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A Method for the Solution of Coupled System of Emden–Fowler–Type Equations

Aishah A. Alsulami, Mariam AL-Mazmumy, Huda O. Bakodah and Nawal Alzaid *

Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah 80327, Saudi Arabia; aalsulami1183.stu@uj.edu.sa (A.A.A.); mhalmazmumy@uj.edu.sa (M.A.-M.); hobakodah@uj.edu.sa (H.O.B.)
* Correspondence: naalzaid@uj.edu.sa

Abstract: A dependable semi-analytical method via the application of a modified Adomian Decomposition Method (ADM) to tackle the coupled system of Emden–Fowler-type equations has been proposed. More precisely, an effective differential operator together with its corresponding inverse is successfully constructed. Moreover, this operator is able to navigate to the closed-form solution easily without resorting to converting the coupled system to a system of Volterra integral equations; as in the case of a well-known reference in the literature. Lastly, the effectiveness of the method is demonstrated on some coupled systems of the governing model, and a speedier convergence rate was noted.

Keywords: modified ADM; coupled ODEs; singular ODEs; Emden-Fowler equations; Lane-Emden equations

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1. Introduction

Ordinary differential equations arise in different areas of applied sciences, such as engineering, physics, and applied science. Numerous methods have been used to determine solutions of these problems. The solutions to ordinary differential equations exhibit symmetries and this property can be exploited to find those solutions. Many real-life problems are mathematically modeled through Initial-Value Problems (IVPs) of nonlinear Ordinary Differential Equations (ODEs), including, for instance, models, such as the Emden–Fowler equation [1–4]. The Emden–Fowler equation is a singular second-order nonlinear ODE that arises in modeling various phenomena in thermodynamics and astrophysics to mention a few. This important equation in the presence of inhomogeneous term $g(x)$ reads the following equation [5]

$$u'' + \frac{r}{x}u' + f(u) = g(x), \tag{1}$$

where $r > 0$ is a positive constant, and $g(x)$ and $f(u)$ are prescribed functions in x and u, sequentially.

Moreover, Equation (1) reduces to the famous Lane–Emden [6] equation when

$$r = 2, \qquad f(u) = u^m, \qquad g(x) = 0. \tag{2}$$

This equation is one of the fundamental equations that are used to study stellar structures. The model equally has applications in modeling thermal and isothermal behaviors of spherical gas clouds, and also works magically in the theory and application of harmonic current to mention a few; see [1–6] and the references therein for other vital methods to treat such singular nonlinear models. Additionally, one may find various approaches in both the recent and past literature to study these types of models, including, for instance, an analytical procedure via the combination of the Laplace transform and homotopy perturbation approach to study a class of Lane–Emden equations by Eltayeb [7], the homotopy perturbation technique for the class of Emden–Fowler equations by Chowdhury

and Hashim [8], the Homotopy analysis procedure for the solution of Emden–Fowler-type models by Bataineh et al. [9], and the Harr wavelet numerical process for the solution of Emden–Fowler equations by Singh et al. [10]; see also methods for the solution of time-dependent Emden–Fowler type and Lane–Emden–Fowler equations, and the generalized Thomos–Fermi equations in [11,12], respectively.

Further, there exist several variants of Emden–Fowler equation given in Equation (1), including the coupled system of Emden–Fowler-type equations that reads as follows [5,13–15]

$$\begin{aligned} u_1'' + \frac{r_1}{x}u_1' + f_1(u_1, u_2) &= g_1(x), \\ u_2'' + \frac{r_2}{x}u_2' + f_2(u_1, u_2) &= g_2(x), \end{aligned} \tag{3}$$

where $r_1 > 0$, $r_2 > 0$ are real constants, and $g_1(x)$ and $g_2(x)$ are given functions of x; while $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ are prescribed analytic nonlinear functions of u_1 and u_2. This equation arises in certain physical processes, such as population growth, pattern formation, and chemical reaction, among others. More so, one would see the application of the various methods to study such a coupled model, such as the modified ADM to solve certain systems of Emden–Fowler type equations by Biazar [13], the variational iteration process for the systems of Emden–Fowler equations by Wazwaz [14]; and, lastly, an analytical procedure by Singh [15] to treat certain systems of Lane–Emden–Fowler type equations, among others.

Furthermore, the literature is full of diverse methods to study ODEs, such as the classical Adomian Decomposition Method (ADM) [16,17] and its different reliable extensions and modifications [18–20]. No wonder, the ADM procedure and its modifications have been greatly used in both the past and recent times to solve different types of differential equations, integral equations, and mixed integro-differential equations. In fact, this is associated with the flexibility of the method in handling both linear and nonlinear problems via the application of domain decomposition. Various forms of tricky models have been successfully solved through the application of this method and its modifications. To state a few, we recall how certain nonlinear equations were solved using integral transform-ADM methods in [21,22], the ADM modification method for the solution of KdV equation [23], solution of nonlinear wave propagation model via ADM [24], and the study of fractional heat diffusion model in the nonlocal setting by Laplace–ADM approach [25], to mention a few. However, the current study aims at studying the coupled system of the Emden–Fowler-type equations by utilizing a dependable semi-analytical method. This method is based upon the application of the modified ADM by Hasan [3]. Additionally, the effectiveness of the method will be demonstrated on certain systems of the coupled model. These systems will be taken from the existing literature as test problems. What is more, the current paper is arranged in the following way: Section 2 outlines the method of the study; while Section 3 demonstrates the presented method given in Section 2 on certain test problems, and Section 4 presents some concluding remarks.

2. Methodology

This section gives a mathematical algorithm for the determination of recursive solution of the inhomogeneous coupled system of Emden–Fowler-type equations. This algorithm is based upon the modified ADM procedure for the solution of various functional equations. Different differential operators and their corresponding inverse integral operators will be recalled and thereafter used to treat certain forms of the coupled equations under consideration.

Let us consider the inhomogeneous coupled system of Emden–Fowler-type equations

$$\begin{cases} u_j'' + \frac{(2h_j+r_j)}{x}u_j' + \frac{(h_j-1)(h_j+r_j)}{x^2}u_j + N_j(u_1, u_2) = g_j(x), \\ \qquad\qquad j = 1, 2, \quad h_j \geq 1, \quad r_j \geq -1, \end{cases} \tag{4}$$

subject to the following prescribed initial data

$$u_j(0) = \xi_j, \;\; u_j'(0) = \zeta_j, \quad j = 1, 2, \tag{5}$$

where h_j and r_j are real constants, N_j are the nonlinear functions of u_j; while $g_j(x)$ are known functions, all for $j = 1, 2$.

So, rewriting the system given in Equation (4) through a differential operator notation L becomes

$$Lu_j = g_j(x) - N_j(u_1, u_2), \qquad j = 1, 2, \tag{6}$$

where the differential operator L and its corresponding two-fold inverse integral operator L^{-1} are considered in this study based on the ADM modification by Hassan [3] as follows

$$\begin{aligned} L(.) &= x^{-h_j} \frac{d}{dx}\left(x^{-r_j} \frac{d}{dx} x^{h_j + r_j}\right)(.), \\ L^{-1}(.) &= x^{-(h_j + r_j)} \int_0^x x^{r_j} \int_0^x x^{h_j}(.) dx dx. \end{aligned} \tag{7}$$

More so, these operators are specifically devised in the present examination to study the coupled Emden–Fowler-type equations. What is more, applying the inverse operator L^{-1} to the first three terms $u_j'' + \frac{(2h_j + r_j)}{x} u_j' + \frac{(h_j - 1)(h_j + r_j)}{x^2} u_j$ of Equation (6) yields the following

$$u_j = u_j(0) + u_j'(0)x + L^{-1}(g_j(x)) - L^{-1} N_j(u_1, u_2). \tag{8}$$

Therefore, the ADM decomposes the solutions $u_j(x)$ and the nonlinear functions $N_j(u_1, u_2)$ for $j = 1, 2$, through infinite series of the following forms

$$u_j(x) = \sum_{n=0}^{\infty} u_{jn}(x), \quad j = 1, 2, \tag{9}$$

and

$$N_j(u_1, u_2) = \sum_{n=0}^{\infty} A_{jn}, \quad j = 1, 2, \tag{10}$$

where the components $u_{jn}(x)$ are recursively computed; while the Adomian polynomials A_{jn}'s corresponding to the nonlinear functions N_j are acquired through the following relation [16,17]

$$A_{jn} = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{k=0}^{n} \lambda^k u_{jn} \right) \right]_{\lambda = 0}, \quad j = 1, 2, \quad n = 0, 1, 2, \ldots \tag{11}$$

So, substituting Equations (9) and (10) into Equation (8) yields

$$\sum_{n=0}^{\infty} u_{jn} = \xi_j + \zeta_j x + L^{-1}(g_j(x)) - L^{-1} \sum_{n=0}^{\infty} A_{jn}, \tag{12}$$

of which the components $u_{jn}(x)$ are recursively obtained via the ADM process as follows

$$\begin{cases} u_{j0} = \xi_j + \zeta_j x + L^{-1}(g_j(x)), \\ u_{j(n+1)} = -L^{-1} A_{jn}, \quad n \geq 0, \end{cases} \tag{13}$$

for $j = 1, 2$.

We, therefore, remark here that this method that is based on the modification of the standards ADM and presented on the coupled system of Emden–Fowler-type equations has numerous advantages over the approach presented in [10]. However, the most notable advantage of the method is its ability to reveal a convergent series solution without resorting

to converting the coupled system to a system of Volterra integral equations; as in the case of the algorithm that was presented in Wazwaz et al. [20].

In addition, the convergence of the ADM was discussed by Cherruault [26], Cherruault and Adomian [27], Abbaoui and Cherruault [28]. Cherruault [26] has given the first proof of convergence of the ADM using the fixed point theorems for abstract functional equations. In [27], Cherruault and Adomian have avoided this type of hypothesis which is difficult to satisfy and to verify in physical problems. Additionally, in [29], it has been proven that the Adomian polynomials A_n depend only on $u_0, u_1, ..., u_n$. Furthermore, Gabet in [30] generalized the convergence results obtained by Cherruaulet in Banach space; while Babolian and Biazar [31] used the Cherruaulet's definition and considered the order of convergence of the method.

Thus, in what follows, we make consideration to several numerical test problems featuring both the linear and nonlinear coupled systems of Emden–Fowler-type equations.

3. Applications

The present section examines the application of the proposed algorithms on different test singular problems of the coupled system of Emden–Fowler-type equations.

Example 1. *Consider the coupled system of Lane–Emden-type equations when* $r_1 = 3, r_2 = 2$ *as follows [5,20]*

$$\begin{cases} u_1'' + \frac{3}{x}u_1' - 4(u_1 + u_2) = 0, \\ u_2'' + \frac{2}{x}u_2' + 3(u_1 + u_2) = 0, \end{cases} \tag{14}$$

with initial conditions

$$\begin{cases} u_1(0) = 1 = u_2(0), \\ u_1'(0) = 0 = u_2'(0). \end{cases} \tag{15}$$

Accordingly, we make use of $2h_1 + r_1 = 3$ and $(h_1 - 1)(h_1 + r_1) = 0$ in the first ODE to obtain $h_1 = 1, r_1 = 1$. Therefore, substituting these values into Equation (7) gives the following differential operator L and its inverse L^{-1} as follows

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(x^{-1}\frac{d}{dx}x^2\right)(.), \\ L^{-1}(.) &= x^{-2}\int_0^x x\int_0^x x(.)dxdx. \end{aligned} \tag{16}$$

Furthermore, we use the relations $2h_2 + r_2 = 2$ and $(h_2 - 1)(h_2 + r_2) = 0$ in the second ODE to obtain $h_2 = 1, r_2 = 0$. This yields from Equation (7) the following operators

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(\frac{d}{dx}x\right)(.), \\ L^{-1}(.) &= x^{-1}\int_0^x \int_0^x x(.)dxdx. \end{aligned} \tag{17}$$

Therefore, Equation (14) in an operator form is expressed as

$$\begin{cases} Lu_1 = 4(u_1 + u_2), \\ Lu_2 = -3(u_1 + u_2), \end{cases} \tag{18}$$

such that after operating L^{-1} on the above equation gives the following recursive schemes

$$\begin{cases} u_{10} = 1, \\ u_{1(n+1)} = 4L^{-1}(u_1 + u_2), \quad n \geq 0, \end{cases}$$

and

$$\begin{cases} u_{20} = 1, \\ u_{2(n+1)} = -3L^{-1}(u_1 + u_2), \ \ n \geq 0. \end{cases}$$

Therefore, we express some of the components of the above recurrent relations as follows

$$u_{10} = 1, \qquad u_{20} = 1,$$

$$u_{11} = 4L^{-1}(u_{10} + u_{20}) = x^2, \qquad u_{21} = -3L^{-1}(u_{10} + u_{20}) = -x^2,$$

$$u_{12} = 4L^{-1}(u_{11} + u_{21}) = 0, \qquad u_{22} = -3L^{-1}(u_{11} + u_{21}) = 0,$$

leading to the closed-form solution for the system as follows

$$\begin{cases} u_1 = 1 + x^2, \\ u_2 = 1 - x^2. \end{cases} \tag{19}$$

Example 2. *Consider the coupled system of Lane–Emden-type equations when* $r_1 = 1, r_2 = 3$ *as follows [20]*

$$\begin{cases} u_1'' + \frac{1}{x}u_1' - u_2^3(u_1^2 + 1) = 0, \\ \\ u_2'' + \frac{3}{x}u_2' + u_2^5(u_1^2 + 3) = 0, \end{cases} \tag{20}$$

with initial conditions

$$\begin{cases} u_1(0) = 1 = u_2(0), \\ u_1'(0) = 0 = u_2'(0). \end{cases} \tag{21}$$

Here, from the first ODE, let us make a transformation using $2h_1 + r_1 = 1$ and $(h_1 - 1)(h_1 + r_1) = 0$. This gives that $h_1 = 1$, $r_1 = -1$. Therefore, we devise the following differential operator L and its inverse L^{-1} by substituting these values into Equation (7) as follows substitution

$$\begin{aligned} L(.) =& x^{-1}\frac{d}{dx}\left(x\frac{d}{dx}\right)(.), \\ L^{-1}(.) =& \int_0^x x^{-1}\int_0^x x(.)dxdx. \end{aligned} \tag{22}$$

Similarly, from the second ODE, we use the following relations $2h_2 + r_2 = 3$ and $(h_2 - 1)(h_2 + r_2) = 0$. This results in getting $h_2 = 1$, $r_2 = 1$ such that the following operators are obtained from Equation (7) as follows

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(x^{-1}\frac{d}{dx}x^2\right)(.), \\ L^{-1}(.) &= x^{-2}\int_0^x x\int_0^x x(.)dxdx. \end{aligned} \tag{23}$$

So, the system given in Equation (20) becomes in an operator form the following

$$\begin{cases} Lu_1 = u_2^3(u_1^2 + 1), \\ Lu_2 = -u_2^5(u_1^2 + 3), \end{cases} \tag{24}$$

such that after operating L^{-1} on the above equation gives the following recursive schemes

$$\begin{cases} u_{10} = 1, \\ u_{1(n+1)} = L^{-1}(A_{1n}), \ \ n \geq 0, \end{cases} \tag{25}$$

and

$$\begin{cases} u_{20} = 1, \\ u_{2(n+1)} = -L^{-1}(A_{2n}), \ \ n \geq 0, \end{cases} \tag{26}$$

where A_{1n}'s and A_{2n}'s are the Adomian polynomials corresponding to the nonlinear terms $u_2^3(u_1^2+1)$ and $u_2^5(u_1^2+3)$ given, respectively, as follows

$$\begin{aligned} &A_{10} = u_{20}^3(u_{10}^2+1), \\ &A_{11} = (2u_{20}^3u_{10}u_{11} + 3u_{20}^2u_{10}^2u_{21} + 3u_{20}^2u_{21}), \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} &A_{20} = u_{20}^5(u_{10}^2+3), \\ &A_{21} = (2u_{20}^5u_{10}u_{11} + 5u_{20}^4u_{10}^2u_{21} + 15u_{20}^4u_{21}), \\ &\vdots \end{aligned}$$

Then, we express some of the components of the above recurrent relations as follows

$$\begin{aligned} &u_{10} = 1, && u_{20} = 1, \\ &u_{11} = L^{-1}(A_{10}) = \frac{1}{2}x^2, && u_{21} = -L^{-1}(A_{20}) = -\frac{1}{2}x^2, \\ &u_{12} = L^{-1}(A_{11}) = -\frac{1}{8}x^4, && u_{22} = -L^{-1}(A_{21}) = \frac{3}{8}x^4, \\ &u_{13} = L^{-1}(A_{12}) = \frac{1}{16}x^6, && u_{23} = -L^{-1}(A_{22}) = -\frac{5}{16}x^6, \\ &\vdots && \vdots \end{aligned}$$

leading to the following series solution

$$\begin{cases} u_1 = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots \\ u_2 = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots \end{cases} \tag{27}$$

which subsequently yields the closed-form solution for the coupled system as follows

$$(u_1(x), u_2(x)) = \left(\sqrt{1+x^2}, \frac{1}{\sqrt{1+x^2}}\right). \tag{28}$$

Example 3. *Consider the coupled system of Emden–Fowler-type equations with $r_1 = 5, r_2 = 3$ as follows [20]*

$$\begin{cases} u_1'' + \frac{5}{x}u_1' + 8(e^{u_1} + 2e^{-\frac{u_2}{2}}) = 0, \\ u_2'' + \frac{3}{x}u_2' - 8(e^{-u_2} + e^{\frac{u_1}{2}}) = 0, \end{cases} \tag{29}$$

with initial conditions

$$\begin{cases} u_1(0) = 0 = u_2(0), \\ u_1'(0) = 0 = u_2'(0). \end{cases} \tag{30}$$

Accordingly, from the first ODE, we set $2h_1 + r_1 = 5$ and $(h_1 - 1)(h_1 + r_1) = 0$ to obtain $h_1 = 1, r_1 = 3$ and further lead to the following operator from Equation (7), together with its corresponding inverse integral operator

$$\begin{aligned} L(.) =& x^{-1}\frac{d}{dx}\left(x^{-3}\frac{d}{dx}x^4\right)(.), \\ L^{-1}(.) =& x^{-4}\int_0^x x^3\int_0^x x(.)dxdx. \end{aligned} \tag{31}$$

Similarly, we set $2h_2 + r_2 = 3$ and $(h_2 - 1)(h_2 + r_2) = 0$ in the second ODE to get $h_2 = 1, r_2 = 1$. This yields the following operators via Equation (7)

$$\begin{aligned} L(.) =& x^{-1}\frac{d}{dx}\left(x^{-1}\frac{d}{dx}x^2\right)(.), \\ L^{-1}(.) =& x^{-2}\int_0^x x\int_0^x x(.)dxdx. \end{aligned} \tag{32}$$

Therefore, the given coupled model in these new operators becomes

$$\begin{cases} Lu_1 = -8(e^{u_1} + 2e^{-\frac{u_2}{2}}), \\ Lu_2 = 8(e^{-u_2} + e^{\frac{u_1}{2}}), \end{cases} \tag{33}$$

such that after operating the inverse operator L^{-1} on the above respective equations gives the following recursive schemes

$$\begin{cases} u_{10} = 0, \\ u_{1(n+1)} = -8L^{-1}(A_{1n}), \ \ n \geq 0, \end{cases}$$

and

$$\begin{cases} u_{20} = 0, \\ u_{2(n+1)} = 8L^{-1}(A_{2n}), \ \ n \geq 0, \end{cases}$$

where A_{1n}'s and A_{2n}'s are the Adomian polynomials corresponding to the nonlinear terms $(e^{u_1} + 2e^{-\frac{u_2}{2}})$ and $(e^{-u_2} + e^{\frac{u_1}{2}})$ given, respectively, as follows

$$\begin{aligned} A_{10} &= e^{u_{10}} + e^{-\frac{u_{20}}{2}}, \\ A_{11} &= u_{11}e^{u_{10}} - u_{21}e^{-\frac{u_{20}}{2}}, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} A_{20} &= (e^{-u_{20}} + e^{\frac{u_{10}}{2}}), \\ A_{21} &= -u_{21}e^{-u_{20}} + \frac{1}{2}u_{11}e^{\frac{u_{10}}{2}}, \\ &\vdots \end{aligned}$$

Thus, we express some of the components of the above recursive relations as follows

$$\begin{aligned}
u_{10} &= 0, & u_{20} &= 0,\\
u_{11} &= -8L^{-1}(A_{10}) = -2x^2, & u_{21} &= 8L^{-1}(A_{20}) = 2x^2,\\
u_{12} &= -8L^{-1}(A_{11}) = x^4, & u_{22} &= 8L^{-1}(A_{21}) = -x^4,\\
u_{13} &= -8L^{-1}(A_{12}) = -\frac{2}{3}x^6, & u_{23} &= 8L^{-1}(A_{22}) = \frac{2}{3}x^6,\\
u_{14} &= -8L^{-1}(A_{13}) = \frac{1}{2}x^8, & u_{24} &= 8L^{-1}(A_{23}) = -\frac{1}{2}x^8,\\
&\vdots & &\vdots
\end{aligned}$$

that lead to the following series solution

$$\begin{cases} u_1 = -2x^2 + x^4 - \frac{2}{3}x^6 + \frac{1}{2}x^8 + \cdots \\ u_2 = 2x^2 - x^4 + \frac{2}{3}x^6 - \frac{1}{2}x^8 + \cdots \end{cases} \tag{34}$$

which subsequently yields the closed-form solution for the coupled system as follows

$$(u_1(x), u_2(x)) = (-2\ln(1+x^2), 2\ln(1+x^2)). \tag{35}$$

Example 4. *Consider the coupled system of Emden–Fowler-type equations when* $r_1 = 8, r_2 = 4$ *as follows [20]*

$$\begin{cases} u_1'' + \frac{8}{x}u_1' + 18u_1 - 4u_1 \ln u_2 = 0, \\ u_2'' + \frac{4}{x}u_2' - 10u_2 + 4u_2 \ln u_1 = 0, \end{cases} \tag{36}$$

with initial conditions

$$\begin{cases} u_1(0) = 1 = u_2(0), \\ u_1'(0) = 0 = u_2'(0). \end{cases} \tag{37}$$

As preceded, in the first ODE, we get $2h_1 + r_1 = 8$ and $(h_1 - 1)(h_1 + r_1) = 0$. This gives $h_1 = 1, r_1 = 6$. So, the operators in Equation (7) become

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(x^{-6}\frac{d}{dx}x^7\right)(.), \\ L^{-1}(.) &= x^{-7}\int_0^x x^6 \int_0^x x(.)dxdx. \end{aligned} \tag{38}$$

Additionally, we put $2h_2 + r_2 = 4$ and $(h_2 - 1)(h_2 + r_2) = 0$ from the second ODE to get $h_2 = 1, r_2 = 2$ such that the operators in Equation (7) yield

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(x^{-2}\frac{d}{dx}x^3\right)(.), \\ L^{-1}(.) &= x^{-3}\int_0^x x^2 \int_0^x x(.)dxdx. \end{aligned} \tag{39}$$

Furthermore, Equation (36) in these new operators becomes

$$\begin{cases} Lu_1 = -18u_1 + 4u_1 \ln u_2, \\ Lu_2 = 10u_2 - 4u_2 \ln u_1, \end{cases} \tag{40}$$

of which the application of the inversions L^{-1} on Equation (40) reveals the following recursive schemes

$$\begin{cases} u_{10} = 1, \\ u_{1(n+1)} = -18L^{-1}(u_1) + 4L^{-1}(A_{1n}), \ \ n \geq 0, \end{cases}$$

and

$$\begin{cases} u_{20} = 1, \\ u_{2(n+1)} = 10L^{-1}(u_2) - 4L^{-1}(A_{2n}), \ \ n \geq 0, \end{cases}$$

where A_{1n}'s and A_{2n}'s are the Adomian polynomials corresponding to the nonlinear terms $u_1 \ln u_2$ and $u_2 \ln u_1$, correspondingly.

Then, we express some of the components of the above recursive relations as follows

$$\begin{aligned} &u_{10} = 1, && u_{20} = 1, \\ &u_{11} = -18L^{-1}(u_{10}) + 4L^{-1}(A_{10}) = -x^2, && u_{21} = 10L^{-1}(u_{20}) - 4L^{-1}(A_{20}) = x^2, \\ &u_{12} = -18L^{-1}(u_{10}) + 4L^{-1}(A_{10}) = \frac{1}{2}x^4, && u_{22} = 10L^{-1}(u_{20}) - 4L^{-1}(A_{20}) = \frac{1}{2}x^4, \\ &u_{13} = -18L^{-1}(u_{10}) + 4L^{-1}(A_{10}) = -\frac{1}{6}x^6, && u_{23} = 10L^{-1}(u_{20}) - 4L^{-1}(A_{20}) = \frac{1}{6}x^6, \\ &\vdots && \vdots \end{aligned}$$

that lead to the following series solutions

$$\begin{cases} u_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \\ u_2 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \end{cases} \tag{41}$$

which subsequently yields the following closed-form solution of the system

$$(u_1(x), u_2(x)) = (e^{-x^2}, e^{x^2}). \tag{42}$$

Example 5. *Consider the coupled system of Emden–Fowler-type equations when* $r_1 = 1, r_2 = 2$ *as follows [13]*

$$\begin{cases} u_1'' + \frac{1}{x}u_1' + u_1^2 u_2 - (4x^2 + 5)u_1 = 0, \\ u_2'' + \frac{2}{x}u_2' + u_1 u_2^2 - (4x^2 - 5)u_2 = 0, \end{cases} \tag{43}$$

with initial conditions

$$\begin{cases} u_1(0) = 1 = u_2(0), \\ u_1'(0) = 0 = u_2'(0). \end{cases} \tag{44}$$

As in the preceding examples, the first ODE admits $2h_1 + r_1 = 1$ and $(h_1 - 1)(h_1 + r_1) = 0$. This gives $h_1 = 1, r_1 = -1$. So, the operators in Equation (7) become

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(x\frac{d}{dx}\right)(.), \\ L^{-1}(.) &= \int_0^x x^{-1}\int_0^x x(.)dxdx \end{aligned} \tag{45}$$

Additionally, the second ODE admits $2h_2 + r_2 = 2$ and $(h_2 - 1)(h_2 + r_2) = 0$. This gives $h_2 = 1, r_2 = 0$. So, the operators in Equation (7) become

$$\begin{aligned} L(.) &= x^{-1}\frac{d}{dx}\left(\frac{d}{dx}x\right)(.), \\ L^{-1}(.) &= x^{-1}\int_0^x \int_0^x x(.)dxdx \end{aligned} \tag{46}$$

Furthermore, Equation (43) in these new operators becomes

$$\begin{cases} Lu_1 = -u_1^2u_2 + (4x^2 + 5)u_1, \\ Lu_2 = -u_1u_2^2 + (4x^2 - 5)u_2, \end{cases} \tag{47}$$

of which the application of the inversions L^{-1} on Equation (47) reveals the following recursive schemes

$$\begin{cases} u_{10} = 1, \\ u_{1(n+1)} = L^{-1}((4x^2 + 5)u_{1n}) - L^{-1}(A_{1n}), \ \ n \geq 0, \end{cases}$$

and

$$\begin{cases} u_{20} = 1, \\ u_{2(n+1)} = L^{-1}((4x^2 - 5)u_{2n}) - L^{-1}(A_{2n}), \ \ n \geq 0, \end{cases}$$

where A_{1n}'s and A_{2n}'s are the respective Adomian polynomials of the nonlinear terms $u_1^2u_2$and $u_1u_2^2$, correspondingly.

What is more, we express some of the components of the above recurrent relations as follows

$$u_{10} = 1, \qquad u_{20} = 1,$$

$$\begin{aligned} u_{11} &= L^{-1}((4x^2 + 5)u_{10}) - L^{-1}(A_{10}), \\ &= x^2 + \frac{1}{4}x^4, \end{aligned} \qquad \begin{aligned} u_{21} &= L^{-1}((4x^2 - 5)u_{20}) - L^{-1}(A_{20}), \\ &= -x^2 + \frac{1}{5}x^4, \end{aligned}$$

$$\begin{aligned} u_{12} &= L^{-1}((4x^2 + 5)u_{11}) - L^{-1}(A_{11}), \\ &= \frac{1}{4}x^4 + \cdots, \\ &\vdots \end{aligned} \qquad \begin{aligned} u_{22} &= L^{-1}((4x^2 - 5)u_{21}) - L^{-1}(A_{21}), \\ &= \frac{3}{10}x^4 + \cdots \\ &\vdots \end{aligned}$$

that lead to the following series solution

$$\begin{cases} u_1 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \\ \\ u_2 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots \end{cases} \tag{48}$$

and subsequently yields the following closed-form solution of the system

$$(u_1(x), u_2(x)) = (e^{x^2}, e^{-x^2}). \tag{49}$$

4. Conclusions

In conclusion, the current paper examined some important IVPs of the coupled system of Emden–Fowler-type equations. Emden–Fowler equation is a generalization of the Lane–Emden equation that arises in modeling a variety of phenomena in physics and engineering. The present study proposed a method via the application of the modified ADM by Hassan [3] to construct a generalized differential operator together with its

corresponding integral inverse operator. In fact, this operator was able to navigate to the closed-form solution easily, as against the methods presented in [5,20]. It is pertinent to recall here that an ADM procedure coupled with an integral transform was utilized in [5]; while [20] presented a modification of ADM via the application of Volterra integral equations. Amazingly, our devised method rapidly gets hold of the closed-form solutions once the proposed differential operator is applied. Lastly, the effectiveness of the devised method was further evaluated taking into account the noted speedier convergence rate and the level of exactitude with the exact analytical solutions in comparison with the highlighted references.

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