symmetry

MDPI

Article

On a Unique Solution of a Class of Stochastic Predator–Prey Models with Two-Choice Behavior of Predator Animals

Reny George [1,2,†], Zoran D. Mitrović [3,†], Ali Turab [4,*,†], Ana Savić [5,†] and Wajahat Ali [6,†]

1 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia; renygeorge02@yahoo.com
2 Department of Mathematics and Computer Science, St. Thomas College, Bhilai 490006, India
3 Faculty of Electrical Engineering, University of Banja Luka, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina; zoran.mitrovic@etf.unibl.org
4 Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathum Thani 12120, Thailand
5 School of Electrical and Computer Engineering, Academy of Technical and Art Applied Studies, 11000 Belgrade, Serbia; ana.savic@viser.edu.rs
6 School of Science, Nanjing University of Science and Technology, Nanjing 210094, China; wajahatali.kh@gmail.com
* Correspondence: taurusnoor@yahoo.com
† These authors contributed equally to this work.

Abstract: Simple birth–death phenomena are frequently examined in mathematical modeling and probability theory courses since they serve as an excellent foundation for stochastic modeling. Such mechanisms are inherent stochastic extensions of the deterministic population paradigm for population expansion of a particular species in a habitat with constant resource availability and many other organisms. Most animal behavior research differentiates such circumstances into two different events when it comes to two-choice scenarios. On the other hand, in this kind of research, the reward serves a significant role, because, depending on the chosen side and food placement, such situations may be divided into four groups. This article presents a novel stochastic equation that may be used to describe the vast majority of models discussed in the current studies. It is noteworthy that they are connected to the symmetry of the progression of a solution of stochastic equations. The techniques of fixed point theory are employed to explore the existence, uniqueness, and stability of solutions to the proposed functional equation. Additionally, some examples are offered to emphasize the significance of our findings.

Keywords: stochastic predator–prey model; stability; fixed points

check for updates

Citation: George, R.; Mitrović, Z.D.; Turab, A.; Savić, A.; Ali, W. On a Unique Solution of a Class of Stochastic Predator–Prey Models with Two-Choice Behavior of Predator Animals. *Symmetry* **2022**, *14*, 846. https://doi.org/10.3390/sym14050846

Academic Editor: Dumitru Baleanu

Received: 28 March 2022
Accepted: 15 April 2022
Published: 19 April 2022

1. Introduction and Preliminaries

The dynamics of populations is a highly contentious issue in biomathematics. The study of the development of diverse habitats has always piqued our attention, beginning with individuals of a single species and progressing to more complex systems, in which many species coexist in the same environment.

Many times, symmetry has appeared in mathematical interpretations, and it has been shown that it is vital for solving issues or progressing studies. It helps to make it possible to find high-quality work that uses significant mathematics and associated topologies to address critical concerns in various domains.

Since Volterra's groundbreaking work [1], numerous predator–prey models have been developed to comprehend population evolution and dynamics better (see [2,3]). Several variations of the Lotka–Volterra model have been suggested and researched in various fields, including mathematical biology (see [4–6]), ecology (see [7,8]), and economics (see [9,10]), among others (see [11–18] and references therein), in the last half-century. The so-called functional reaction involves the quantity of prey captured per predator and it substantially influences the kinetic characteristics. It is directly tied to several aspects, including prey density, handling time and attack efficiency (see [1,19]).

The stochastic averaging method helps to explore linear/nonlinear systems triggered by stochastic mechanism. It was originally used on nonlinear systems with Gaussian white noise stimuli (see [20–22]), and later it was extended to nonlinear systems with various forms of stochastic excitation (see [23,24]). It has been used to investigate stationary PDFs (see [25,26] and references therein) and it is the best way to manage species densities in habitats with low self- and stochastic simulations. Furthermore, the average stochastic technique has yet to be used to solve dynamic ecosystem behavior under continuous and random jump excitation (see [27]).

The learning stage in an animal or a human organism, on the other hand, is often seen as a series of options among many possible answers. It is also useful to look for structural changes in the possibilities that indicate variations in the corresponding event probability. The majority of learning research, from this viewpoint, indicates the likelihood of a trial-to-test emergence, which is a hallmark of stochastic processes. As a result, it is not a new concept. In [28,29], the authors established a notion of "reward" based on animals selecting the right side in a two-choice scenario and separated it into four categories: left-reward, right-reward, right-non-reward, and left non-reward. They utilized the following operators to monitor such behavior by relying on four occurrences between a predator and its prey choices:

A few researchers observed the responses of various animals in a two-choice scenario (see [30–36]) using the aforementioned operators (given in Table 1). Recently, in [37], the author used such operators to examine the two-choice behavior of rhesus monkeys in a non-contingent environment. The author focused on the chosen side of the animal rather than the food placement.

Table 1. Some operators and their outcomes presented in [28,29].

Operators for reinforcement-extinction model		
Animal's Response	**Outcome (Left side)**	**Outcome (Right side)**
Reinforcement	rx	$rx+1-r$
Non-reinforcement	$sx+1-s$	sx
Operators for habit formation model		
Animal's Response	**Outcome (Left side)**	**Outcome (Right side)**
Reinforcement	rx	$rx+1-r$
Non-reinforcement	sx	$sx+1-s$

In contrast to the above work, here, we extend the model by adding two extra compartments discussed in [28,29] to the model with the corresponding probabilities:

$$\mathfrak{Z}(x) = \wp_1\hbar_1(x)\mathfrak{Z}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{Z}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{Z}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{Z}(\varrho_4(x)), \quad (1)$$

for all $x \in [\mu, v]$, $\wp_1 = (\gamma - \mu)\Delta\tilde{v}^{-1}$ and $\wp_2 = (v - \gamma)\Delta\tilde{v}^{-1}$, where $0 \le \gamma \le 1$, $\Delta\tilde{v} = v - \mu$, $\mathfrak{Z} : [\mu, v] \to \mathbb{R}$ is an unknown function and $\varrho_1 - \varrho_4 : [\mu, v] \to [\mu, v]$ are given mappings. Moreover, $\hbar_1, \hbar_2 : [\mu, v] \to \mathbb{R}$ are defined by

$$\begin{cases} \hbar_1(x) = (x-\mu)\Delta\tilde{v}^{-1}, \\ \hbar_2(x) = (v-x)\Delta\tilde{v}^{-1}, \end{cases} \quad (2)$$

The physical meaning of the parameters/operators are given in Table 2.

Table 2. Physical meaning of the parameters/operators.

Parameter/Operator	Physical Meaning
$[\mu, v]$	State space
r, s	Learning-rate parameters
γ	Probability of a chosen side
$\varrho_1, \varrho_2, \varrho_3, \varrho_4$	Transition operators
$\mathfrak{Z}$	Final probability

The presented functional Equation (1) with (2) has great importance in mathematical biology and learning theory. Such equations are used to investigate the response of animals in a two-choice situation, and the solution exists when a predator is fixed to one type of prey (see Figure 1).

(a) (b)

Figure 1. (**a**): A predator with two choices of prey [38]; (**b**): a predator fixed to one type of prey [39].

On the other hand, the fixed point approach is regarded as a fundamental component and is a very effective method in nonlinear analysis due to its many critical implementations in various fields, including physics, engineering, computer science, biology, economics, and chemistry. This approach is widely used in mathematics to examine game-theoretic models, dynamical systems, statistical models, and differential equations. More specifically, this approach is mainly used to analyze certain integro-differential equations, functional equations, differential and integral equations, and fractional equations, which simplifies the process of obtaining computational solutions to such problems (for the details, see [40–44] and references therein).

In this work, our aim was to use the appropriate fixed point technique to demonstrate the existence of a unique solution to Equation (1) with (2). After that, we considered the stability of solutions to the suggested stochastic Equation (1) under the Hyers–Ulam (HU) and Hyers–Ulam–Rassias type (HUR) stability results. We provide three examples to emphasize the significance of our main conclusions.

Progress of this work requires the accomplishment of the following stated result, which guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to find those fixed points.

Theorem 1 (Banach fixed point theorem)**.** *Let $(\mathscr{J}, d)$ be a complete metric space and $\mathfrak{Z} : \mathscr{J} \to \mathscr{J}$ be a Banach contraction mapping (shortly, BCM) defined by*

$$d(\mathfrak{Z}\mu, \mathfrak{Z}v) \leq \Upsilon d(\mu, v) \tag{3}$$

for some $\Upsilon < 1$ and for all $\mu, v \in \mathscr{J}$. Then $\mathfrak{Z}$ has one fixed point. Furthermore, the Picard iteration $\{\mu_n\}$ in $\mathscr{J}$ which is defined by $\mu_n = \mathfrak{Z}\mu_{n-1}$ for all $n \in \mathbb{N}$, where $\mu_0 \in \mathscr{J}$, converges to the unique fixed point of $\mathfrak{Z}$.

Proof. For the proof, we refer [45,46]. □

2. Main Results

Let $\mathscr{J} = [\mu, v]$ with $\mu < v$, where $\mu, v \in \mathbb{R}$. We denote a class $\mathscr{T}$ having the continuous real-valued functions $\mathfrak{z} : \mathscr{J} \to \mathbb{R}$ with $\mathfrak{z}(\mu) = 0$ and $\sup_{\xi \neq \chi} \frac{|\mathfrak{z}(\xi) - \mathfrak{z}(\chi)|}{|\xi - \chi|} < \infty$, where

$$\|\mathfrak{z}\| = \sup_{\xi \neq \chi} \frac{|\mathfrak{z}(\xi) - \mathfrak{z}(\chi)|}{|\xi - \chi|} \tag{4}$$

for all $\mathfrak{z} \in \mathscr{T}$.

We shall use the following conditions to prove the main results:

($\mathcal{A}_1$) There is a nonempty subset $\mathscr{C}$ of $\mathscr{S} := \{\mathfrak{z} \in \mathscr{T} | \mathfrak{z}(v) \leq v\}$ such that $(\mathscr{C}, \|\cdot\|)$ is a Banach space (for detail, see [34]), where $\|\cdot\|$ is given in (4).

($\mathcal{A}_2$) The mappings $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ are Banach contraction mappings with contractive coefficients $\varpi_1 - \varpi_4$, respectively, and satisfy the following conditions

$$\begin{cases} \varrho_1(v) = v = \varrho_2(v), & \text{and} \\ \varrho_3(\mu) = \mu = \varrho_4(\mu). \end{cases} \tag{5}$$

($\mathcal{A}_3$) For a function $\varphi : \mathscr{C} \to [0, \infty)$, we have that for every $\mathfrak{z} \in \mathscr{C}$ with $d(\mathscr{F}\mathfrak{z}, \mathfrak{z}) \leq \varphi(\mathfrak{z})$, there is a unique $\mathfrak{z}^\star \in \mathscr{C}$ with $\mathscr{F}\mathfrak{z}^\star = \mathfrak{z}^\star$ and $d(\mathfrak{z}, \mathfrak{z}^\star) \leq \varsigma\varphi(\mathfrak{z})$ for some $\varsigma > 0$.

($\mathcal{A}_4$) For $\varpi > 0$, we have that for every $\mathfrak{z} \in \mathscr{C}$ with $d(\mathscr{F}\mathfrak{z}, \mathfrak{z}) \leq \varpi$, there is a unique $\mathfrak{z}^\star \in \mathscr{C}$ with $\mathscr{F}\mathfrak{z}^\star = \mathfrak{z}^\star$ and $d(\mathfrak{z}, \mathfrak{z}^\star) \leq \varsigma\varpi$, for some $\varsigma > 0$.

Now, we begin with the outcome stated below.

Theorem 2. *Consider the stochastic functional Equation (1) associated with (2). Suppose that the conditions ($\mathcal{A}_1$) and ($\mathcal{A}_2$) are satisfied and there exists an $\Upsilon < 1$, where*

$$\Upsilon := |\wp_1(2\varpi_1 + 2\varpi_3) + \wp_2(2\varpi_2 + 2\varpi_4)|. \tag{6}$$

Then, for each $\mathfrak{z} \in \mathscr{T}$ and for all $x \in \mathscr{J}$, a self-mapping $\mathscr{F}$ from $\mathscr{C}$ to $\mathscr{C}$ is a BCM which is defined by

$$(\mathscr{F}\mathfrak{z})(x) = \wp_1\hbar_1(x)\mathfrak{z}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{z}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{z}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{z}(\varrho_4(x)). \tag{7}$$

Proof. Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathscr{C}$. For each $\tau_1, \tau_2 \in \mathscr{J}$ such that $\tau_1 \neq \tau_2$, $\Delta\tau = \tau_1 - \tau_2$ and $\Delta\mathfrak{z} = \mathfrak{z}_1 - \mathfrak{z}_2$, we obtain

$$\begin{aligned}
& |\mathscr{F}(\Delta\mathfrak{z})(\tau_1) - \mathscr{F}(\Delta\mathfrak{z})(\tau_2)||\Delta\tau|^{-1} \\
= & \Big|\Delta\tau^{-1}[\wp_1\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_1(\tau_1)) + \wp_2\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_2(\tau_1)) + \wp_1\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_3(\tau_1)) \\
& + \wp_2\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_4(\tau_1)) - \wp_1\hbar_1(\tau_2)\Delta\mathfrak{z}(\varrho_1(\tau_2)) - \wp_2\hbar_1(\tau_2)\Delta\mathfrak{z}(\varrho_2(\tau_2)) \\
& - \wp_1\hbar_2(\tau_2)\Delta\mathfrak{z}(\varrho_3(\tau_2)) - \wp_2\hbar_2(\tau_2)\Delta\mathfrak{z}(\varrho_4(\tau_2))]\Big| \\
= & \Big|\Delta\tau^{-1}[\wp_1\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_1(\tau_1)) - \wp_1\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_1(\tau_2))] + \Delta\tau^{-1}[\wp_2\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_2(\tau_1)) \\
& - \wp_2\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_2(\tau_2))] + \Delta\tau^{-1}[\wp_1\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_3(\tau_1)) - \wp_1\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_3(\tau_2))] \\
& + \Delta\tau^{-1}[\wp_2\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_4(\tau_1)) - \wp_2\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_4(\tau_2))] + \Delta\tau^{-1}[\wp_1\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_1(\tau_2)) \\
& - \wp_1\hbar_1(\tau_2)\Delta\mathfrak{z}(\varrho_1(\tau_2))] + \Delta\tau^{-1}[\wp_2\hbar_1(\tau_1)\Delta\mathfrak{z}(\varrho_2(\tau_2)) - \wp_2\hbar_1(\tau_2)\Delta\mathfrak{z}(\varrho_2(\tau_2))] \\
& \Delta\tau^{-1}[\wp_1\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_3(\tau_2)) - \wp_1\hbar_2(\tau_2)\Delta\mathfrak{z}(\varrho_3(\tau_2))] + \Delta\tau^{-1}[\wp_2\hbar_2(\tau_1)\Delta\mathfrak{z}(\varrho_4(\tau_2)) \\
& - \wp_2\hbar_2(\tau_2)\Delta\mathfrak{z}(\varrho_4(\tau_2))]\Big|.
\end{aligned}$$

From the above equation, we can write

$$
\begin{aligned}
& |\mathscr{F}(\Delta\mathfrak{Z})(\tau_1) - \mathscr{F}(\Delta\mathfrak{Z})(\tau_2)||\Delta\tau|^{-1} \\
\leq \;& |\Delta\tau|^{-1}|\wp_1\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_1(\tau_1)) - \wp_1\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_1(\tau_2))| + |\Delta\tau|^{-1}|\wp_2\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_2(\tau_1)) \\
& - \wp_2\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_2(\tau_2))| + |\Delta\tau|^{-1}|\wp_1\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_3(\tau_1)) - \wp_1\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_3(\tau_2))| \\
& +|\Delta\tau|^{-1}|\wp_2\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_4(\tau_1)) - \wp_2\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_4(\tau_2))| + |\Delta\tau|^{-1}|\wp_1\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_1(\tau_2)) \\
& -\wp_1\hbar_1(\tau_2)\Delta\mathfrak{Z}(\varrho_1(\tau_2))| + |\Delta\tau|^{-1}|\wp_2\hbar_1(\tau_1)\Delta\mathfrak{Z}(\varrho_2(\tau_2)) - \wp_2\hbar_1(\tau_2)\Delta\mathfrak{Z}(\varrho_2(\tau_2))| \\
& |\Delta\tau|^{-1}|\wp_1\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_3(\tau_2)) - \wp_1\hbar_2(\tau_2)\Delta\mathfrak{Z}(\varrho_3(\tau_2))| + |\Delta\tau|^{-1}|\wp_2\hbar_2(\tau_1)\Delta\mathfrak{Z}(\varrho_4(\tau_2)) \\
& - \wp_2\hbar_2(\tau_2)\Delta\mathfrak{Z}(\varrho_4(\tau_2))|.
\end{aligned}
$$

As $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ are Banach contraction mappings, i.e.,

$$
\begin{aligned}
|\varrho_1(\tau_1) - \varrho_1(\tau_2)| &\leq \varpi_1|\tau_1 - \tau_2|, & |\varrho_2(\tau_1) - \varrho_2(\tau_2)| &\leq \varpi_2|\tau_1 - \tau_2| \\
|\varrho_3(\tau_1) - \varrho_3(\tau_2)| &\leq \varpi_3|\tau_1 - \tau_2|, & |\varrho_4(\tau_1) - \varrho_4(\tau_2)| &\leq \varpi_4|\tau_1 - \tau_2|
\end{aligned}
$$

where $\varpi_1 - \varpi_4$ are contractive coefficients, respectively. Thus, by using the above relation with the definition of norm (4), we have

$$
|\mathscr{F}(\Delta\mathfrak{Z})(\tau_1) - \mathscr{F}(\Delta\mathfrak{Z})(\tau_2)||\Delta\tau|^{-1} \quad \leq \quad \Upsilon\|\Delta\mathfrak{Z}\|,
$$

where Υ is given in (6). This gives that

$$
d(\mathscr{F}\mathfrak{Z}_1, \mathscr{F}\mathfrak{Z}_2) = \|\mathscr{F}\mathfrak{Z}_1 - \mathscr{F}\mathfrak{Z}_2\| \leq \Upsilon\|\mathfrak{Z}_1 - \mathfrak{Z}_2\| = \Upsilon d(\mathfrak{Z}_1, \mathfrak{Z}_2).
$$

It follows from $0 < \Upsilon < 1$ that $\mathscr{F}$ is a BCM. This completes the proof. □

Theorem 3. *Consider the Equation (1) with (2). Suppose that the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ are satisfied and there exists an $\Upsilon < 1$, where Υ is given in (6). The mapping $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is a BCM, which is defined in (7). Thus, the proposed problem (1) associated with (2) has a unique solution in $\mathscr{C}$. Moreover, the iteration $\mathfrak{Z}_n$ in $\mathscr{C}$ ($\forall n \in \mathbb{N}$ and $\mathfrak{Z}_0 \in \mathscr{C}$) given by*

$$
\begin{aligned}
(\mathfrak{Z}_n)(x) \quad = \quad & \wp_1\hbar_1(x)\mathfrak{Z}_{n-1}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{Z}_{n-1}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{Z}_{n-1}(\varrho_3(x)) \\
& +\wp_2\hbar_2(x)\mathfrak{Z}_{n-1}(\varrho_4(x)) \qquad (8)
\end{aligned}
$$

converges to the unique solution of (1).

Proof. As $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is a BCM, we obtain the outcome of this result by combining Theorem 2 with the Banach fixed point theorem. □

Here, we shall look at different conditions. If $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ are Banach contraction mappings with contractive coefficients $\varpi_1 \leq \varpi_2 \leq \varpi_3 \leq \varpi_4$, respectively, then by Theorems 2 and 3, the outcomes are as follows.

Corollary 1. *Consider the stochastic Equation (1) associated with (2). Assume that the condition $(\mathcal{A}_1)$ is satisfied and there exists an $\tilde{\Upsilon} := 4\varpi_4 < 1$. Then, for each $\mathfrak{Z} \in \mathscr{T}$ and for all $x \in \mathscr{J}$, a self-mapping $\mathscr{F}$ from $\mathscr{C}$ to $\mathscr{C}$ is a BCM, which is defined by*

$$
(\mathscr{F}\mathfrak{Z})(x) = \wp_1\hbar_1(x)\mathfrak{Z}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{Z}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{Z}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{Z}(\varrho_4(x)). \qquad (9)
$$

Corollary 2. *Consider the Equations (1) and (2). Assume that the condition $(\mathcal{A}_1)$ is satisfied and there exists an $\tilde{\Upsilon} := 4\varpi_4 < 1$. The mapping $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is a BCM, which is defined in (9). Thus, the proposed problem (1) associated with (2) has a unique solution in $\mathscr{C}$. Additionally, the iteration $\mathfrak{Z}_n$ in $\mathscr{C}$ ($\forall n \in \mathbb{N}$ and $\mathfrak{Z}_0 \in \mathscr{C}$) is defined as*

$$(\mathfrak{z}_n)(x) = \wp_1\hbar_1(x)\mathfrak{z}_{n-1}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{z}_{n-1}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{z}_{n-1}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{z}_{n-1}(\varrho_4(x)) \quad (10)$$

converges to the unique solution of (1).

Remark 1. *The suggested stochastic Equation (1) is a generalization of the equations discussed in [30–34].*

3. Stability Analysis

Here, we shall discuss the stability of the solution to the Equation (1) (see [47–51] for the details).

Theorem 4. *In light of Theorem 2's assumptions, the equation $\mathscr{F}\mathfrak{z} = \mathfrak{z}$, where $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is defined as*

$$(\mathscr{F}\mathfrak{z})(x) = \wp_1\hbar_1(x)\mathfrak{z}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{z}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{z}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{z}(\varrho_4(x)), \quad (11)$$

for all $\mathfrak{z} \in \mathscr{C}$ and $x \in \mathscr{J}$, has HUR stability (defined in $(\mathcal{A}_3)$).

Proof. Let $\mathfrak{z} \in \mathscr{C}$ such that $d(\mathscr{F}\mathfrak{z}, \mathfrak{z}) \leq \varphi(\mathfrak{z})$. By using Theorem 2, we have a unique $\mathfrak{z}^\star \in \mathscr{C}$, such that $\mathscr{F}\mathfrak{z}^\star = \mathfrak{z}^\star$. Thus, we obtain

$$\begin{aligned} d(\mathfrak{z}, \mathfrak{z}^\star) &\leq d(\mathfrak{z}, \mathscr{F}\mathfrak{z}) + d(\mathscr{F}\mathfrak{z}, \mathfrak{z}^\star) \\ &\leq \varphi(\mathfrak{z}) + d(\mathscr{F}\mathfrak{z}, \mathscr{F}\mathfrak{z}^\star) \\ &\leq \varphi(\mathfrak{z}) + \Upsilon d(\mathfrak{z}, \mathfrak{z}^\star) \end{aligned}$$

where Υ is defined in (6), and so by $\varsigma := \dfrac{1}{1-\Upsilon}$, we have

$$d(\mathfrak{z}, \mathfrak{z}^\star) \leq \varsigma\varphi(\mathfrak{z}).$$

□

We gain the following conclusion about the HU stability from the aforementioned investigation.

Corollary 3. *In light of Theorem 2's assumptions, the equation $\mathscr{F}\mathfrak{z} = \mathfrak{z}$, where $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is given by*

$$(\mathscr{F}\mathfrak{z})(x) = \wp_1\hbar_1(x)\mathfrak{z}(\varrho_1(x)) + \wp_2\hbar_1(x)\mathfrak{z}(\varrho_2(x)) + \wp_1\hbar_2(x)\mathfrak{z}(\varrho_3(x)) + \wp_2\hbar_2(x)\mathfrak{z}(\varrho_4(x)), \quad (12)$$

for all $\mathfrak{z} \in \mathscr{C}$ and $x \in \mathscr{J}$, has HU stability (defined in $(\mathcal{A}_4)$).

4. Some Illustrative Examples

Here, we provide the following examples to justify our findings.

Example 1. *Consider the stochastic functional equation given below*

$$\begin{aligned} \mathfrak{z}(x) = {} & \wp_1\hbar_1(x)\mathfrak{z}\Big(v(k-\mu)\Delta\tilde{v}^{-1} + (v-k)\Delta\tilde{v}^{-1}x\Big) \\ & + \wp_2\hbar_1(x)\mathfrak{z}\Big(v(v-k)\Delta\tilde{v}^{-1} + (k-\mu)\Delta\tilde{v}^{-1}x\Big) \\ & + \wp_1\hbar_2(x)\mathfrak{z}\Big((v-\ell)(x-\mu)\Delta\tilde{v}^{-1} + \mu\Big) \end{aligned}$$

$$+\wp_2\hbar_2(x)\mathfrak{z}\Big((\ell-\mu)(x-\mu)\Delta\tilde{\upsilon}^{-1}+\mu\Big). \tag{13}$$

for all $x \in \mathscr{J}$ *with* $\mu < k, \ell < \upsilon$ *and* $\mathfrak{z} \in \mathscr{T}$. *If we set the mappings* $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ *by*

$$\begin{cases} \varrho_1(x) = \upsilon(k-\mu)\Delta\tilde{\upsilon}^{-1} + (\upsilon-k)\Delta\tilde{\upsilon}^{-1}x, \\ \varrho_2(x) = \upsilon(\upsilon-k)\Delta\tilde{\upsilon}^{-1} + (k-\mu)\Delta\tilde{\upsilon}^{-1}x, \\ \varrho_3(x) = (\upsilon-\ell)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu, \\ \varrho_4(x) = (\ell-\mu)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu, \end{cases}$$

for all $x \in \mathscr{J}$, *so* (1) *decreases to the Equation* (13). *It is easy to see that the mappings* $\varrho_1 - \varrho_4$ *satisfy* ($\mathcal{A}_2$), *i.e.,*

$$\begin{cases} |\varrho_1(x) - \varrho_1(y)| \le (\upsilon-k)\Delta\tilde{\upsilon}^{-1}|x-y|, \\ |\varrho_2(x) - \varrho_2(y)| \le (k-\mu)\Delta\tilde{\upsilon}^{-1}|x-y|, \\ |\varrho_3(x) - \varrho_3(y)| \le (\upsilon-\ell)\Delta\tilde{\upsilon}^{-1}|x-y|, \\ |\varrho_4(x) - \varrho_4(y)| \le (\ell-\mu)\Delta\tilde{\upsilon}^{-1}|x-y|, \end{cases}$$

for all $x, y \in \mathscr{J}$, *where* $\varpi_1 = (\upsilon-k)\Delta\tilde{\upsilon}^{-1}$, $\varpi_2 = (k-\mu)\Delta\tilde{\upsilon}^{-1}$, $\varpi_3 = (\upsilon-\ell)\Delta\tilde{\upsilon}^{-1}$, $\varpi_4 = (\ell-\mu)\Delta\tilde{\upsilon}^{-1}$ *are contractive coefficients, respectively. Here, if* ($\mathcal{A}_1$) *is satisfied with*

$$\Upsilon := \Big|\wp_1(4\upsilon-2k-2\ell)\Delta\tilde{\upsilon}^{-2} + \wp_2(2k+2\ell-4\mu)\Delta\tilde{\upsilon}^{-2}\Big| < 1,$$

then the mapping $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ *defined on* (13) *is a BCM. Thus, all conditions of Theorem 2 are fulfilled and, therefore, we obtain the existence of a solution to the functional Equation* (13).

For a unique solution of (13), *we define* $\mathfrak{z}_0 = \mathbf{I} \in \mathscr{C}$ *as a starting approximation (whereas* $\mathbf{I}$ *is an identity function), then by Theorem 3, we obtain the convergence of the following iteration process:*

$$\begin{aligned}
\mathfrak{z}_1(x) &= \Delta\tilde{\upsilon}^{-3}\left[\begin{array}{c} (2\ell\gamma - 2\gamma k + k\mu - \ell\mu + k\upsilon - \ell\upsilon)x^2 \\ +(-\mu^3 - k\mu^2 + \ell\mu^2 + 2\gamma k\mu - 2\ell\gamma\mu + \upsilon^3 - k\upsilon^2 \\ -3\mu^2\upsilon - 2k\mu\upsilon + 2\gamma k\upsilon - 2\ell\gamma\upsilon + 2\ell\mu\upsilon)x \\ +(k\mu\upsilon^2 - \ell\mu\upsilon^2 + k\mu^2\upsilon - 2\gamma k\mu\upsilon + 2\ell\gamma\mu\upsilon - \ell\mu^2\upsilon) \end{array}\right], \\
\mathfrak{z}_2(x) &= \wp_1\hbar_1(x)\mathfrak{z}_1\Big(\upsilon(k-\mu)\Delta\tilde{\upsilon}^{-1} + (\upsilon-k)\Delta\tilde{\upsilon}^{-1}x\Big) \\
&\quad +\wp_2\hbar_1(x)\mathfrak{z}_1\Big(\upsilon(\upsilon-k)\Delta\tilde{\upsilon}^{-1} + (k-\mu)\Delta\tilde{\upsilon}^{-1}x\Big) \\
&\quad +\wp_1\hbar_2(x)\mathfrak{z}_1\Big((\upsilon-\ell)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu\Big) \\
&\quad +\wp_2\hbar_2(x)\mathfrak{z}_1\Big((\ell-\mu)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu\Big), \\
&\ \ \vdots \\
\mathfrak{z}_n(x) &= \wp_1\hbar_1(x)\mathfrak{z}_{n-1}\Big(\upsilon(k-\mu)\Delta\tilde{\upsilon}^{-1} + (\upsilon-k)\Delta\tilde{\upsilon}^{-1}x\Big) \\
&\quad +\wp_2\hbar_1(x)\mathfrak{z}_{n-1}\Big(\upsilon(\upsilon-k)\Delta\tilde{\upsilon}^{-1} + (k-\mu)\Delta\tilde{\upsilon}^{-1}x\Big) \\
&\quad +\wp_1\hbar_2(x)\mathfrak{z}_{n-1}\Big((\upsilon-\ell)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu\Big) \\
&\quad +\wp_2\hbar_2(x)\mathfrak{z}_{n-1}\Big((\ell-\mu)(x-\mu)\Delta\tilde{\upsilon}^{-1} + \mu\Big),
\end{aligned}$$

for all $n \in \mathbb{N}$.

On the other hand, as $0 \le \Upsilon < 1$, *we obtain*

$$\varsigma := \frac{1}{1-\Upsilon} > 0.$$

If a function $\mathfrak{z} \in \mathscr{C}$ satisfies the inequality

$$d(\mathscr{F}\mathfrak{z}, \mathfrak{z}) \leq \varphi(\mathfrak{z}), \quad \text{for some } \varphi(\mathfrak{z}) > 0,$$

then Theorem 4 implies that there exists a unique $\mathfrak{z}^\star \in \mathscr{C}$, such that

$$\mathscr{F}\mathfrak{z}^\star = \mathfrak{z}^\star \quad \text{and} \quad d(\mathfrak{z}, \mathfrak{z}^\star) \leq \varsigma\varphi(\mathfrak{z}).$$

Example 2. *Consider the functional equation given below*

$$\begin{aligned} \mathfrak{z}(x) \;=\; & \wp_1 \hbar_1(x)\mathfrak{z}\Big((k-\mu)\Delta\tilde{v}^{-1}x + (1-(k-\mu)\Delta\tilde{v}^{-1})v\Big) \\ & + \wp_2 \hbar_1(x)\mathfrak{z}\Big((\ell-\mu)\Delta\tilde{v}^{-1}x + (1-(\ell-\mu)\Delta\tilde{v}^{-1})v\Big) \\ & + \wp_1 \hbar_2(x)\mathfrak{z}\Big((m-\mu)\Delta\tilde{v}^{-1}x + (v-m)\Delta\tilde{v}^{-1}\mu\Big) \\ & + \wp_2 \hbar_2(x)\mathfrak{z}\Big((p-\mu)\Delta\tilde{v}^{-1}x + (v-p)\Delta\tilde{v}^{-1}\mu\Big), \end{aligned} \tag{14}$$

for all $x \in \mathscr{J}$ with $\mu < k \leq \ell \leq m \leq p < v$ and $\mathfrak{z} \in \mathscr{T}$. If we set the mappings $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ by

$$\begin{cases} \varrho_1(x) = (k-\mu)\Delta\tilde{v}^{-1}x + (1-(k-\mu)\Delta\tilde{v}^{-1})v, \\ \varrho_2(x) = (\ell-\mu)\Delta\tilde{v}^{-1}x + (1-(\ell-\mu)\Delta\tilde{v}^{-1})v, \\ \varrho_3(x) = (m-\mu)\Delta\tilde{v}^{-1}x + (v-m)\Delta\tilde{v}^{-1}\mu, \\ \varrho_4(x) = (p-\mu)\Delta\tilde{v}^{-1}x + (v-p)\Delta\tilde{v}^{-1}\mu, \end{cases}$$

for all $x \in \mathscr{J}$, so (1) decreases to the Equation (14). It is easy to see that the mappings $\varrho_1 - \varrho_4$ satisfy ($\mathcal{A}_2$), i.e.,

$$\begin{cases} |\varrho_1(x) - \varrho_1(y)| \leq (k-\mu)\Delta\tilde{v}^{-1}|x-y|, \\ |\varrho_2(x) - \varrho_2(y)| \leq (\ell-\mu)\Delta\tilde{v}^{-1}|x-y|, \\ |\varrho_3(x) - \varrho_3(y)| \leq (m-\mu)\Delta\tilde{v}^{-1}|x-y|, \\ |\varrho_4(x) - \varrho_4(y)| \leq (p-\mu)\Delta\tilde{v}^{-1}|x-y|, \end{cases}$$

for all $x, y \in \mathscr{J}$, where $\varpi_1 = (k-\mu)\Delta\tilde{v}^{-1}$, $\varpi_2 = (\ell-\mu)\Delta\tilde{v}^{-1}$, $\varpi_3 = (m-\mu)\Delta\tilde{v}^{-1}$, $\varpi_4 = (p-\mu)\Delta\tilde{v}^{-1}$ are contractive coefficients, respectively. Here, if ($\mathcal{A}_1$) is satisfied with

$$\tilde{\Upsilon} = 4\Big|(p-\mu)\Delta\tilde{v}^{-1}\Big| < 1,$$

then the mapping $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ defined on (14) is a BCM. Thus, it fulfills all the conditions of Corollary 1 and, therefore, we obtain the results related to the existence of a solution to the functional Equation (14).

If we define $\mathfrak{z}_0 = \mathbf{I} \in \mathscr{C}$ as a starting approximation (whereas $\mathbf{I}$ is an identity function), then by Corollary 2, we have a unique solution of (14) followed by the iteration process stated below:

$$
\begin{aligned}
\mathfrak{z}_1(x) &= \Delta\tilde{\upsilon}^{-3}\left[\begin{array}{c}(p\gamma - \upsilon p - \gamma m + \ell\upsilon - \ell\gamma + \gamma k + m\mu - k\mu)x^2 \\ +(\gamma\upsilon p + \upsilon^2 p + nm\upsilon - \ell\upsilon^2 + \upsilon^3 + \ell\gamma\upsilon - \gamma k\upsilon - \gamma\mu p + \mu\upsilon p + nm\mu \\ -m\mu\upsilon - \ell\mu\upsilon + \ell\gamma\mu + k\mu\upsilon - 3\mu\upsilon^2 - \gamma k\mu - m\mu^2 + 3\mu^2\upsilon + k\mu^2 - \mu^3)x \\ +(\gamma\mu\upsilon p - \mu\upsilon^2 p - \gamma m\mu\upsilon - \ell\gamma\mu\upsilon + \gamma k\mu\upsilon + \ell\mu\upsilon^2 + m\mu^2\upsilon - k\mu^2\upsilon)\end{array}\right], \\
\mathfrak{z}_2(x) &= \wp_1\hbar_1(x)\mathfrak{z}_1\Big((k-\mu)\Delta\tilde{\upsilon}^{-1}x + (1-(k-\mu)\Delta\tilde{\upsilon}^{-1})\upsilon\Big) \\
&\quad +\wp_2\hbar_1(x)\mathfrak{z}_1\Big((\ell-\mu)\Delta\tilde{\upsilon}^{-1}x + (1-(\ell-\mu)\Delta\tilde{\upsilon}^{-1})\upsilon\Big) \\
&\quad +\wp_1\hbar_2(x)\mathfrak{z}_1\Big((m-\mu)\Delta\tilde{\upsilon}^{-1}x + (\upsilon-m)\Delta\tilde{\upsilon}^{-1}\mu\Big) \\
&\quad +\wp_2\hbar_2(x)\mathfrak{z}_1\Big((p-\mu)\Delta\tilde{\upsilon}^{-1}x + (\upsilon-p)\Delta\tilde{\upsilon}^{-1}\mu\Big), \\
&\ \ \vdots \\
\mathfrak{z}_n(x) &= \wp_1\hbar_1(x)\mathfrak{z}_{n-1}\Big((k-\mu)\Delta\tilde{\upsilon}^{-1}x + (1-(k-\mu)\Delta\tilde{\upsilon}^{-1})\upsilon\Big) \\
&\quad +\wp_2\hbar_1(x)\mathfrak{z}_{n-1}\Big((\ell-\mu)\Delta\tilde{\upsilon}^{-1}x + (1-(\ell-\mu)\Delta\tilde{\upsilon}^{-1})\upsilon\Big) \\
&\quad +\wp_1\hbar_2(x)\mathfrak{z}_{n-1}\Big((m-\mu)\Delta\tilde{\upsilon}^{-1}x + (\upsilon-m)\Delta\tilde{\upsilon}^{-1}\mu\Big) \\
&\quad +\wp_2\hbar_2(x)\mathfrak{z}_{n-1}\Big((p-\mu)\Delta\tilde{\upsilon}^{-1}x + (\upsilon-p)\Delta\tilde{\upsilon}^{-1}\mu\Big),
\end{aligned}
$$

for all $n \in \mathbb{N}$.

As $0 \le \tilde{\Upsilon} < 1$, we have

$$\varsigma := \frac{1}{1-\tilde{\Upsilon}} > 0.$$

If a function $\mathfrak{Z} \in \mathscr{C}$ satisfies the inequality

$$d(\mathscr{F}\mathfrak{Z}, \mathfrak{Z}) \le \varphi(\mathfrak{Z}), \quad \text{for some } \varphi(\mathfrak{Z}) > 0,$$

then Theorem 4 implies that there exists a unique $\mathfrak{Z}^\star \in \mathscr{C}$, such that

$$\mathscr{F}\mathfrak{Z}^\star = \mathfrak{Z}^\star \quad \text{and} \quad d(\mathfrak{Z}, \mathfrak{Z}^\star) \le \varsigma\varphi(\mathfrak{Z}).$$

Example 3. *Consider the functional equation given below*

$$\mathfrak{Z}(x) = \gamma x\mathfrak{Z}\left(\frac{x+2}{3}\right) + (1-\gamma)x\mathfrak{Z}\left(\frac{x+6}{7}\right) + \gamma(1-x)\mathfrak{Z}\left(\frac{x}{11}\right) + (1-\gamma)(1-x)\mathfrak{Z}\left(\frac{2x}{17}\right), \tag{15}$$

for all $x \in \mathscr{J} = [0,1]$ and $\mathfrak{Z} \in \mathscr{T}$. If we set the mappings $\varrho_1 - \varrho_4 : \mathscr{J} \to \mathscr{J}$ by

$$\varrho_1(x) = \frac{x+2}{3}, \quad \varrho_2(x) = \frac{x+6}{7}, \quad \varrho_3(x) = \frac{x}{11}, \quad \varrho_4(x) = \frac{2x}{17},$$

for all $x \in \mathscr{J}$, so (1) decreases to the Equation (15). It is easy to see that the mappings $\varrho_1 - \varrho_4$ satisfy ($\mathcal{A}_2$), i.e.,

$$\begin{cases} |\varrho_1(x) - \varrho_1(y)| \le \dfrac{1}{3}|x-y|, \\ |\varrho_2(x) - \varrho_2(y)| \le \dfrac{1}{7}|x-y|, \\ |\varrho_3(x) - \varrho_3(y)| \le \dfrac{1}{11}|x-y|, \\ |\varrho_4(x) - \varrho_4(y)| \le \dfrac{2}{17}|x-y|, \end{cases}$$

for all $x, y \in \mathscr{J}$, *where* $\varpi_1 = \frac{1}{3}$, $\varpi_2 = \frac{1}{7}$, $\varpi_3 = \frac{1}{11}$, $\varpi_4 = \frac{2}{17}$ *are contractive coefficients, respectively. Here, if* $(\mathcal{A}_1)$ *is satisfied with*

$$\Upsilon = \frac{1286}{3927}x + \frac{62}{119} < 1,$$

then the mapping $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ *defined on (15) is a BCM. Thus it fulfills all the conditions of Theorem 2 and, therefore, we obtain the results related to the existence of a solution to the functional Equation (15).*

If we define $\mathfrak{z}_0 = \mathbf{I} \in \mathscr{C}$ *as a starting approximation (whereas* $\mathbf{I}$ *is an identity function), then by Theorem 2, we have a unique solution of (15) followed by the iteration process stated below:*

$$\begin{aligned}
\mathfrak{z}_1(x) &= \frac{1}{3927}\Big[853\gamma x^2 - 853\gamma x + 99x^2 + 3828x\Big], \\
\mathfrak{z}_2(x) &= \frac{1}{4943637468}\begin{bmatrix} 1265337942\gamma x^2 + 95473731\gamma^2 x^3 - 94357154\gamma^2 x^2 \\ -209916476\gamma^2 x + 26046658\gamma x^3 - 1930757796\gamma\, x \\ +279494028x^2 + 1736955x^3 + 5510579580x \end{bmatrix} \\
&\vdots \\
\mathfrak{z}_n(x) &= \gamma x \mathfrak{z}_{n-1}\left(\frac{x+2}{3}\right) + (1-\gamma)x\mathfrak{z}_{n-1}\left(\frac{x+6}{7}\right) + \gamma(1-x)\mathfrak{z}_{n-1}\left(\frac{x}{11}\right) \\
&\quad + (1-\gamma)(1-x)\mathfrak{z}_{n-1}\left(\frac{2x}{17}\right),
\end{aligned}$$

for all $n \in \mathbb{N}$.

As $0 \leq \Upsilon < 1$, *we have*

$$\varsigma := \frac{1}{1-\Upsilon} = \frac{3927}{1891 - 1286\gamma} > 0.$$

If a function $\mathfrak{z} \in \mathscr{C}$ *satisfies the inequality*

$$d(\mathscr{F}\mathfrak{z}, \mathfrak{z}) \leq \varphi(\mathfrak{z}), \quad for\ some\ \varphi(\mathfrak{z}) > 0,$$

then Theorem 4 implies that there exists a unique $\mathfrak{z}^\star \in \mathscr{C}$, *such that*

$$\mathscr{F}\mathfrak{z}^\star = \mathfrak{z}^\star \quad and \quad d(\mathfrak{z}, \mathfrak{z}^\star) \leq \left(\frac{3927}{1891 - 1286\gamma}\right)\varphi(\mathfrak{z}).$$

5. Conclusions

In population biology, predator–prey or host–parasite relationships are arguably the most often simulated phenomena. According to such models, a predator has two prey options, and the solution is determined when the predator is drawn to a certain prey type. In this research, we presented a generic functional equation that can cover numerous learning theory models in the current study. Additionally, we analyzed the solution to the proposed stochastic equation for its existence, uniqueness, and stability. To demonstrate the significance of our findings, we presented two examples. Our method is novel and can be applied to many mathematical models associated with mathematical psychology and learning theory.

Finally, we present the following open problems for those who are interested in this research.

Question 1: what would happen if a predator does not approach any prey and remains stuck to its original position?

Question 2: is there another way to establish the conclusions from Theorems 2 and 3?

Author Contributions: Conceptualization, A.T., W.A. and Z.D.M.; methodology, A.T., W.A., A.S. and R.G.; validation, A.T., R.G., Z.D.M. and W.A.; formal analysis, A.T. and R.G.; writing—original draft preparation, A.T., W.A., A.S. and Z.D.M.; funding acquisition, A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bazykin, A.D. *Nonlinear Dynamics of Interacting Populations*; World Scientific: Singapore, 1998.
2. Ma, Z.; Wang, W. Asymptotic behavior of predator–Prey system with time dependent coefficients. *Appl. Anal.* **1989**, *34*, 79–90.
3. Chen, F.; Shi, C. Global attractivity in an almost periodic multi-species nonlinear ecological model. *Appl. Math. Comput.* **2006**, *180*, 376–392. [CrossRef]
4. Yuan, S.; Wu, D.; Lan, G.; Wang, H. Noise-induced transitions in a nonsmooth Producer–Grazer model with stoichiometric constraints. *Bull. Math. Biol.* **2020**, *82*, 1–22. [CrossRef] [PubMed]
5. Yu, X.; Yuan, S.; Zhang, T. Survival and ergodicity of a stochastic phytoplankton–zooplankton model with toxin-producing phytoplankton in an impulsive polluted environment. *Appl. Math. Comput.* **2019**, *347*, 249–264. [CrossRef]
6. Zhang, T.; Liu, X.; Meng, X.; Zhang, T. Spatio-temporal dynamics near the steady state of a planktonic system. *Comput. Math. Appl.* **2018**, *12*, 4490–4504. [CrossRef]
7. Xu, C.; Yuan, S.; Zhang, T. Global dynamics of a predator–prey model with defense mechanism for prey. *Appl. Math. Lett.* **2016**, *62*, 42–48. [CrossRef]
8. Tian, Y.; Zhang, T.; Sun, K. Dynamics analysis of a pest management prey–predator model by means of interval state monitoring and control. *Nonlinear Anal. Hybrid Syst.* **2017**, *23*, 122–141. [CrossRef]
9. Liu, M.; He, X.; Yu, J. Dynamics of a stochastic regime-switching predator–prey model with harvesting and distributed delays. *Nonlinear Anal. Hybrid Syst.* **2018**, *28*, 87–104. [CrossRef]
10. Yu, X.; Yuan, S.; Zhang, T. The effects of toxin-producing phytoplankton and environmental fluctuations on the planktonic blooms. *Nonlinear Dyn.* **2018**, *91*, 1653–1668. [CrossRef]
11. Zhao, S.; Yuan, S.; Wang, H. Threshold behavior in a stochastic algal growth model with stoichiometric constraints and seasonal variation. *J. Differ. Equ.* **2020**, *9*, 5113–5139. [CrossRef]
12. Zhu, G.; Meng, X.; Chen, L. The dynamics of a mutual interference age structured predator–prey model with time delay and impulsive perturbations on predators. *Appl. Math. Comput.* **2010**, *216*, 308–316. [CrossRef]
13. Diz-Pita, É.; Otero-Espinar, M.V. Predator–Prey Models: A Review of Some Recent Advances. *Mathematics* **2021**, *9*, 1783. [CrossRef]
14. Banerjee, M.; Mukherjee, N.; Volpert, V. Prey-predator model with a nonlocal bistable dynamics of prey. *Mathematics* **2018**, *6*, 41. [CrossRef]
15. Yang, R.; Zhao, X.; An, Y. Dynamical analysis of a delayed diffusive predator–prey model with additional food provided and anti-predator behavior. *Mathematics* **2022**, *10*, 469. [CrossRef]
16. Bai, D.; Zhang, X. Dynamics of a predator–prey model with the additive predation in prey. *Mathematics* **2022**, *10*, 655. [CrossRef]
17. Iqbal, N.; Wu, R. Pattern formation by fractional cross-diffusion in a predator-prey model with Beddington-DeAngelis type functional response. *Int. J. Mod. Phys. A* **2019**, *33*, 1950286. [CrossRef]
18. Jia, W.; Xu, Y.; Li, D.; Hu, R. Stochastic analysis of predator-prey models under combined Gaussian and poisson white noise via stochastic averaging method. *Entropy* **2021**, *23*, 1208. [CrossRef]
19. Rosenzweig, M.L.; Macarthur, R.H. Graphical representation and stability conditions of predator-prey Interactions. *Am. Nat.* **1963**, *97*, 209–223. [CrossRef]
20. Zhu, W.Q.; Yang, Y.Q. Stochastic averaging of quasi-nonintegrable-Hamiltonian systems. *J. Appl. Mech.-Trans. ASME* **1997**, *64*, 157–164. [CrossRef]
21. Roberts, J.B.; Spanos, P.D. Stochastic averaging: An approximate method of solving random vibration problems. *Int. J. Non-Linear Mech.* **1986**, *21*, 111–134. [CrossRef]
22. Zhu, W. Stochastic averaging methods in random vibration. *Appl. Mech. Rev.* **1988**, *41*, 189–199. [CrossRef]
23. Huang, Z.L.; Zhu, W.Q. Stochastic averaging of quasi-integrable Hamiltonian systems under combined harmonic and white noise excitations. *Int. J. Non-Linear Mech.* **2004**, *39*, 1421–1434. [CrossRef]
24. Jia, W.T.; Xu, Y.; Liu, Z.H.; Zhu, W.Q. An asymptotic method for quasi-integrable Hamiltonian system with multi-time-delayed feedback controls under combined Gaussian and Poisson white noises. *Nonlinear Dyn.* **2017**, *90*, 2711–2727. [CrossRef]
25. Pan, S.S.; Zhu, W.Q. Dynamics of a prey-predator system under Poisson white noise excitation. *Acta Mech. Sin.* **2014**, *30*, 739–745. [CrossRef]

26. Jia, W.T.; Xu, Y.; Li, D.X. Stochastic dynamics of a time-delayed ecosystem driven by Poisson white noise excitation. *Entropy* **2018**, *20*, 143. [CrossRef] [PubMed]
27. Gu, X.D.; Zhu, W.Q. Stochastic optimal control of predator-prey ecosystem by using stochastic maximum principle. *Nonlinear Dyn.* **2016**, *85*, 1177–1184. [CrossRef]
28. Bush, A.A.; Wilson, T.R. Two-choice behavior of paradise fish. *J. Exp. Psych.* **1956**, *51*, 315–322. [CrossRef]
29. Bush, R.; Mosteller, F. *Stochastic Models for Learning*; Wiley: New York, NY, USA, 1955.
30. Istrăţescu, V.I. On a functional equation. *J. Math. Anal. Appl.* **1976**, *56*, 133–136. [CrossRef]
31. Berinde, V.; Khan, A.R. On a functional equation arising in mathematical biology and theory of learning. *Creat. Math. Inform.* **2015**, *24*, 9–16. [CrossRef]
32. Turab, A.; Sintunavarat, W. On the solutions of the two preys and one predator type model approached by the fixed point theory. *Sādhanā* **2020**, *45*, 211. [CrossRef]
33. Turab, A.; Sintunavarat, W. On the solution of the traumatic avoidance learning model approached by the Banach fixed point theorem. *J. Fixed Point Theory Appl.* **2020**, *22*, 50. [CrossRef]
34. Turab, A.; Sintunavarat, W. On analytic model for two-choice behavior of the paradise fish based on the fixed point method. *J. Fixed Point Theory Appl.* **2019**, *21*, 56; Erratum in *J. Fixed Point Theory Appl.* **2020**, *22*, 82. [CrossRef]
35. Turab, A.; Ali, A.; Park, C. A unified fixed point approach to study the existence and uniqueness of solutions to the generalized stochastic functional equation emerging in the psychological theory of learning. *AIMS Math.* **2022**, *7*, 5291–5304. [CrossRef]
36. Turab, A.; Park, W.-G.; Ali, W. Existence, uniqueness, and stability analysis of the probabilistic functional equation emerging in mathematical biology and the theory of learning. *Symmetry* **2021**, *13*, 1313. [CrossRef]
37. Debnath, P. A mathematical model using fixed point theorem for two-choice behavior of rhesus monkeys in a noncontingent environment. In *Metric Fixed Point Theory*; Debnath, P., Konwar, N., Radenović, S., Eds.; Forum for Interdisciplinary Mathematics; Springer: Singapore, 2021. [CrossRef]
38. FUTURITY. 2019. Available online: https://www.futurity.org/predator-prey-cycles-coexistence-2238732/ (accessed on 25 March 2022).
39. Geek Reply. 2015. Available online: https://geekreply.com/science/2015/09/09/predator-prey-ratio-may-reveal-a-new-law-of-nature (accessed on 25 March 2022).
40. Aydi, H.; Karapinar, E.; Rakocevic, V. Nonunique fixed point theorems on b-metric spaces via simulation functions. *Jordan J. Math. Stat.* **2019**, *12*, 265–288.
41. Karapinar, E. Ciric type nonunique fixed points results: A review. *Appl. Comput. Math. Int. J.* **2019**, *1*, 3–21.
42. Alsulami, H.H.; Karapinar, E.; Rakocevic, V. Ciric type nonunique fixed point theorems on b-metric spaces. *Filomat* **2017**, *31*, 3147–3156. [CrossRef]
43. Gopal, D.; Abbas, M.; Patel, D.K.; Vetro, C. Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation. *Acta Math. Sci.* **2016**, *36*, 957–970. [CrossRef]
44. Lakzian, H.; Gopal, D.; Sintunavarat, W. New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations. *J. Fixed Point Theory Appl.* **2016**, *18*, 251–266. [CrossRef]
45. Banach, S. Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
46. Agarwal, P.; Jleli, M.; Samet, B. Banach contraction principle and applications. In *Fixed Point Theory in Metric Spaces*; Springer: Singapore, 2018; pp. 1–23. [CrossRef]
47. Hyers, D.H.; Isac, G.; Rassias, T.M. *Stability of Functional Equations in Several Variables*; Birkhauser: Basel, Switzerland, 1998.
48. Morales, J.S.; Rojas, E.M. Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay. *Int. J. Nonlinear Anal. Appl.* **2011**, *2*, 1–6.
49. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
50. Bae, J.H.; Park, W.G. A fixed point approach to the stability of a Cauchy-Jensen functional equation. *Abstr. Appl. Anal.* **2012**, *2012*, 205160. [CrossRef]
51. Gachpazan, M.; Bagdani, O. Hyers-Ulam stability of nonlinear integral equation. *Fixed Point Theory Appl.* **2010**, 927640, 1–6. [CrossRef]