s_\ell\in[0,x_\ell]$, where x_ℓ are positive real numbers. Let Φ_ℓ, α_ℓ, α'_ℓ, α, and α' be as in Equation (4). Then

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{\ell=1}^n\Phi_\ell(f(s_\ell))}{\left(\sum_{\ell=1}^n\alpha'_\ell s_\ell\right)^{\alpha'}}ds_n\ldots ds_1 \leqslant L(x_1,\ldots,x_n)\prod_{\ell=1}^n\left(\int_0^{x_\ell}(x_\ell-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{f'(s_\ell)}{p(s_\ell)}\right)^{\frac{1}{\alpha_\ell}}ds_\ell\right)^{\alpha_\ell}, \tag{5}$$

where

$$L(x_1,\ldots,x_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{\ell=1}^n\left(\int_0^{x_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\alpha'_\ell}}ds_\ell\right)^{\alpha'_\ell}.$$

In [4], Pachpatte established the following Hilbert-type integral inequalities under the conditions: If $h\geqslant 1$, $l\geqslant 1$, and $f(t)\geqslant 0$, $g(t)\geqslant 0$, for $t\in(0,x)$ and $\tau\in(0,y)$, where x and y are positive real numbers and define $F(s)=\int_0^s f(t)dt$ and $G(t)=\int_0^t g(\tau)d\tau$, for $s\in(0,x)$ and $t\in(0,y)$. Let P, Q, Φ, Ψ and α' are defined as in (3), then

$$\int_0^x\int_0^y\frac{F^h(s)G^l(t)}{s+t}dsdt \leqslant \frac{1}{2}hl(xy)^{\frac{1}{2}}\left(\int_0^x(x-s)\left(F^{h-1}(s)f(s)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^y(y-t)\left(G^{l-1}g(t)\right)^2 dt\right)^{\frac{1}{2}}, \tag{6}$$

and

$$\int_0^x\int_0^y \frac{\Phi(F(s))\Psi(G(t))}{s+t}dsdt \leqslant L(x,y)\left(\int_0^x(x-s)\left(p(s)\Phi\left(\frac{f(s)}{p(s)}\right)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^y(y-t)\left(q(t)\Psi\left(\frac{g(t)}{q(t)}\right)\right)^2 dt\right)^{\frac{1}{2}} \tag{7}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\Phi(P(s))}{P(s)}\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^y\left(\frac{\Psi(Q(t))}{Q(t)}\right)^2 dt\right)^{\frac{1}{2}},$$

and

$$\int_0^x\int_0^y \frac{P(s)Q(t)\Phi(F(s))\Psi(G(t))}{s+t}dsdt \leqslant \frac{1}{2}(xy)^{\frac{1}{2}}\left(\int_0^x(x-s)\left(p(s)\Phi\left(f(s)\right)\right)^2 ds\right)^{\frac{1}{2}} \times\left(\int_0^y(y-t)\left(q(t)\Psi\left(g(t)\right)\right)^2 dt\right)^{\frac{1}{2}}. \tag{8}$$

and

$$\begin{aligned}&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{\ell=1}^n \Phi_\ell(F_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n \frac{1}{\alpha_\ell}(s_\ell)(t_\ell)\right)^{\frac{1}{\alpha}}}ds_n dt_n\ldots ds_1 dt_1\\ &\geqslant L(x_1y_1,\ldots,x_ny_n)\\ &\times\prod_{\ell=1}^n\left(\int_0^{x_\ell}\int_0^{y_\ell}(x_\ell-s_\ell)(y_\ell-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell\left(\frac{f_\ell(s_\ell,t_\ell)}{p_\ell(s_\ell,t_\ell)}\right)\right)^{\beta_\ell}ds_\ell dt_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned} \tag{9}$$

where

$$L(x_1y_1,\ldots,x_ny_n)=\prod_{\ell=1}^n\left(\int_0^{x_\ell}\int_0^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\right)^{\alpha_\ell}ds_\ell dt_\ell\right)^{\frac{1}{\alpha_\ell}}.$$

Over the past decade, a great number of dynamic Hilbert-type inequalities on timescales has been established by many researchers who were motivated by some applications, see the papers [5–17].

A timescale $\mathbb{T}$ is an arbitrary, nonempty closed subset of the set of real numbers $\mathbb{R}$. Throughout the article, we assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ for any $t\in\mathbb{T}$ by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\},$$

and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ for any $t\in\mathbb{T}$ by

$$\rho(t):=\sup\{s\in\mathbb{T}:s<t\}.$$

In the preceding two definitions, we set $\inf\emptyset=\sup\mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t)=t$) and $\sup\emptyset=\inf\mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t)=t$), where $\emptyset$ denotes the empty set. For more details on time scales calculus see [11].

A point $t\in\mathbb{T}$ with $\inf\mathbb{T}<t<\sup\mathbb{T}$ is said to be right-scattered if $\sigma(t)>t$, right-dense if $\sigma(t)=t$, left-scattered if $\rho(t)<t$, and left-dense if $\rho(t)=t$. Points that are simultaneously right-dense and left-dense are said to be dense points, whereas points that are simultaneously right-scattered and left-scattered are said to be isolated points.

The forward graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined for any $t\in\mathbb{T}$ by $\mu(t):=\sigma(t)-t$.

If $F:\mathbb{T}\to\mathbb{R}$ is a function, then the function $F^\sigma:\mathbb{T}\to\mathbb{R}$ is defined by $F^\sigma(t)=F(\sigma(t)),\forall t\in\mathbb{T}$, that is $F^\sigma=F\circ\sigma$. Similarly, the function $F^\rho:\mathbb{T}\to\mathbb{R}$ is defined by $F^\rho(t)=g(\rho(t)),\forall t\in\mathbb{T}$, that is $F^\rho=F\circ\rho$.

The interval $[a,b]$ in $\mathbb{T}$ is defined by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

We define open intervals and half-closed intervals similarly.

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) if F is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

In a similar manner, a function $F : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous) if F is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist at all right-dense points in $\mathbb{T}$.

The delta integration by parts on timescales is given by the following formula:

$$\int_a^b g^{\Delta}(t)F(t)\Delta t = g(b)F(b) - g(a)F(a) - \int_a^b g^{\sigma}(t)F^{\Delta}(t)\Delta t, \tag{10}$$

whereas the nabla integration by parts on timescales is given by

$$\int_a^b g^{\nabla}(t)F(t)\nabla t = g(b)F(b) - g(a)F(a) - \int_a^b g^{\rho}(t)F^{\nabla}(t)\nabla t. \tag{11}$$

The following relationships will be used.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered} \sigma(t) = \rho(t) = t, \quad \mu(t) = \nu(t) = 0, \quad F^{\Delta}(t) = F^{\nabla}(t) = F'(t), \\ \int_a^b F(t)\Delta t = \int_a^b F(t)\nabla t = \int_a^b F(t)dt. \end{gathered} \tag{12}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered} \sigma(t) = t+1, \quad \rho(t) = t-1, \quad \mu(t) = \nu(t) = 1, \\ F^{\Delta}(t) = \Delta F(t), \quad F^{\nabla}(t) = \nabla F(t), \\ \int_a^b F(t)\Delta t = \sum_{t=a}^{b-1} F(t), \quad \int_a^b F(t)\nabla t = \sum_{t=a+1}^{b} F(t), \end{gathered} \tag{13}$$

where Δ and ∇ are the forward and backward difference operators, respectively.

Next, we write Hölder's inequality and Jensen's inequality on timescales, where CC_{ld} denotes the set of all ld-continuous functions $F(x,y)$ in x and y, and CC_{ld}^1 is the set of all functions in CC_{ld} for which both the first partial derivative ∇_1 and ∇_2 exist in CC_{ld}. Similarly we can define CC_{ld}^2.

Lemma 1 (Dynamic Hölder's Inequality [7])**.** *Suppose u, $v \in \mathbb{T}$ with $u < v$. Assume F, $g \in CC_{ld}^1([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ then*

$$\begin{aligned} \int_u^v \int_u^v |F(r,t)g(r,t)|\nabla r \nabla t \quad &\leq \left[\int_u^v \int_u^v |F(r,t)|^p \nabla r \nabla t \right]^{\frac{1}{p}} \\ &\times \left[\int_u^v \int_u^v |g(r,t)|^q \nabla r \nabla t \right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2 (Dynamic Jensen's inequality [7])**.** *Let $r,\ t \in R$ and $-\infty \leqslant m,n \leqslant \infty$. If $F \in CC^1_{ld}(\mathbb{R},(m,n))$ and $\Phi : (m,n) \longrightarrow \mathbb{R}$ is convex then*

$$\phi\left(\frac{\int_u^v \int_\omega^s F(r,t)\nabla r\nabla t}{\int_u^v \int_\omega^s \nabla r\nabla t}\right) \leqslant \frac{\int_u^v \int_\omega^s \phi(F(r,t))\nabla r\nabla t}{\int_u^v \int_\omega^s \nabla r\nabla t}. \tag{15}$$

This inequality is reversed if $\phi \in C_{ld}\big((c,d),\mathbb{R}\big)$ is concave.

Definition 1. *Φ is called a super-multiplicative function on $[0,\infty)$ if*

$$\Phi(xy) \geqslant \Phi(x)\Phi(y),\ \text{for all}\ \ x,y \geqslant 0. \tag{16}$$

In this paper, we prove some extensions of the nabla integral Hardy–Hilbert inequality to a general timescale. As special cases of our results, we will recover some dynamic integral and discrete inequalities known in the literature. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

2. Main Results

In this section, we state and prove the main results that extend several results in the literature.

Theorem 1. *Let $\mathbb{T}$ be timescales with $t_0, x_\ell, y_\ell, s_\ell, t_\ell \in \mathbb{T}$, $(\ell = 1,\ldots,n)$. Assume $\omega_\ell(s_\ell,t_\ell) \in C^2_{ld}\big([t_0,x_\ell)_\mathbb{T} \times [t_0,y_\ell)_\mathbb{T},[0,\infty)\big)$ $(\ell = 1,\ldots,n)$, where x_ℓ and y_ℓ are positive real numbers. Define $\Omega_\ell(s_\ell,t_\ell) = \int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} \omega_\ell(\xi_\ell,\tau_\ell)\nabla\xi_\ell\nabla\tau_\ell$. Let $p_\ell(\xi_\ell,\tau_\ell)$ be n positive, left-dense continuous functions defined for $\xi_\ell \in (t_0,s_\ell)_\mathbb{T}$, $\tau_\ell \in (t_0,t_\ell)_\mathbb{T}$ and define $P_\ell(s_\ell,t_\ell) = \int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(\xi_\ell,\tau_\ell)\nabla\xi_\ell\nabla\tau_\ell$. Let Φ_ℓ $(\ell = 1,\ldots,n)$ be n real-valued, non-negative concave and super-multiplicative functions defined on $(0,\infty)$. Then for $s_\ell \in [t_0,x_\ell)_\mathbb{T}$, $t_\ell \in [t_0,y_\ell)_\mathbb{T}$, $\frac{1}{\alpha_\ell}+\frac{1}{\beta_\ell} = 1$, $0 < \beta_\ell < 1$, and $\sum_{\ell=1}^n \frac{1}{\alpha_\ell} = \frac{1}{\alpha}$ we have that*

$$\begin{aligned}
&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n} \frac{\prod_{\ell=1}^n \Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n \frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\
&\geqslant L(x_1y_1,\ldots,x_ny_n)\\
&\times\prod_{\ell=1}^n\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell)-s_\ell)(\rho(y_\ell)-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell\left(\frac{\omega_\ell(s_\ell,t_\ell)}{p_\ell(s_\ell,t_\ell)}\right)\right)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}
\end{aligned}\tag{17}$$

where

$$L(x_1y_1,\ldots,x_ny_n) = \prod_{\ell=1}^n\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\right)^{\alpha_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\alpha_\ell}}.$$

Proof. From the hypotheses of Theorem 1, it is easy to observe that

$$\begin{aligned}
\Phi_\ell(\Omega_\ell(s_\ell,t_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell,t_\ell)\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell}{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\nabla t_\ell\nabla\tau_\ell}\right)\\
&\geqslant \Phi_\ell(P_\ell(s_\ell,t_\ell))\Phi_\ell\left(\frac{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell}{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\nabla t_\ell\nabla\tau_\ell}\right).
\end{aligned}\tag{18}$$

By using inverse Jensen dynamic inequality, we obtain that

$$\Phi_\ell(\Omega_\ell(s_\ell,t_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell. \tag{19}$$

Applying the inverse Hölder's inequality on the left-hand side of (19) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\Omega_\ell(s_\ell,t_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left[(s_\ell-t_0)(t_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \quad (20)$$

By using the following inequality on the term $\left[(s_\ell-t_0)(t_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}$,

$$\prod_{\ell=1}^{n} m_\ell^{\frac{1}{\alpha_\ell}} \geqslant \left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}m_\ell\right)^{\frac{1}{\alpha}}, \quad (21)$$

we obtain that

$$\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}} \geqslant \prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \quad (22)$$

Integrating both sides of (22) over s_ℓ, t_ℓ from t_0 to x_ℓ, y_ℓ ($\ell=1,\dots,n$), we obtain that

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1\\ &\geqslant \prod_{\ell=1}^{n}\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}}\nabla s_\ell\nabla t_\ell.\end{aligned} \quad (23)$$

Applying the inverse Hölder's inequality on the left-hand side of (23) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1\\ &\geqslant \prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\right)^{\alpha_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\alpha_\ell}}\\ &\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell(t_\ell,\tau_\ell)}{p_\ell(t_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned} \quad (24)$$

Using Fubini's theorem, we observe that

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1\\ &\geqslant L(x_1y_1,\dots,x_ny_n)\\ &\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(x_\ell-s_\ell)(y_\ell-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell\left(\frac{\omega_\ell(s_\ell,t_\ell)}{p_\ell(s_\ell,t_\ell)}\right)\right)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}$$

By using the fact $x_\ell \geqslant \rho(x_\ell)$, and $y_\ell \geqslant \rho(y_\ell)$, we obtain that

$$\begin{aligned}
&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\
&\geqslant L(x_1y_1,\ldots,x_ny_n)\\
&\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell)-s_\ell)(\rho(y_\ell)-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell\left(\frac{\omega_\ell(s_\ell,t_\ell)}{p_\ell(s_\ell,t_\ell)}\right)\right)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}.
\end{aligned}$$

This completes the proof. □

Remark 1. *In Theorem 1, if $\mathbb{T}=\mathbb{R}$, we obtain the result due to Zhao et al.'s [9], Theorem 2.*

As a special case of Theorem 1, when $\mathbb{T}=\mathbb{Z}$, we have $\rho(n)=n-1$, we obtain the following result.

Corollary 1. *Let $\{a_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\}$ and $\{p_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\}$, $(\ell=1,\ldots,n)$ be n sequences of non-negative numbers defined for $m_{s_\ell}=1,\ldots,k_{s_\ell}$, and $m_{t_\ell}=1,\ldots,k_{t_\ell}$, and define*

$$\begin{aligned}
A_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}} &= \textstyle\sum_{m_{t_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{t_\ell}} a_{s_\ell,t_\ell,m_{t_\ell},m_{\eta_\ell}}\\
P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}} &= \textstyle\sum_{m_{t_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{t_\ell}} p_{s_\ell,t_\ell,m_{t_\ell},m_{\eta_\ell}}.
\end{aligned}\tag{25}$$

Then

$$\begin{aligned}
&\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{t_1}}^{k_{t_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{t_n}}^{k_{t_n}}\frac{\prod_{\ell=1}^{n}\Phi_\ell(A_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}})}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(m_{s_\ell}m_{t_\ell})\right)^{\frac{1}{\alpha}}}\\
&\geqslant C(k_{s_1}k_{t_1},\ldots,k_{s_n}k_{t_n})\\
&\times\prod_{\ell=1}^{n}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{t_\ell}}^{k_{t_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{t_\ell}-(m_{t_\ell}-1))\left(P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\Phi_\ell\left(\frac{a_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}}{P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}}\right)\right)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}}
\end{aligned}$$

where

$$C(k_{s_1}k_{t_1},\ldots,k_{s_n}k_{t_n})=\prod_{\ell=1}^{n}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{t_\ell}}^{k_{t_\ell}}\left(\frac{\Phi_\ell(P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}})}{P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}}\right)\right)^{\beta_\ell}\Bigg)^{\frac{1}{\beta_\ell}}.$$

Remark 2. *Let $\omega_\ell(t_\ell,\tau_\ell)$, $p_\ell(t_\ell,\tau_\ell)$, $P_\ell(t_\ell,\tau_\ell)$, and $\Omega_\ell(t_\ell,\tau_\ell)$ change to $\omega_\ell(t_\ell)$, $p_\ell(t_\ell)$, $P_\ell(s_\ell)$ and $\Omega_\ell(s_\ell)$, respectively, and with suitable changes, we have the following new corollary:*

Corollary 2. *Let $\omega_\ell(\xi_\ell)\in C^1_{ld}[t_0,x_\ell]_{\mathbb{T}}$, $(\ell=1,\ldots,n)$, where x_ℓ positive real number, and define $\Omega_\ell(s_\ell)=\int_{t_0}^{s_\ell}\omega_\ell(\xi_\ell)\nabla\xi_\ell$, then for $s_\ell\in[t_0,x_\ell)_{\mathbb{T}}$, $\frac{1}{\alpha_\ell}+\frac{1}{\beta_\ell}=1$, $0<\beta_\ell<1$, and $\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}=\frac{1}{\alpha}$. Let $p_\ell(\xi_\ell)$ be n positive functions defined for $\xi_\ell\in(t_0,x_\ell)_{\mathbb{T}}$ $(\ell=1,\ldots,n)$ and define $P_\ell(s_\ell)=\int_{t_0}^{s_\ell}p_\ell(\xi_\ell)\nabla\xi_\ell$, where x_ℓ are positive real number and let Φ_ℓ be n real-valued non-negative, concave, and super-multiplicative function defined on $(0,\infty)$. Then*

$$\begin{aligned}
&\textstyle\int_{t_0}^{x_1}\cdots\int_{t_0}^{x_n}\dfrac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell)}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\ldots\nabla s_1\\
&\geqslant L^*(x_1,\ldots,x_n)\textstyle\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}(\rho(x_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\omega_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\beta_\ell}\nabla s_\ell\right)^{\frac{1}{\beta_\ell}},
\end{aligned}\tag{26}$$

where

$$L^{*}(x_1,\ldots,x_n)=\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\alpha_\ell}\nabla s_\ell\right)^{\frac{1}{\alpha_\ell}}.$$

Corollary 3. *In Corollary 2, if we take $n=2$, $\beta_\ell=\frac{1}{2}$ then the inequality (26) changes to*

$$\begin{aligned}\int_{t_0}^{x_1}\int_{t_0}^{x_2}\frac{\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{((s_1-t_0)+(s_2-t_0))^{-2}}\nabla s_1\nabla s_2 \quad &\geqslant L^{**}(x_1,x_2)\left(\int_{t_0}^{x_1}(\rho(x_1)-s_1)\left(p_1(s_1)\Phi\left(\frac{\omega_1(s_1)}{p_1(s_1)}\right)\right)^2\nabla s_1\right)^{\frac{1}{2}}\\ &\times\left(\int_{t_0}^{x_2}(\rho(x_2)-s_2)\left(p_2(s_2)\Psi\left(\frac{\omega_2(s_2)}{p_2(s_2)}\right)\right)^2\nabla s_2\right)^{\frac{1}{2}}\end{aligned}\tag{27}$$

where

$$L^{**}(x_1,x_2)=4\left(\int_{t_0}^{x_1}\left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}\nabla s_1\right)^{-1}\left(\int_{t_0}^{x_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}\nabla s_2\right)^{-1}$$

Remark 3. *In Corollary 3, if we take $\mathbb{T}=\mathbb{R}$, then the inequality (27) changes to*

$$\begin{aligned}\int_{0}^{x_1}\int_{0}^{x_2}\frac{\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{(s_1+s_2)^{-2}}ds_1 ds_2 \quad &\geqslant L^{**}(x_1,x_2)\left(\int_{0}^{x_1}(x_1-s_1)\left(p_1(s_1)\Phi\left(\frac{\omega_1(s_1)}{p_1(s_1)}\right)\right)^2 ds_1\right)^{\frac{1}{2}}\\ &\times\left(\int_{0}^{x_2}(x_2-s_2)\left(p_2(s_2)\Psi\left(\frac{\omega_2(s_2)}{p_2(s_2)}\right)\right)^2 ds_2\right)^{\frac{1}{2}}\end{aligned}\tag{28}$$

where

$$L^{**}(x_1,x_2)=4\left(\int_{0}^{x_1}\left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_{0}^{x_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}$$

This is an inverse of the inequality (7) which was proved by Pachpatte [4].

Corollary 4. *In Corollary 2, if we take $\beta_\ell=\frac{n-1}{n}$ the inequality (26) becomes*

$$\begin{aligned}&\int_{t_0}^{x_1}\cdots\int_{t_0}^{x_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Omega_\ell(s_\ell)}{\left(\sum_{\ell=1}^{n}(s_\ell-t_0)\right)^{\frac{-n}{n-1}}}\nabla s_n\ldots\nabla s_1\\ &\geqslant L^{*}(x_1,\ldots,x_n)\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}(\rho(x_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\omega_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{n-1}{n}}\nabla s_\ell\right)^{\frac{n}{n-1}}\end{aligned}$$

where

$$L^{*}(x_1,\ldots,x_n)=n^{\frac{n}{n-1}}\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{-(n-1)}\nabla s_\ell\right)^{\frac{-1}{n-1}}.$$

Theorem 2. *Let $\mathbb{T}$ be timescales with t_0, x_ℓ, y_ℓ, s_ℓ, $t_\ell\in\mathbb{T}$, $(\ell=1,\ldots,n)$. Let $\omega_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$, α_ℓ, and β_ℓ be as Theorem 1 and define $\Omega_\ell(s_\ell,t_\ell)=\frac{1}{P_\ell(\xi_\ell,\tau_\ell)}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(\xi_\ell,\tau_\ell)\omega_\ell(\xi_\ell,\tau_\ell)\nabla\xi_\ell\nabla\tau_\ell$ for $\xi_\ell,s_\ell\in(t_0,x_\ell)_{\mathbb{T}}$, $\tau_\ell,t_\ell\in(t_0,y_\ell)_{\mathbb{T}}$, where x_ℓ and y_ℓ are positive real numbers. Let Φ_ℓ $(\ell=1,\ldots,n)$ be n real-valued non-negative concave and super-multiplicative functions defined on $(0,\infty)$. Then*

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\ &\geqslant\prod_{\ell=1}^{n}\Big[(x_\ell-t_0)(y_\ell-t_0)\Big]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell)-s_\ell)(\rho(y_\ell)-t_\ell)\big(p_\ell(s_\ell,t_\ell)\Phi_\ell(\omega_\ell(s_\ell,t_\ell))\big)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}\end{aligned}\tag{29}$$

Proof. From the hypotheses of Theorem 2, and by using inverse Jensen dynamic inequality, we have

$$\begin{aligned} \Phi_\ell(\omega_\ell(s_\ell,t_\ell)) &= \Phi_\ell\left(\frac{1}{P_\ell(s_\ell,t_\ell)}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(t_\ell,\tau_\ell)\Omega_\ell(t_\ell,\tau_\ell)\nabla t_\ell\nabla\tau_\ell\right) \\ &\geqslant \frac{1}{P_\ell(s_\ell,t_\ell)}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell} p_\ell(\sigma_\ell,\tau_\ell)\Phi_\ell(\omega_\ell(t_\ell,\tau_\ell))\nabla t_\ell\nabla\tau_\ell. \end{aligned} \tag{30}$$

Applying the inverse Hölder's inequality on the left-hand side of (30) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\Omega_\ell(s_\ell,t_\ell)) \geqslant \frac{1}{P_\ell(s_\ell,t_\ell)}\left[(s_\ell-t_0)(t_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell(\omega_\ell(t_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the inequality (21), on the term $\left[(s_\ell-t_0)(t_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}$ we obtain that

$$\frac{P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}} \geqslant \left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell(\omega_\ell(t_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}} \tag{31}$$

Integrating both sides of (31) over s_ℓ, t_ℓ from t_0 to x_ℓ, y_ℓ $(\ell=1,\dots,n)$, we obtain that

$$\begin{aligned} &\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1 \\ &\geqslant \prod_{\ell=1}^n\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell(\omega_\ell(t_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla\sigma_\ell\nabla\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \end{aligned}$$

Applying the inverse Hölder's inequality on the left-hand side of (32) with indices α_ℓ and β_ℓ, it is easy to observe that

$$\begin{aligned} &\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1 \\ &\geqslant \prod_{\ell=1}^n\left[(x_\ell-t_0)(y_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell,\tau_\ell)\Phi_\ell(\omega_\ell(t_\ell,\tau_\ell))\right)^{\beta_\ell}\nabla t_\ell\nabla\tau_\ell\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}. \end{aligned} \tag{32}$$

By using Fubini's theorem, we observe that

$$\begin{aligned} &\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^n\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\dots\nabla s_1\nabla t_1 \\ &\geqslant \prod_{\ell=1}^n\left[(x_\ell-t_0)(y_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(x_\ell-s_\ell)(y_\ell-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell(\omega_\ell(s_\ell,t_\ell))\right)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}. \end{aligned}$$

By using the fact $x_\ell \geqslant \rho(x_\ell)$, and $y_\ell \geqslant \rho(y_\ell)$, we obtain that

$$\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell,t_\ell)\Phi_\ell(\Omega_\ell(s_\ell,t_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)(t_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1$$
$$\geqslant\prod_{\ell=1}^{n}\left[(x_\ell-t_0)(y_\ell-t_0)\right]^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell)-s_\ell)(\rho(y_\ell)-t_\ell)\left(p_\ell(s_\ell,t_\ell)\Phi_\ell(\omega_\ell(s_\ell,t_\ell))\right)^{\beta_\ell}\nabla s_\ell\nabla t_\ell\right)^{\frac{1}{\beta_\ell}}$$

This completes the proof. □

Remark 4. *In Theorem 2, if $\mathbb{T}=\mathbb{R}$, we obtain the result due to Zhao et al. [9], Theorem 3.*

As a special case of Theorem 2, when $\mathbb{T}=\mathbb{Z}$, we have $\rho(n)=n-1$, we obtain the following result.

Corollary 5. *Let $\{a_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\}$ and $\{p_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\}$, $(\ell=1,\ldots,n)$ be n sequences of non-negative numbers defined for $m_{s_\ell}=1,\ldots,k_{s_\ell}$, and $m_{t_\ell}=1,\ldots,k_{t_\ell}$, and define*

$$\begin{aligned}A_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}} &= \frac{1}{P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}}\sum_{m_{t_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{t_\ell}}a_{s_\ell,t_\ell,m_{t_\ell},m_{\eta_\ell}}p_{s_\ell,t_\ell,m_{t_\ell},m_{\eta_\ell}},\\ P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}} &= \sum_{m_{t_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{t_\ell}}p_{s_\ell,t_\ell,m_{t_\ell},m_{\eta_\ell}}.\end{aligned}\tag{33}$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{t_1}}^{k_{t_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{t_n}}^{k_{t_n}}\frac{\prod_{\ell=1}^{n}P_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\Phi_\ell(A_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}})}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(m_{s_\ell}m_{t_\ell})\right)^{\frac{1}{\alpha}}}$$
$$\geqslant\prod_{\ell=1}^{n}(k_{s_\ell}k_{t_\ell})^{\frac{1}{\alpha_\ell}}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{t_\ell}}^{k_{t_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{t_\ell}-(m_{t_\ell}-1))\left(p_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\Phi_\ell\left(a_{s_\ell,t_\ell,m_{s_\ell},m_{t_\ell}}\right)\right)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}}.$$

Remark 5. *Let $\omega_\ell(t_\ell,\tau_\ell)$, $p_\ell(t_\ell,\tau_\ell)$, $P_\ell(t_\ell,\tau_\ell)$ be defined as above and*

$$\Omega_\ell(s_\ell,t_\ell)=\frac{1}{P_\ell(s_\ell,t_\ell)}\int_{t_0}^{s_\ell}\int_{0}^{t_\ell}p_\ell(t_\ell,\tau_\ell)\omega_\ell(t_\ell,\tau_\ell)\nabla t_\ell\nabla\tau_\ell$$

changes to $\omega_\ell(t_\ell)$, $p_\ell(t_\ell)$, $P_\ell(s_\ell)$, and

$$\Omega_\ell(s_\ell)=\frac{1}{P_\ell(s_\ell)}\int_{t_0}^{s_\ell}p_\ell(t_\ell)\omega_\ell(t_\ell)\nabla t_\ell.$$

respectively and with suitable changes, we have the following new corollary:

Corollary 6. *Let $\omega_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(\xi_\ell)$, α_ℓ and β_ℓ be as Corollary 2 and define $\Omega_\ell(s_\ell)=\frac{1}{P(s_\ell)}\int_{t_0}^{s_\ell}p_\ell(\xi_\ell)\omega_\ell(\xi_\ell)\nabla\xi_\ell$ for ξ_ℓ $s_\ell\in(t_0,x_\ell)_{\mathbb{T}}$, where x_ℓ are positive real numbers. Let Φ_ℓ be n real-valued, non-negative, concave, and super-multiplicative function defined on $(0,\infty)$. Then*

$$\begin{aligned}&\int_{t_0}^{x_1}\cdots\int_{t_0}^{x_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell)\Phi_\ell(\Omega_\ell(s_\ell))}{\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}(s_\ell-t_0)\right)^{\frac{1}{\alpha}}}\nabla s_n\ldots\nabla s_1\\ &\geqslant\prod_{\ell=1}^{n}(x_\ell-t_0)^{\frac{1}{\alpha_\ell}}\left(\int_{t_0}^{x_\ell}(\rho(x_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell(\omega_\ell(s_\ell))\right)^{\beta_\ell}\nabla s_\ell\right)^{\frac{1}{\beta_\ell}}.\end{aligned}\tag{34}$$

Corollary 7. *In Corollary 6, if we take $n = 2$, $\beta_\ell = \frac{1}{2}$ then the inequality (26) changes to*

$$\begin{aligned} &\int_{t_0}^{x_1}\int_{t_0}^{x_2} \frac{P_1(s_1)P_2(s_2)\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{\big((s_1-t_0)+(s_2-t_0)\big)^{-2}}\nabla s_1\nabla s_2 \geqslant 4\big[(x_1-t_0)(x_2-t_0)\big]^{-1} \\ &\quad\times\bigg(\int_{t_0}^{x_1}(\rho(x_1)-s_1)\Big(p_1(s_1)\Phi_1(\omega_1(s_1))\Big)^2\nabla s_1\bigg)^{\frac{1}{2}}\bigg(\int_{t_0}^{x_2}(\rho(x_2)-s_2)\Big(p_2(s_2)\Phi_2\Big(\omega_2(s_2)\Big)\Big)^2\nabla s_2\bigg)^{\frac{1}{2}}. \end{aligned} \tag{35}$$

Remark 6. *In Corollary 7, if we take $\mathbb{T} = \mathbb{R}$, then the inequality (35) changes to*

$$\begin{aligned} &\int_{0}^{x_1}\int_{0}^{x_2} \frac{P_1(s_1)P_2(s_2)\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[x_1x_2\big]^{-1} \\ &\quad\times\bigg(\int_{0}^{x_1}(x_1-s_1)\Big(p_1(s_1)\Phi_1(\omega_1(s_1))\Big)^2 ds_1\bigg)^{\frac{1}{2}}\bigg(\int_{0}^{x_2}(x_2-s_2)\Big(p_2(s_2)\Phi_2\Big(\omega_2(s_2)\Big)\Big)^2 ds_2\bigg)^{\frac{1}{2}}. \end{aligned} \tag{36}$$

This is an inverse of the inequality (8) which was proved by Pachpatte [4].

Corollary 8. *In Corollary 7, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore, the inequality (35) changes to*

$$\begin{aligned} &\int_{t_0}^{x_1}\int_{t_0}^{x_2} \frac{\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{(s_1s_2)^{-1}\big((s_1-t_0)+(s_2-t_0)\big)^{-2}}\nabla s_1\nabla s_2 \geqslant 4\big[(x_1-t_0)(x_2-t_0)\big]^{-1} \\ &\quad\times\bigg(\int_{t_0}^{x_1}(\rho(x_1)-s_1)\Big(\Phi_1(\omega_1(s_1))\Big)^2\nabla s_1\bigg)^{\frac{1}{2}}\bigg(\int_{t_0}^{x_2}(\rho(x_2)-s_2)\Big(\Phi_2\Big(\omega_2(s_2)\Big)\Big)^2\nabla s_2\bigg)^{\frac{1}{2}}. \end{aligned} \tag{37}$$

Remark 7. *In Corollary 8, if we take $\mathbb{T} = \mathbb{R}$, then the inequality (37) change to*

$$\begin{aligned} &\int_{0}^{x_1}\int_{0}^{x_2} \frac{\Phi_1(\Omega_1(s_1))\Phi_2(\Omega_2(s_2))}{(s_1s_2)^{-1}(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[x_1x_2\big]^{-1} \\ &\quad\times\bigg(\int_{0}^{x_1}(x_1-s_1)\Big(\Phi_1(\omega_1(s_1))\Big)^2 ds_1\bigg)^{\frac{1}{2}}\bigg(\int_{0}^{x_2}(x_2-s_2)\Big(\Phi_2\Big(\omega_2(s_2)\Big)\Big)^2 ds_2\bigg)^{\frac{1}{2}}. \end{aligned}$$

This is an inverse inequality of the following inequality which was proved by Pachpatte [9].

$$\begin{aligned} &\int_{0}^{x}\int_{0}^{y} \frac{\Phi(\Omega(s))\Psi(G(t))}{(st)^{-1}(s+t)}dsdt \leqslant \frac{1}{2}\big[xy\big]^{\frac{1}{2}} \\ &\quad\times\bigg(\int_{0}^{x}(x-s_1)\Big(\Phi(\omega(s))\Big)^2 ds\bigg)^{\frac{1}{2}}\bigg(\int_{0}^{y}(y-t)\Big(\Psi\Big(g(t)\Big)\Big)^2 dt\bigg)^{\frac{1}{2}}. \end{aligned}$$

Corollary 9. *In Corollary 6, if we take $\beta_\ell = \frac{n-1}{n}$ $(\ell = 1, \dots, n)$ the inequality (34) becomes.*

$$\begin{aligned} &\int_{t_0}^{x_1}\cdots\int_{t_0}^{x_n} \frac{\prod_{\ell=1}^{n} P_\ell(s_\ell)\Phi_\ell(\Omega_\ell(s_\ell))}{\Big(\sum_{\ell=1}^{n}(s_\ell-t_0)\Big)^{\frac{-n}{n-1}}}\nabla s_n\ldots\nabla s_1 \\ &\geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^{n}(x_\ell-t_0)^{\frac{-1}{n-1}}\bigg(\int_{t_0}^{x_\ell}(\rho(x_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(\omega_\ell(s_\ell))\Big)^{\frac{n-1}{n}}\nabla s_\ell\bigg)^{\frac{n}{n-1}}. \end{aligned}$$

Theorem 3. *Let $\mathbb{T}$ be timescales with t_0, x_ℓ, y_ℓ, s_ℓ, $t_\ell \in \mathbb{T}$, $(\ell = 1, \dots, n)$. Assume $\omega_\ell(s_\ell, t_\ell)$ $(\ell = 1, \dots, n)$ are non-negative, left-dense continuous functions defined on $[t_0, x_\ell)_\mathbb{T} \times [t_0, y_\ell)_\mathbb{T}$, where x_ℓ and y_ℓ are positive real numbers and with $\omega_\ell(t_0, t_\ell) = \omega_\ell(s_\ell, t_0) = 0$, $(\ell = 1, \dots, n)$.*

Let $p_\ell(\xi_\ell)$ and $q_\ell(\tau_\ell)$ be positive left-dense, continuous functions defined for $\xi_\ell \in (t_0, s_\ell)_{\mathbb{T}}$, $\tau_\ell \in (t_0, t_\ell)_{\mathbb{T}}$. Set

$$P_\ell(s_\ell, t_\ell) = \int_{t_0}^{t_\ell}\int_{t_0}^{s_\ell} p_\ell(\xi_\ell)q_\ell(\tau_\ell)\nabla\xi_\ell\nabla\tau_\ell. \tag{38}$$

The functions $\omega_\ell(s_\ell, t_\ell)$ have partial ∇- derivatives $\omega_\ell^{\nabla_1}(s_\ell, t_\ell)$ and $\omega_\ell^{\nabla_2}(s_\ell, t_\ell)$ with respect s_ℓ and t_ℓ respectively and $\omega_\ell^{\nabla_1\nabla_2}(s_\ell, t_\ell) = \omega_\ell^{\nabla_2\nabla_1}(s_\ell, t_\ell)$. Let Φ_ℓ $(\ell = 1, \ldots, n)$ be n real-valued non-negative concave and super-multiplicative functions defined on $(0, \infty)$. Let $\alpha_\ell \in (1, \infty)$. Set $\alpha'_\ell = 1 - \alpha_\ell$ $(\ell = 1, \ldots, n)$, $\alpha = \sum_{\ell=1}^n \alpha_\ell$, and $\alpha' = \sum_{\ell=1}^n \alpha'_\ell = n - \alpha$. Then for $s_\ell \in [t_0, x_\ell)_{\mathbb{T}}$ and $t_\ell \in [t_0, y_\ell)_{\mathbb{T}}$, we have that

$$\begin{aligned}
&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^n \frac{\Phi_\ell(\omega_\ell(s_\ell, t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^n \alpha'_\ell(s_\ell - t_0)(t_\ell - t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1 \\
&\geqslant G(x_1y_1, \ldots, x_ny_n) \\
&\times \prod_{\ell=1}^n\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell) - s_\ell)(\rho(y_\ell) - t_\ell)\left(p_\ell(s_\ell)q_\ell(t_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(s_\ell, t_\ell)}{p_\ell(s_\ell)q_\ell(t_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha_\ell}
\end{aligned} \tag{39}$$

where

$$G(x_1y_1, \ldots, x_ny_n) = \prod_{\ell=1}^n\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell, t_\ell))}{P_\ell(s_\ell, t_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha'_\ell}.$$

Proof. From the hypotheses of Theorem 3, we obtain

$$\omega_\ell(s_\ell, t_\ell) = \int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\omega_\ell^{\nabla_2\nabla_1}(t_\ell, \tau_\ell)\nabla t_\ell\nabla\tau_\ell. \tag{40}$$

From (40) and S_8, it is easy to observe that

$$\begin{aligned}
\Phi_\ell(\omega_\ell(s_\ell, t_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell, t_\ell)\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(t_\ell)q_\ell(\tau_\ell)\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell, \tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell}{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(t_\ell)q_\ell(\tau_\ell)\nabla t_\ell\nabla\tau_\ell}\right) \\
&\geqslant \Phi_\ell(P_\ell(s_\ell, t_\ell))\Phi_\ell\left(\frac{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(t_\ell)q_\ell(\tau_\ell)\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell, \tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell}{\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(t_\ell)q_\ell(\tau_\ell)\nabla t_\ell\nabla\tau_\ell}\right).
\end{aligned} \tag{41}$$

By using an inverse Jensen's dynamic inequality, we obtain that

$$\Phi_\ell(\omega_\ell(s_\ell, t_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell, t_\ell))}{P_\ell(s_\ell, t_\ell)}\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell, \tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\nabla t_\ell\nabla\tau_\ell \tag{42}$$

Applying the inverse Hölder's inequality on the left-hand side of (42) with indices $1/\alpha_\ell$ and $1/\alpha'_\ell$, we obtain

$$\begin{aligned}
\Phi_\ell(\omega_\ell(s_\ell, t_\ell)) \geqslant\ & \frac{\Phi_\ell(P_\ell(s_\ell, t_\ell))}{P_\ell(s_\ell, t_\ell)}\left[(s_\ell - t_0)(t_\ell - t_0)\right]^{\alpha'_\ell} \\
&\times\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell, \tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla t_\ell\nabla\tau_\ell\right)^{\alpha_\ell}.
\end{aligned} \tag{43}$$

Using the following inequality on the term $\left[(s_\ell - t_0)(t_\ell - t_0)\right]^{\alpha'_\ell}$, where $\alpha'_\ell < 0$ and $\lambda_\ell > 0$.

$$\prod_{\ell=1}^n \lambda_\ell^{\alpha'_\ell} \geqslant \left(\frac{1}{\alpha'}\left(\sum_{\ell=1}^n \alpha'_\ell\lambda_\ell\right)\right)^{\alpha'}, \tag{44}$$

we obtain that

$$\begin{aligned}\prod_{\ell=1}^{n}\Phi_\ell(\omega_\ell(s_\ell,t_\ell)) \geqslant{}& \prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}\\ &\times\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell,\tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla t_\ell\nabla\tau_\ell\right)^{\alpha_\ell}.\end{aligned}\tag{45}$$

From (45), we have that

$$\begin{aligned}&\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\\ &\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell,\tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla t_\ell\nabla\tau_\ell\right)^{\alpha_\ell}.\end{aligned}\tag{46}$$

Integrating both sides of (46) over s_ℓ, t_ℓ from t_0 to x_ℓ, y_ℓ $(\ell=1,\ldots,n)$, we obtain that

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\ &\geqslant\prod_{\ell=1}^{n}\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell,\tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla t_\ell\nabla\tau_\ell\right)^{\alpha_\ell}\nabla s_\ell\nabla t_\ell\end{aligned}\tag{47}$$

Applying the inverse Hölder's inequality on the left-hand side of (47) with indices $1/\alpha_\ell$ and $1/\alpha'_\ell$, we obtain

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\ &\geqslant\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha'_\ell}\\ &\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\int_{t_0}^{s_\ell}\int_{t_0}^{t_\ell}\left(p_\ell(t_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(t_\ell,\tau_\ell)}{p_\ell(t_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla t_\ell\nabla\tau_\ell\right)\nabla s_\ell\nabla t_\ell\right)^{\alpha_\ell}\end{aligned}\tag{48}$$

By using Fubini's theorem, we observe that

$$\begin{aligned}&\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1\\ &\geqslant G(x_1y_1,\ldots,x_ny_n)\\ &\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(x_\ell-s_\ell)(y_\ell-t_\ell)\left(p_\ell(s_\ell)q_\ell(t_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(s_\ell,t_\ell)}{p_\ell(s_\ell)q_\ell(t_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha_\ell}.\end{aligned}\tag{49}$$

By using the fact $x_\ell\geqslant\rho(x_\ell)$, and $y_\ell\geqslant\rho(y_\ell)$, we obtain that

$$\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1$$
$$\geqslant G(x_1y_1,\ldots,x_ny_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\rho(x_\ell)-s_\ell)(\rho(y_\ell)-t_\ell)\left(p_\ell(s_\ell)q_\ell(t_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(s_\ell,t_\ell)}{p_\ell(s_\ell)q_\ell(t_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha_\ell}.$$

This completes the proof. □

Remark 8. *In Theorem 3, if $\mathbb{T}=\mathbb{Z}$, we obtain the result due to Zhao et al.'s [10], Theorem 1.5.*

Remark 9. *In Theorem 3, if we take $\mathbb{T}=\mathbb{R}$, we obtain the result due to Zhao et al.'s [10], Theorem 1.6.*

Remark 10. *Let $\omega_\ell(s_\ell,t_\ell)$, x_ℓ, y_ℓ, $\omega_\ell(t_0,t_\ell)$, $\omega_\ell(s_\ell,t_0)$, $\omega_\ell^{\nabla_1}(s_\ell,t_\ell)$, $\omega_\ell^{\nabla_2}(s_\ell,t_\ell)$, $\omega_\ell^{\nabla_2\nabla_1}(s_\ell,t_\ell)$, $p_\ell(\xi_\ell)$, $q_\ell(\tau_\ell)$ and $P(s_\ell,t_\ell)$ be as in Theorem 3. Let Φ_ℓ, α_ℓ, α'_ℓ, α, and α' be the same as in Theorem 4. Similar to the proof of Theorem 3, we have*

$$\int_{t_0}^{x_1}\int_{t_0}^{y_1}\cdots\int_{t_0}^{x_n}\int_{t_0}^{y_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell,t_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)(t_\ell-t_0)\right)^{\alpha'}}\nabla s_n\nabla t_n\ldots\nabla s_1\nabla t_1$$
$$\leqslant G^*(x_1y_1,\ldots,x_ny_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}(\sigma(x_\ell)-s_\ell)(\sigma(y_\ell)-t_\ell)\left(p_\ell(s_\ell)q_\ell(t_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla_2\nabla_1}(s_\ell,t_\ell)}{p_\ell(s_\ell)q_\ell(t_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha_\ell}.$$

where

$$G^*(x_1y_1,\ldots,x_ny_n)=\frac{1}{(\alpha')^{\alpha'}}\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\int_{t_0}^{y_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,t_\ell))}{P_\ell(s_\ell,t_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla s_\ell\nabla t_\ell\right)^{\alpha'_\ell}.$$

This is an inverse form of the inequality (39).

Corollary 10. *Let $\mathbb{T}$ be a timescale with t_0, x_ℓ, $s_\ell\in\mathbb{T}$. Let $\omega_\ell\in C_{ld}[t_0,k_\ell]_{\mathbb{T}}$, $(\ell=1\ldots,n)$ with $\omega_\ell(t_0)=0$, let $p_\ell(\xi_\ell)$ be n positive functions defined for $\xi_\ell\in[t_0,s_\ell]_{\mathbb{T}}$. Set $P_\ell(s_\ell)=\int_{t_0}^{s_\ell}p_\ell(\xi_\ell)\nabla\xi_\ell$ for $s_\ell\in[t_0,x_\ell]_{\mathbb{T}}$, where x_ℓ are positive real numbers. Let Φ_ℓ, α_ℓ, α'_ℓ, α, and α' be as in Theorem 3. Then*

$$\int_{t_0}^{x_1}\cdots\int_{t_0}^{x_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\omega_\ell(s_\ell))}{\left(\frac{1}{\alpha'}\sum_{\ell=1}^{n}\alpha'_\ell(s_\ell-t_0)\right)^{\alpha'}}\nabla s_n\ldots\nabla s_1$$
$$\geqslant G^{**}(x_1,\ldots,x_n) \tag{50}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}(\rho(x_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\omega_\ell^{\nabla}(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{1}{\alpha_\ell}}\nabla s_\ell\right)^{\alpha_\ell}.$$

where

$$G^{**}(x_1,\ldots,x_n)=\prod_{\ell=1}^{n}\left(\int_{t_0}^{x_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\alpha'_\ell}}\nabla s_\ell\right)^{\alpha'_\ell}.$$

Remark 11. *In Corollary 10, if we take $\mathbb{T}=\mathbb{Z}$, we obtain an inverse form of inequality (4), which was given by Handley et al.*

Remark 12. *In Corollary 10, if we take $\mathbb{T} = \mathbb{R}$, we obtain an inverse form of inequality (5), which was given by Handley et al.*

Remark 13. *In inequality (50) taking $n = 2$, $\alpha_1 = \alpha_2 = 2$, then $\alpha_1' = \alpha_2' = -1$,, we have*

$$\begin{aligned}
&\int_{t_0}^{x_1}\int_{t_0}^{x_2}\prod_{\ell=1}^{n}\frac{\Phi_1(\omega_1(s_1))\Phi_1(\omega_2(s_2))}{\Big((s_1-t_0)+(s_2-t_0)\Big)^{-2}}\nabla s_1\nabla s_2 \\
&\geqslant D(x_1,x_2)\left(\int_{t_0}^{x_1}(\rho(x_1)-s_1)\left(p_1(s_1)\Phi_1\left(\frac{\omega_1^{\nabla}(s_1)}{p_1(s_1)}\right)\right)^{\frac{1}{2}}\nabla s_1\right)^2 \qquad (51)\\
&\times\left(\int_{t_0}^{x_2}(\rho(x_2)-s_2)\left(p_2(s_2)\Phi_2\left(\frac{\omega_2^{\nabla}(s_2)}{p_2(s_2)}\right)\right)^{\frac{1}{2}}\nabla s_2\right)^2.
\end{aligned}$$

where

$$D(x_1,x_2) = 4\left(\int_{t_0}^{x_1}\left(\frac{\Phi_1(P_1(s_1))}{P_2(s_1)}\right)^{-1}\nabla s_1\right)^{-1}\left(\int_{t_0}^{x_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}\nabla s_2\right)^{-1}.$$

Remark 14. *If we take $\mathbb{T} = \mathbb{Z}$, the inequality (51) is an inverse of inequality due to Pachpatte [2].*

Remark 15. *If we take $\mathbb{T} = \mathbb{R}$, the inequality (51) is an inverse of inequality due to Pachpatte [2].*

3. Conclusions

In this article, we introduced some investigations of the nabla Hilbert inequality on a general timescale, some dynamic integral and discrete inequalities, known in the literature, are extended as special cases of our results. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

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