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New Conditions for Testing the Oscillation of Fourth-Order Differential Equations with Several Delays

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Abstract: In this paper, we establish oscillation theorems for all solutions to fourth-order neutral differential equations using the Riccati transformation approach and some inequalities. Some new criteria are established that can be used in cases where known theorems fail to apply. The approach followed depends on finding conditions that guarantee the exclusion of positive solutions, and as a result of the symmetry between the positive and negative solutions of the studied equation, we therefore exclude negative solutions. An illustrative example is given.

Keywords: Riccati transformation; neutral differential equations; oscillation theorems

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1. Introduction

In this paper, we are concerned with the oscillation of solutions of the fourth-order neutral differential equation

$$\left(\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha}\right)' + \sum_{j=1}^{m} q_j(\mathfrak{r}) x^{\beta}\left(\zeta_j(\mathfrak{r})\right) = 0, \tag{1}$$

where $\mathfrak{r} \geq \mathfrak{r}_0$ and $V(\mathfrak{r}) = x(\mathfrak{r}) + p(\mathfrak{r})x(\xi(\mathfrak{r}))$. In this work, we assume α and β are quotients of odd positive integers and $\beta \geq \alpha$, $\mathfrak{a}, \xi \in C^1[\mathfrak{r}_0, \infty)$, $p, q_j, \zeta_j \in C[\mathfrak{r}_0, \infty)$, $\mathfrak{a}(\mathfrak{r}) > 0$, $\mathfrak{a}'(\mathfrak{r}) \geq 0$, $q_j(\mathfrak{r}) > 0$, $0 \leq p(\mathfrak{r}) < p_0 < \infty$, $\xi'(\mathfrak{r}) \geq \xi_0 > 0$, $\zeta_j \circ \xi = \xi \circ \zeta_j$, $\zeta_j(\mathfrak{r}) \leq \xi(\mathfrak{r}) \leq \mathfrak{r}$, $\lim_{\mathfrak{r}\to\infty} \xi(\mathfrak{r}) = \lim_{\mathfrak{r}\to\infty} \zeta_j(\mathfrak{r}) = \infty$, and

$$\int_{\mathfrak{r}_0}^{\infty} \frac{1}{\mathfrak{a}^{1/\alpha}(s)} \mathrm{d}s = \infty. \tag{2}$$

By a solution of (1) we mean a function $x \in C^3[\mathfrak{r}_x, \infty)$, $\mathfrak{r}_x \geq \mathfrak{r}_0$, which has the property $\mathfrak{a}(V''')^{\alpha} \in C^1[\mathfrak{r}_x, \infty)$, and satisfies (1) on $[\mathfrak{r}_x, \infty)$. We consider only those solutions x of (1) which satisfy $\sup\{|x(\mathfrak{r})| : \mathfrak{r} \geq \mathfrak{r}\} > 0$, for all $\mathfrak{r} \geq \mathfrak{r}_x$. A solution x of (1) is said to be nonoscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

The differential and functional differential equations arise in many applied problems in natural sciences and engineering; see Hale [1].

The oscillation theory has become a significant numerical mathematical tool for many disciplines and high technologies. The subject of finding oscillation criteria for certain functional DEs has been a highly active study area in recent decades; for example, see [2–14] and the references cited therein. In what follows, we briefly comment on some closely related results that motivated our study.

Baculikova et al. [15] studied the oscillatory behavior of solutions to the even-order neutral differential equation

$$\left(\left(V^{(n-1)}(\mathfrak{r})\right)^{\alpha}\right)' + q(\mathfrak{r})x^{\alpha}(\zeta(\mathfrak{r})) = 0,$$

where $\alpha \geq 1$. They established some oscillation results.

Agarwal et al. [16] concerned the even-order neutral differential equation

$$(V(\mathfrak{r}))^{(n)} + q(\mathfrak{r})x(\zeta(\mathfrak{r})) = 0. \tag{3}$$

They established some sufficient conditions for oscillation by using the Riccati transformation technique.

Bazighifan et al. [17] investigated the oscillation of fourth-order nonlinear differential equation with neutral delay

$$\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)' + q(\mathfrak{r})x^{\beta}(\zeta(\mathfrak{r})) = 0.$$

They obtained some oscillation criteria for the equation by the theory of comparison.

Li and Rogovchenko [18] studied oscillation for (3). They used comparison with the first-order delay equation to obtain the following result:

Theorem 1. *Assume that there exist functions $\eta \in C[\mathfrak{r}_0, \infty)$ and $\delta \in C^1[\mathfrak{r}_0, \infty)$ satisfying*

$$\eta(\mathfrak{r}) \leq \zeta(\mathfrak{r}),\ \eta(\mathfrak{r}) < \xi(\mathfrak{r}),\ \delta(\mathfrak{r}) \leq \zeta(\mathfrak{r}),\ \delta(\mathfrak{r}) < \xi(\mathfrak{r}),\ \delta'(\mathfrak{r}) \geq 0$$

and

$$\lim_{\mathfrak{r}\to\infty} \eta(\mathfrak{r}) = \lim_{\mathfrak{r}\to\infty} \delta(\mathfrak{r}) = \infty.$$

If

$$\frac{1}{(n-1)!}\liminf_{\mathfrak{r}\to\infty} \int_{\xi^{-1}(\eta(\mathfrak{r}))}^{\mathfrak{r}} q(s)p_*(\zeta(s))\left(\xi^{-1}(\eta(s))\right)^{n-1}\mathrm{d}s > \frac{1}{\mathrm{e}}$$

and

$$\frac{1}{(n-3)!}\liminf_{\mathfrak{r}\to\infty} \int_{\xi^{-1}(\delta(\mathfrak{r}))}^{\mathfrak{r}} \left(\int_s^{\infty} (\varkappa - s)^{n-3} q(\varkappa)p^*(\zeta(\varkappa))\mathrm{d}\varkappa\right)\xi^{-1}(\delta(s))\mathrm{d}s > \frac{1}{\mathrm{e}},$$

then (3) is oscillatory, where

$$p_*(\mathfrak{r}) := \frac{1}{p(\xi^{-1}(\mathfrak{r}))}\left(1 - \frac{\left(\xi^{-1}(\xi^{-1}(\mathfrak{r}))\right)^{n-1}}{(\xi^{-1}(\mathfrak{r}))^{n-1}p(\xi^{-1}(\xi^{-1}(\mathfrak{r})))}\right)$$

and

$$p^*(\mathfrak{r}) = \frac{1}{p(\xi^{-1}(\mathfrak{r}))}\left(1 - \frac{\xi^{-1}(\xi^{-1}(\mathfrak{r}))}{\xi^{-1}(\mathfrak{r})p(\xi^{-1}(\xi^{-1}(\mathfrak{r})))}\right).$$

The purpose of this article is to give sufficient conditions for the oscillatory behavior of (1). Based on introducing a new Riccati substitution, we obtain an improved criteria without requiring the existence of the unknown function.

We will need the following lemmas to discuss our main results:

Lemma 1 ([19])**.** *If the function x satisfies $x^{(i)}(\mathfrak{r}) > 0$, $i = 0, 1, \ldots, n$, and $x^{(n+1)}(\mathfrak{r}) < 0$, then*

$$x(\mathfrak{r}) \geq \frac{\lambda}{n}\mathfrak{r}x'(\mathfrak{r}),$$

for every $\lambda \in (0, 1)$ eventually.

Lemma 2. *Assume that* $\varkappa, \varrho \geq 0$ *and* β *is a positive real number. Then*

$$(\varkappa + \varrho)^{\beta} \leq 2^{\beta-1}\left(\varkappa^{\beta} + \varrho^{\beta}\right), \text{ for } \beta \geq 1$$

and

$$(\varkappa + \varrho)^{\beta} \leq \left(\varkappa^{\beta} + \varrho^{\beta}\right), \text{ for } \beta \leq 1.$$

Lemma 3 ([20])**.** *Let* α *be a ratio of two odd positive integers. Then*

$$K\varrho - L\varrho^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{K^{\alpha+1}}{L^{\alpha}},\ L > 0.$$

Lemma 4. *Assume that (2) holds and* x *is an eventually positive solution of (1). Then,* $(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha})' < 0$ *and there are the following two possible cases eventually:*

$$\begin{aligned} (\mathbf{C}_1)\ V(\mathfrak{r}) &> 0,\ V'(\mathfrak{r}) > 0,\ V''(\mathfrak{r}) > 0,\ V'''(\mathfrak{r}) > 0,\ V^{(4)}(\mathfrak{r}) < 0, \\ (\mathbf{C}_2)\ V(\mathfrak{r}) &> 0,\ V'(\mathfrak{r}) > 0,\ V''(\mathfrak{r}) < 0,\ V'''(\mathfrak{r}) > 0. \end{aligned}$$

2. Main Results

In the sequel, we will adopt the following notation:

$$\eta_1(\mathfrak{r}, \mathfrak{r}_1) = \int_{\mathfrak{r}_1}^{\mathfrak{r}} \frac{1}{\mathfrak{a}^{1/\alpha}(s)} \mathrm{d}s,$$

$$\eta_{k+1}(\mathfrak{r}, \mathfrak{r}_1) = \int_{\mathfrak{r}_1}^{\mathfrak{r}} \eta_k(\varkappa, \mathfrak{r}_1) \mathrm{d}\varkappa,\ \ k = 1, 2,$$

$$\zeta(\mathfrak{r}) := \min\{\zeta_j(\mathfrak{r}) : j = 1, 2, ..., m\}$$

and

$$Q(\mathfrak{r}) = \min\{q_j(\mathfrak{r}), q_j(\xi(\mathfrak{r})) : j = 1, 2, ..., m\}.$$

Lemma 5. *Assume that* x *is a positive solution of (1). Then*

$$\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)' + \frac{p_0^{\beta}}{\xi_0}\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)' + \frac{Q(\mathfrak{r})}{2^{\beta-1}}\sum_{j=1}^{m} V^{\beta}(\zeta_j(\mathfrak{r})) \leq 0. \tag{4}$$

Proof. Assume that x is a positive solution of (1). From (1), we obtain

$$\begin{aligned} 0 &= \left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)' + \frac{p_0^{\beta}}{\xi_0}\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)' + \sum_{j=1}^{m} q_j(\mathfrak{r})x^{\beta}(\zeta_j(\mathfrak{r})) \\ &\quad + p_0^{\beta}\sum_{j=1}^{m} q_j(\xi(\mathfrak{r}))x^{\beta}(\zeta_j(\xi(\mathfrak{r}))) \\ &\geq \left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)' + \frac{p_0^{\beta}}{\xi_0}\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)' \\ &\quad + Q(\mathfrak{r})\sum_{j=1}^{m}\left(x^{\beta}(\zeta_j(\mathfrak{r})) + p_0^{\beta}x^{\beta}(\zeta_j(\xi(\mathfrak{r})))\right), \end{aligned}$$

which follows from Lemma 2 and $\zeta_j \circ \xi = \xi \circ \zeta_j$ that

$$\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)' + \frac{p_0^{\beta}}{\xi_0}\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)' + \frac{Q(\mathfrak{r})}{2^{\beta-1}}\sum_{j=1}^{m} V^{\beta}(\zeta_j(\mathfrak{r})) \leq 0.$$

The proof is complete. □

Lemma 6. *Assume that x is a positive solution of (1). If (C_1) holds, then*

$$V(\mathfrak{r}) \geq \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\eta_3(\mathfrak{r},\mathfrak{r}_1). \tag{5}$$

Proof. Assume that x is a positive solution of (1). Let (C_1) hold. Since $\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^\alpha\right)' \leq 0$. Then we get

$$\begin{aligned} V''(\mathfrak{r}) &\geq V''(\mathfrak{r}) - V''(\mathfrak{r}_1) = \int_{\mathfrak{r}_1}^{\mathfrak{r}} \frac{\left(\mathfrak{a}(s)(V'''(s))^\alpha\right)^{1/\alpha}}{\mathfrak{a}^{1/\alpha}(s)} \mathrm{d}s \\ &\geq \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\eta_1(\mathfrak{r},\mathfrak{r}_1), \end{aligned}$$

integrating the above inequality from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V'(\mathfrak{r}) \geq \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\eta_2(\mathfrak{r},\mathfrak{r}_1), \tag{6}$$

integrating (6) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we get

$$V(\mathfrak{r}) \geq \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\eta_3(\mathfrak{r},\mathfrak{r}_1).$$

The proof is complete. □

Lemma 7. *Assume that x is a positive solution of (1). If (C_2) holds, then*

$$V''(\mathfrak{r}) \leq -\left(\frac{\xi_0}{\xi_0 + p_0^\beta}\right)^{1/\alpha} V^{\beta/\alpha}(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)} \int_{\xi^{-1}(\varkappa)}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}\varkappa. \tag{7}$$

Proof. Assume that x is a positive solution of (1). Integrating (4) from $\mathfrak{r}$ to ∞ and using $\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^\alpha\right)' \leq 0$, we obtain

$$-\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^\alpha - \frac{p_0^\beta}{\xi_0}\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha \leq -\int_{\mathfrak{r}}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} V^\beta(\zeta_j(s))\mathrm{d}s. \tag{8}$$

From Lemma 1 and (8), we have

$$-\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^\alpha - \frac{p_0^\beta}{\xi_0}\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha \leq -\int_{\mathfrak{r}}^{\infty} \frac{Q(s)}{2^{\beta-1}} V^\beta(s) \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s,$$

that is,

$$\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^\alpha + \frac{p_0^\beta}{\xi_0}\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha \geq V^\beta(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s,$$

since $\xi(\mathfrak{r}) \leq \mathfrak{r}$ and $\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^\alpha\right)' \leq 0$, then, we have

$$\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha + \frac{p_0^\beta}{\xi_0}\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha \geq V^\beta(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s,$$

that is,

$$\mathfrak{a}(\xi(\mathfrak{r}))\left(V'''(\xi(\mathfrak{r}))\right)^\alpha \geq \left(\frac{\xi_0}{\xi_0 + p_0^\beta}\right) V^\beta(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s \tag{9}$$

or

$$\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha} \geq \left(\frac{\xi_0}{\xi_0 + p_0^{\beta}}\right) V^{\beta}\left(\xi^{-1}(\mathfrak{r})\right) \int_{\xi^{-1}(\mathfrak{r})}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s,$$

since $\xi^{-1}(\mathfrak{r}) > \mathfrak{r}$ then $V\left(\xi^{-1}(\mathfrak{r})\right) > V(\mathfrak{r})$. From the above inequality, we have

$$\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha} \geq \left(\frac{\xi_0}{\xi_0 + p_0^{\beta}}\right) V^{\beta}(\mathfrak{r}) \int_{\xi^{-1}(\mathfrak{r})}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s.$$

Integrating the above inequality from $\mathfrak{r}$ to ∞, we obtain

$$V''(\mathfrak{r}) \leq -\left(\frac{\xi_0}{\xi_0 + p_0^{\beta}}\right)^{1/\alpha} V^{\beta/\alpha}(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)} \int_{\xi^{-1}(\varkappa)}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}\varkappa.$$

The proof is complete. □

Theorem 2. *Let $\beta \geq 1$, $\zeta_j(\mathfrak{r}) \in C^1([\mathfrak{r}_0, \infty))$, $\zeta_j' > 0$, and $\zeta_j(\mathfrak{r}) \leq \xi(\mathfrak{r})$. Assume that there exists a functions $\rho(\mathfrak{r}), \theta(\mathfrak{r}) \in C^1([\mathfrak{r}_0, \infty), (0, \infty))$, for all sufficiently large $\mathfrak{r}_1 \geq \mathfrak{r}_0$, there is a $\mathfrak{r}_2 > \mathfrak{r}_1$ such that*

$$\limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_2}^{\mathfrak{r}} \left(m\rho(s)\frac{Q(s)}{2^{\beta-1}} M^{\beta-\alpha} - \left(1 + \frac{p_0^{\beta}}{\xi_0}\right) \frac{(\alpha+1)^{-(\alpha+1)}\left(\rho_+'(s)\right)^{\alpha+1}}{\left(\rho(s)\eta_2(\zeta(s), \mathfrak{r}_1)\zeta'(s)\right)^{\alpha}}\right) \mathrm{d}s = \infty \tag{10}$$

and

$$\limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_1}^{\mathfrak{r}} \left(\theta(\varrho)\left(\frac{\xi_0}{\xi_0 + p_0^{\beta}}\right)^{1/\alpha} M^{(\beta/\alpha)-1} \int_{\varrho}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)}\Phi(\varkappa)\right)^{1/\alpha} \mathrm{d}\varkappa - \frac{\left(\theta_+'(\varrho)\right)^2}{4\theta(\varrho)}\right) \mathrm{d}\varrho = \infty, \tag{11}$$

where

$$\Phi(\varkappa) = \int_{\xi^{-1}(\varkappa)}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s,$$

$\rho_+'(\mathfrak{r}) = \max\{0, \rho'(\mathfrak{r})\}$ *and* $\theta_+'(\mathfrak{r}) = \max\{0, \theta'(\mathfrak{r})\}$. *Then (1) is oscillatory.*

Proof. Assume that x is a positive solution of (1). It follows from Lemma 4 that there exists two possible cases (C_1) and (C_2). Let (C_1) hold. We define a function $\omega(\mathfrak{r})$ by

$$\omega(\mathfrak{r}) = \rho(\mathfrak{r}) \frac{\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha}}{V^{\alpha}(\zeta(\mathfrak{r}))}, \tag{12}$$

then $\omega(\mathfrak{r}) > 0$. Differentiating (12), we have

$$\omega'(\mathfrak{r}) = \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\omega(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \rho(\mathfrak{r})\frac{\alpha\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha} V'(\zeta(\mathfrak{r}))\zeta'(\mathfrak{r})}{V^{\alpha+1}(\zeta(\mathfrak{r}))}, \tag{13}$$

from (6) and $\zeta(\mathfrak{r}) \leq \zeta_j(\mathfrak{r}) \leq \mathfrak{r}$, we get

$$V'(\zeta(\mathfrak{r})) \geq \mathfrak{a}^{1/\alpha}(\zeta(\mathfrak{r}))V'''(\zeta(\mathfrak{r}))\eta_2(\zeta(\mathfrak{r}), \mathfrak{r}_1) \geq \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\eta_2(\zeta(\mathfrak{r}), \mathfrak{r}_1) \tag{14}$$

and so, (13) can be written as

$$\omega'(\mathfrak{r}) \leq \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\omega(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha\left(V'''(\mathfrak{r})\right)^{\alpha+1}\eta_2(\zeta(\mathfrak{r}), \mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{-1}(\mathfrak{r})\mathfrak{a}^{-(\alpha+1)/\alpha}(\mathfrak{r})V^{\alpha+1}(\zeta(\mathfrak{r}))}. \tag{15}$$

It follows from (12) and (15) that

$$\omega'(\mathfrak{r}) \le \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\omega(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\omega^{(\alpha+1)/\alpha}(\mathfrak{r}). \tag{16}$$

Similarly, define another function ψ by

$$\psi(\mathfrak{r}) = \rho(\mathfrak{r})\frac{\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}}{V^{\alpha}(\zeta(\mathfrak{r}))}, \tag{17}$$

then $\psi(\mathfrak{r}) > 0$. Differentiating (17), we have

$$\psi'(\mathfrak{r}) = \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\psi(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha(V'''(\xi(\mathfrak{r})))^{\alpha}V'(\zeta(\mathfrak{r}))\zeta'(\mathfrak{r})}{\rho^{-1}(\mathfrak{r})\mathfrak{a}^{-1}(\xi(\mathfrak{r}))V^{\alpha+1}(\zeta(\mathfrak{r}))}, \tag{18}$$

from (6) and $\zeta(\mathfrak{r}) \le \zeta_j(\mathfrak{r}) \le \xi(\mathfrak{r})$, we get

$$V'(\zeta(\mathfrak{r})) \ge \mathfrak{a}^{1/\alpha}(\zeta(\mathfrak{r}))V'''(\zeta(\mathfrak{r}))\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1) \ge \mathfrak{a}^{1/\alpha}(\xi(\mathfrak{r}))V'''(\xi(\mathfrak{r}))\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1) \tag{19}$$

and so, (18) can be written as

$$\psi'(\mathfrak{r}) \le \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\psi(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha(V'''(\xi(\mathfrak{r})))^{\alpha+1}\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{-1}(\mathfrak{r})\mathfrak{a}^{-(1+\alpha)/\alpha}(\xi(\mathfrak{r}))V^{\alpha+1}(\zeta(\mathfrak{r}))}. \tag{20}$$

It follows from (17) and (20) that

$$\psi'(\mathfrak{r}) \le \frac{\rho'(\mathfrak{r})}{\rho(\mathfrak{r})}\psi(\mathfrak{r}) + \rho(\mathfrak{r})\frac{\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\psi^{(1+\alpha)/\alpha}(\mathfrak{r}). \tag{21}$$

Using (16) and (21), we get

$$\begin{aligned}\omega'(\mathfrak{r}) + \frac{p_0^{\beta}}{\xi_0}\psi'(\mathfrak{r}) \quad \le \quad & \rho(\mathfrak{r})\left(\frac{\left(\mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))} + \frac{p_0^{\beta}}{\xi_0}\frac{\left(\mathfrak{a}(\xi(\mathfrak{r}))(V'''(\xi(\mathfrak{r})))^{\alpha}\right)'}{V^{\alpha}(\zeta(\mathfrak{r}))}\right) \\ & + \frac{\rho'_+(\mathfrak{r})}{\rho(\mathfrak{r})}\omega(\mathfrak{r}) - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\omega^{(\alpha+1)/\alpha}(\mathfrak{r}) \\ & + \frac{p_0^{\beta}}{\xi_0}\left(\frac{\rho'_+(\mathfrak{r})}{\rho(\mathfrak{r})}\psi(\mathfrak{r}) - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\psi^{(1+\alpha)/\alpha}(\mathfrak{r})\right).\end{aligned} \tag{22}$$

By (4) and (23), we obtain

$$\begin{aligned}\omega'(\mathfrak{r}) + \frac{p_0^{\beta}}{\xi_0}\psi'(\mathfrak{r}) \quad \le \quad & -\rho(\mathfrak{r})\frac{Q(\mathfrak{r})}{2^{\beta-1}}\frac{\sum_{j=1}^{m}V^{\beta}(\zeta_j(\mathfrak{r}))}{V^{\alpha}(\zeta(\mathfrak{r}))} - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\omega^{(\alpha+1)/\alpha}(\mathfrak{r}) \\ & + \frac{p_0^{\beta}}{\xi_0}\left(\frac{\rho'_+(\mathfrak{r})}{\rho(\mathfrak{r})}\psi(\mathfrak{r}) - \frac{\alpha\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r})}{\rho^{1/\alpha}(\mathfrak{r})}\psi^{(1+\alpha)/\alpha}(\mathfrak{r})\right) \\ & + \frac{\rho'_+(\mathfrak{r})}{\rho(\mathfrak{r})}\omega(\mathfrak{r}),\end{aligned} \tag{23}$$

from Lemma 3 and (24), we have

$$\begin{aligned}\omega'(\mathfrak{r}) + \frac{p_0^{\beta}}{\xi_0}\psi'(\mathfrak{r}) \quad \le \quad & -m\rho(\mathfrak{r})\frac{Q(\mathfrak{r})}{2^{\beta-1}}V^{\beta-\alpha}(\zeta(\mathfrak{r})) + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'_+(\mathfrak{r}))^{\alpha+1}}{(\rho(\mathfrak{r})\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r}))^{\alpha}} \\ & + \frac{p_0^{\beta}}{\xi_0}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho'_+(\mathfrak{r}))^{\alpha+1}}{(\rho(\mathfrak{r})\eta_2(\zeta(\mathfrak{r}),\mathfrak{r}_1)\zeta'(\mathfrak{r}))^{\alpha}}.\end{aligned} \tag{24}$$

Since $V'(\mathfrak{r}) > 0$, there exists a $\mathfrak{r}_2 \geq \mathfrak{r}_1$ and a constant $M > 0$ such that

$$V(\mathfrak{r}) > M, \text{ for all } \mathfrak{r} \geq \mathfrak{r}_2, \tag{25}$$

by using (25) and integrating (25) from $\mathfrak{r}_2$ $(\mathfrak{r}_2 \geq \mathfrak{r}_1)$ to $\mathfrak{r}$, we get

$$\begin{aligned}\int_{\mathfrak{r}_2}^{\mathfrak{r}} \left(m\rho(s) \frac{Q(s)}{2^{\beta-1}} M^{\beta-\alpha} - \left(1 + \frac{p_0^\beta}{\xi_0}\right) \frac{(\alpha+1)^{-(\alpha+1)} (\rho'_+(s))^{\alpha+1}}{(\rho(s)\eta_2(\zeta(s),\mathfrak{r}_1)\zeta'(s))^\alpha} \right) \mathrm{d}s \quad &\leq \quad \omega(\mathfrak{r}_2) \\ &+ \frac{p_0^\beta}{\xi_0} \psi(\mathfrak{r}_2),\end{aligned}$$

which contradicts (10).

Let (C_2) hold. We define a function $\varphi(\mathfrak{r})$ by

$$\varphi(\mathfrak{r}) = \theta(\mathfrak{r}) \frac{V'(\mathfrak{r})}{V(\mathfrak{r})}, \tag{26}$$

then $\varphi(\mathfrak{r}) > 0$. Differentiating (26), we have

$$\varphi'(\mathfrak{r}) = \frac{\theta'(\mathfrak{r})}{\theta(\mathfrak{r})} \varphi(\mathfrak{r}) + \theta(\mathfrak{r}) \frac{V''(\mathfrak{r})}{V(\mathfrak{r})} - \theta(\mathfrak{r}) \frac{(V'(\mathfrak{r}))^2}{V^2(\mathfrak{r})}, \tag{27}$$

from (26) and (27), we have

$$\varphi'(\mathfrak{r}) = \frac{\theta'(\mathfrak{r})}{\theta(\mathfrak{r})} \varphi(\mathfrak{r}) + \theta(\mathfrak{r}) \frac{V''(\mathfrak{r})}{V(\mathfrak{r})} - \frac{1}{\theta(\mathfrak{r})} \varphi^2(\mathfrak{r}), \tag{28}$$

from (7) and (28), we have

$$\begin{aligned}\varphi'(\mathfrak{r}) \quad &\leq \quad -\theta(\mathfrak{r}) \left(\frac{\xi_0}{\xi_0 + p_0^\beta} \right)^{1/\alpha} V^{(\beta/\alpha)-1}(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)} \Phi(\varkappa) \right)^{1/\alpha} \mathrm{d}\varkappa \\ &+ \frac{\theta'(\mathfrak{r})}{\theta(\mathfrak{r})} \varphi(\mathfrak{r}) - \frac{1}{\theta(\mathfrak{r})} \varphi^2(\mathfrak{r}).\end{aligned}$$

Thus, we obtain

$$\varphi'(\mathfrak{r}) \leq -\theta(\mathfrak{r}) \left(\frac{\xi_0}{\xi_0 + p_0^\beta} \right)^{1/\alpha} V^{(\beta/\alpha)-1}(\mathfrak{r}) \int_{\mathfrak{r}}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)} \Phi(\varkappa) \right)^{1/\alpha} \mathrm{d}\varkappa + \frac{(\theta'_+(\mathfrak{r}))^2}{4\theta(\mathfrak{r})}, \tag{29}$$

by using (25) and integrating (29) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we get

$$\varphi(\mathfrak{r}_1) \geq \int_{\mathfrak{r}_1}^{\mathfrak{r}} \left(\theta(\varrho) \left(\frac{\xi_0}{\xi_0 + p_0^\beta} \right)^{1/\alpha} M^{(\beta/\alpha)-1} \int_{\varrho}^{\infty} \left(\frac{1}{\mathfrak{a}(\varkappa)} \Phi(\varkappa) \right)^{1/\alpha} \mathrm{d}\varkappa - \frac{(\theta'_+(\varrho))^2}{4\theta(\varrho)} \right) \mathrm{d}\varrho,$$

which contradicts (11). This completes the proof. □

Theorem 3. *Assume that* $0 \leq p(\mathfrak{r}) < 1$ *and* $\alpha = \beta$. *If (11) holds and*

$$\limsup_{\mathfrak{r}\to\infty} \int_{\zeta(\mathfrak{r})}^{\mathfrak{r}} G^\alpha(\zeta(s)) \sum_{j=1}^{m} q_j(s) (1 - p(\zeta_j(s)))^\alpha \mathrm{d}s > 1, \quad \zeta \text{ is nondecreasing} \tag{30}$$

or

$$\liminf_{\mathfrak{r}\to\infty} \int_{\zeta(\mathfrak{r})}^{\mathfrak{r}} G^\alpha(\zeta(s)) \sum_{j=1}^{m} q_j(s) (1 - p(\zeta_j(s)))^\alpha \mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{31}$$

where

$$G(\mathfrak{r})=\eta_3(\mathfrak{r},\mathfrak{r}_1)+\frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{v}\int_{\mathfrak{r}_1}^{u}\eta_1(s,\mathfrak{r}_1)\eta_3^{\alpha}(\zeta(s),\mathfrak{r}_1)\sum_{j=1}^{m}q_j(s)\big(1-p(\zeta_j(s))\big)^{\alpha}\mathrm{d}s\mathrm{d}u\mathrm{d}v,$$

then (1) is oscillatory.

Proof. Assume that x is a positive solution of (1). It follows from Lemma 4 that there exists two possible cases (C_1) and (C_2). Let (C_1) hold. Using the definition of $V(t)$, we get

$$x(\mathfrak{r})=V(\mathfrak{r})-p(\mathfrak{r})x(\xi(\mathfrak{r}))\geq V(\mathfrak{r})-p(\mathfrak{r})V(\xi(\mathfrak{r}))\geq(1-p(\mathfrak{r}))V(\mathfrak{r}), \tag{32}$$

from (1) and (32), we have

$$\begin{aligned}\Big(\mathfrak{a}(\mathfrak{r})\big(V'''(\mathfrak{r})\big)^{\alpha}\Big)' &\leq -\sum_{j=1}^{m}q_j(\mathfrak{r})\big(1-p(\zeta_j(\mathfrak{r}))\big)^{\alpha}V^{\alpha}(\zeta_j(\mathfrak{r}))\\ &\leq -V^{\alpha}(\zeta(\mathfrak{r}))\sum_{j=1}^{m}q_j(\mathfrak{r})\big(1-p(\zeta_j(\mathfrak{r}))\big)^{\alpha}.\end{aligned} \tag{33}$$

On the other hand, it follows from the monotonicity of $\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})$ that

$$V''(\mathfrak{r})=V''(\mathfrak{r}_1)+\int_{\mathfrak{r}_1}^{\mathfrak{r}}\frac{1}{\mathfrak{a}^{1/\alpha}(s)}\mathfrak{a}^{1/\alpha}(s)V'''(s)\mathrm{d}s\geq\eta_1(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}). \tag{34}$$

Integrating (34) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V'(\mathfrak{r})\geq\int_{\mathfrak{r}_1}^{\mathfrak{r}}\eta_1(s,\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(s)V'''(s)\mathrm{d}s\geq\eta_2(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}). \tag{35}$$

Integrating (35) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V(\mathfrak{r})\geq\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{u}\eta_1(s,\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(s)V'''(s)\mathrm{d}s\mathrm{d}u=\eta_3(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}). \tag{36}$$

A simple computation shows that

$$\Big(V''(\mathfrak{r})-\eta_1(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)'=-\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)'. \tag{37}$$

Applying the chain rule, it is easy to see that

$$\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}(\mathfrak{r})\big(V'''(\mathfrak{r})\big)^{\alpha}\Big)'=\alpha\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)^{\alpha-1}\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)'.$$

By virtue of (33), the latter equality yields

$$\begin{aligned}\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)' &\leq -\frac{1}{\alpha}\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)^{1-\alpha}V^{\alpha}(\zeta(\mathfrak{r}))\\ &\quad\times\sum_{j=1}^{m}q_j(\mathfrak{r})\big(1-p(\zeta_j(\mathfrak{r}))\big)^{\alpha}.\end{aligned} \tag{38}$$

Combining (37) and (39), we obtain

$$\begin{aligned}\Big(V''(\mathfrak{r})-\eta_1(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)' &\geq \frac{1}{\alpha}\eta_1(\mathfrak{r},\mathfrak{r}_1)\Big(\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r})\Big)^{1-\alpha}V^{\alpha}(\zeta(\mathfrak{r}))\\ &\quad\times\sum_{j=1}^{m}q_j(\mathfrak{r})\big(1-p(\zeta_j(\mathfrak{r}))\big)^{\alpha}.\end{aligned} \tag{39}$$

Integrating (40) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V''(\mathfrak{r}) \geq \eta_1(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}} \eta_1(s,\mathfrak{r}_1)\left(\mathfrak{a}^{1/\alpha}(s)V'''(s)\right)^{1-\alpha} V^{\beta}(\zeta(s)) \times \sum_{j=1}^{m} q_j(s)(1-p(\zeta_j(s)))^{\beta}\mathrm{d}s. \tag{40}$$

Integrating (40) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V'(\mathfrak{r}) \geq \eta_2(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{u} \eta_1(s,\mathfrak{r}_1)\left(\mathfrak{a}^{1/\alpha}(s)V'''(s)\right)^{1-\alpha} \times V^{\alpha}(\zeta(s))\sum_{j=1}^{m} q_j(s)(1-p(\zeta_j(s)))^{\alpha}\mathrm{d}s\mathrm{d}u. \tag{41}$$

Integrating (41) from $\mathfrak{r}_1$ to $\mathfrak{r}$, we have

$$V(\mathfrak{r}) \geq \eta_3(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{v}\int_{\mathfrak{r}_1}^{u} \eta_1(s,\mathfrak{r}_1)\left(\mathfrak{a}^{1/\alpha}(s)V'''(s)\right)^{1-\alpha} \times V^{\alpha}(\zeta(s))\sum_{j=1}^{m} q_j(s)(1-p(\zeta_j(s)))^{\alpha}\mathrm{d}s\mathrm{d}u\mathrm{d}v.$$

Taking (36) and the monotonicity of $\mathfrak{a}^{1/\alpha}(s)V'''(s)$ into account, we arrive at

$$\begin{aligned}
V(\mathfrak{r}) &\geq \eta_3(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{v}\int_{\mathfrak{r}_1}^{u} \eta_1(s,\mathfrak{r}_1)\left(\mathfrak{a}^{1/\alpha}(s)V'''(s)\right)^{1-\alpha} \\
&\quad \times \eta_3^{\alpha}(\zeta(s),\mathfrak{r}_1)\mathfrak{a}(\zeta(s))(V'''(\zeta(s)))^{\alpha}\sum_{j=1}^{m} q_j(s)\left(1-p\left(\zeta_j(s)\right)\right)^{\alpha}\mathrm{d}s\mathrm{d}u\mathrm{d}v \\
&\geq \eta_3(\mathfrak{r},\mathfrak{r}_1)\mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{v}\int_{\mathfrak{r}_1}^{u} \eta_1(s,\mathfrak{r}_1)\left(\mathfrak{a}^{1/\alpha}(s)V'''(s)\right)^{1-\alpha} \\
&\quad \times \eta_3^{\alpha}(\zeta(s),\mathfrak{r}_1)\mathfrak{a}(s)(V'''(s))^{\alpha}\sum_{j=1}^{m} q_j(s)\left(1-p\left(\zeta_j(s)\right)\right)^{\alpha}\mathrm{d}s\mathrm{d}u\mathrm{d}v \\
&\geq \left(\eta_3(\mathfrak{r},\mathfrak{r}_1) + \frac{1}{\alpha}\int_{\mathfrak{r}_1}^{\mathfrak{r}}\int_{\mathfrak{r}_1}^{v}\int_{\mathfrak{r}_1}^{u} \eta_1(s,\mathfrak{r}_1)\eta_3^{\alpha}(\zeta(s),\mathfrak{r}_1)\sum_{j=1}^{m} q_j(s)\left(1-p\left(\zeta_j(s)\right)\right)^{\alpha}\mathrm{d}s\mathrm{d}u\mathrm{d}v\right) \\
&\quad \times \mathfrak{a}^{1/\alpha}(\mathfrak{r})V'''(\mathfrak{r}).
\end{aligned} \tag{42}$$

Thus, we conclude that

$$V(\zeta(\mathfrak{r})) \geq \mathfrak{a}^{1/\alpha}(\zeta(\mathfrak{r}))V'''(\zeta(\mathfrak{r}))G(\zeta(\mathfrak{r})). \tag{43}$$

Using (43) in (33), by virtue of (C_1), one can see that $y(\mathfrak{r}) := \mathfrak{a}(\mathfrak{r})(V'''(\mathfrak{r}))^{\alpha}$ is a positive solution of the first-order delay differential inequality

$$y'(\mathfrak{r}) + \left(G^{\alpha}(\zeta(\mathfrak{r}))\sum_{j=1}^{m} q_j(\mathfrak{r})(1-p(\zeta_j(\mathfrak{r})))^{\alpha}\right)y(\zeta(\mathfrak{r})) \leq 0. \tag{44}$$

In view of ([21], Theorem 1), the associated delay differential equation

$$y'(\mathfrak{r}) + \left(G^{\alpha}(\zeta(\mathfrak{r}))\sum_{j=1}^{m} q_j(\mathfrak{r})(1-p(\zeta_j(\mathfrak{r})))^{\alpha}\right)y(\zeta(\mathfrak{r})) = 0, \tag{45}$$

also has a positive solution. However, it is well known that condition (30) or condition (31) ensures oscillation of (45), which is a contradiction.

Let (C_2) hold. One proceeds as in the proof of Theorem 2. Therefore, the proof is complete. $\square$

Example 1. *Consider the fourth-order neutral differential equation*

$$\left(x(\mathfrak{r}) + 6x\left(\frac{1}{2}\mathfrak{r}\right)\right)^{(4)} + \frac{q_0}{\mathfrak{r}^4}x\left(\frac{1}{3}\mathfrak{r}\right) + \frac{2q_0}{\mathfrak{r}^4}x\left(\frac{1}{4}\mathfrak{r}\right) = 0, \tag{46}$$

where $q_0 > 0$. We note that $p(\mathfrak{r}) = 6$, $\xi(\mathfrak{r}) = \mathfrak{r}/2$, $\zeta_1(\mathfrak{r}) = \mathfrak{r}/3$, $\zeta_2(\mathfrak{r}) = \mathfrak{r}/4$, $q_1(\mathfrak{r}) = q_0/\mathfrak{r}^4$, and $q_2(\mathfrak{r}) = 2q_0/\mathfrak{r}^4$. It is easy to verify that

$$\begin{aligned} Q(\mathfrak{r}) &= \frac{q_0}{\mathfrak{r}^4}, \\ \Phi(\mathfrak{r}) &= \int_{\xi^{-1}(\mathfrak{r})}^{\infty} \frac{Q(s)}{2^{\beta-1}} \sum_{j=1}^{m} \left(\frac{\zeta_j(s)}{s}\right)^{\beta/\lambda} \mathrm{d}s \\ &= q_0\left(\left(\frac{1}{3^{1/\lambda}}\right) + \left(\frac{1}{4^{1/\lambda}}\right)\right)\frac{1}{3(2\mathfrak{r})^3}, \end{aligned}$$

$$\eta_1(\mathfrak{r}, \mathfrak{r}_1) = \int_{\mathfrak{r}_1}^{\mathfrak{r}} \frac{1}{\mathfrak{a}^{1/\alpha}(s)}\mathrm{d}s, = (\mathfrak{r} - \mathfrak{r}_1)$$

and

$$\eta_2(\zeta(\mathfrak{r}), \mathfrak{r}_1) = \frac{((\mathfrak{r}/4) - \mathfrak{r}_1)^2}{2}.$$

By choosing $\rho(\mathfrak{r}) = \mathfrak{r}^3$ and $\theta(\mathfrak{r}) = \mathfrak{r}$, we obtain

$$\begin{aligned} &\limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_2}^{\mathfrak{r}} \left(m\rho(s)\frac{Q(s)}{2^{\beta-1}}M^{\beta-\alpha} - \left(1 + \frac{p_0^\beta}{\xi_0}\right)\frac{(\alpha+1)^{-(\alpha+1)}(\rho'_+(s))^{\alpha+1}}{(\rho(s)\eta_2(\zeta(s), \mathfrak{r}_1)\zeta'(s))^\alpha}\right)\mathrm{d}s \\ &= \limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_2}^{\mathfrak{r}} \left(2s^3\frac{q_0}{s^4} - \left(1 + \frac{6}{1/2}\right)\frac{(2)^{-(2)}(3s^2)^2}{\left(s^3\frac{((\mathfrak{r}/4)-\mathfrak{r}_1)^2}{2}\frac{1}{4}\right)}\right)\mathrm{d}s \\ &= \limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_2}^{\mathfrak{r}} \left(2\frac{q_0}{s} - \left(1 + \frac{6}{1/2}\right)\frac{4^3 3^2}{2s}\right)\mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} &\limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_1}^{\mathfrak{r}} \left(\theta(\varrho)\left(\frac{\xi_0}{\xi_0 + p_0^\beta}\right)^{1/\alpha} M^{(\beta/\alpha)-1}\int_{\varrho}^{\infty}\left(\frac{1}{\mathfrak{a}(\varkappa)}\Phi(\varkappa)\right)^{1/\alpha}\mathrm{d}\varkappa - \frac{(\theta'_+(\varrho))^2}{4\theta(\varrho)}\right)\mathrm{d}\varrho \\ &= \limsup_{\mathfrak{r}\to\infty} \int_{\mathfrak{r}_1}^{\mathfrak{r}} \left(\frac{1}{\varrho}\left(\frac{(1/2)}{(1/2)+6}\frac{q_0}{2^3 6}\left(\left(\frac{1}{3^{1/\lambda}}\right) + \left(\frac{1}{4^{1/\lambda}}\right)\right)\right) - \frac{1}{4\varrho}\right)\mathrm{d}\varrho. \end{aligned}$$

Thus, conditions (10) and (11) are satisfied if $q_0 > 1872$ and $q_0 > 267.43$, respectively. Therefore, we see that (46) is oscillatory if $q_0 > 1872$.

Remark 1. *Consider the special case*

$$\left(x(\mathfrak{r}) + 6x\left(\frac{1}{2}\mathfrak{r}\right)\right)^{(4)} + \frac{q_0}{\mathfrak{r}^4}x\left(\frac{1}{6}\mathfrak{r}\right) = 0. \tag{47}$$

From Theorem 2, we see that (47) is oscillatory if $q_0 > 12,636$. However, choosing $\eta(\mathfrak{r}) = \mathfrak{r}/6$ and using Theorem 1 we observe that cannot be applied to (47) because $p = 6 < 8$. Therefore, our results improve results in [18].

3. Conclusions

In the canonical case, we established oscillation theorems for all solutions to fourth-order neutral differential equations using the Riccati transformation approach and some inequalities in the case where $0 \leq p(\mathfrak{r}) < p_0 < \infty$. We obtain improved criteria without requiring the existence of the unknown function. It would also be of interest to study the equation

$$\left(\mathfrak{a}(\mathfrak{r})\left(V'''(\mathfrak{r})\right)^{\alpha}\right)' + \sum_{j=1}^{m} q_j(\mathfrak{r}) x^{\beta}\left(\zeta_j(\mathfrak{r})\right) = 0$$

in the non-canonical case.

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