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Hermite–Hadamard Type Inclusions for Interval-Valued Coordinated Preinvex Functions

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Abstract: The connection between generalized convexity and symmetry has been studied by many authors in recent years. Due to this strong connection, generalized convexity and symmetry have arisen as a new topic in the subject of inequalities. In this paper, we introduce the concept of interval-valued preinvex functions on the coordinates in a rectangle from the plane and prove Hermite–Hadamard type inclusions for interval-valued preinvex functions on coordinates. Further, we establish Hermite–Hadamard type inclusions for the product of two interval-valued coordinated preinvex functions. These results are motivated by the symmetric results obtained in the recent article by Kara et al. in 2021 on weighted Hermite–Hadamard type inclusions for products of coordinated convex interval-valued functions. Our established results generalize and extend some recent results obtained in the existing literature. Moreover, we provide suitable examples in the support of our theoretical results.

Keywords: invex set; coordinated preinvex functions; Hermite–Hadamard inequalities; interval-valued functions

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1. Introduction

In recent years, many researchers have made efforts to generalize and extend the classical convexity in different directions and discovered new integral inequalities for this generalized and extended convexity; see, for instance, [1–6]. In 1981, Hanson [7] introduced a useful generalization of convex functions known as invex functions. Craven and Glover [8] showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima. The concept of preinvex functions was introduced by Ben-Israel and Mond [9]. It is well known that preinvex functions are nonconvex functions. This concept inspired a large number of research papers dealing with the analysis and applications of this newly defined nonconvex function in optimization theory and related fields; see [10–12].

Noor [13] obtained Hermite–Hadamard (H-H) inequality for the preinvex functions, which is a generalization of the classical H-H inequality. Dragomir [14] defined the concept of classical convex functions on coordinates and demonstrated H-H type inequalities for these functions. Further, Latif and Dragomir [15] defined preinvex functions on the coordinates and established some H-H type inequalities for functions whose second-order partial derivatives in absolute value are preinvex on the coordinates. Matłoka [16] introduced the class of (h_1, h_2)-preinvex functions on the coordinates and proved H-H and Fejér type inequalities using the symmetricity of the positive function. For more details on preinvex functions and related inequalities, see [17–21].

The concept of interval analysis was first considered by Moore [22]. In 1979, Moore [23] studied the integration of interval-valued functions and investigated interval methods for computing upper and lower bounds on exact values of integrals of interval-valued functions. Bhurjee and Panda [24] presented a general multi-objective fractional programming problem whose parameters in the objective functions and constraints are intervals and developed a methodology to determine its efficient solutions. Zhang et al. [25] extended the concepts of invexity and preinvexity to interval-valued functions and derived KKT optimality conditions for LU-prinvex and invex optimization problems with an interval-valued objective function. Zhao et al. [26] introduced the interval double integral for interval-valued functions and gave Chebyshev type inequalities for interval-valued functions. Practical applications of interval analysis include areas of economics, chemical engineering, beam physics, control circuitry design, global optimization, robotics, error analysis, signal processing, and computer graphics (see [27–31]).

Budak et al. [32] defined interval-valued right-sided Riemann–Liouville fractional integral and derived H-H type inequalities for interval-valued Riemann–Liouville fractional integrals. Sharma et al. [33] introduced interval-valued preinvex function and established fractional H-H type inequalities for these functions. Recently, Zhao et al. [34,35] proposed the notion of interval-valued convex functions on coordinates and established H-H type inequalities for these interval-valued coordinated convex functions. Further, Budak et al. [36] described a new concept of interval-valued fractional integrals on coordinates and investigated H-H type inequalities for interval-valued coordinated convex functions using these fractional integrals. Kara et al. [37] proved H–H–Fejér type inclusions for the product of two interval-valued convex functions on coordinates. For more details of the relationships between the different forms of interval-valued functions and integral inequalities, we refer to [38–43] and references therein.

The work in this research paper is mainly motivated by Zhao et al. [34] and Sharma et al. [33]. We propose the notion of interval-valued preinvex functions on coordinates, which is a generalization of interval-valued convex functions on coordinates, and prove new H-H type inclusions for these interval-valued coordinated preinvex functions. We also present H-H type inclusions for the product of two interval-valued preinvex functions on coordinates. Moreover, we illustrate our results with the help of some suitable examples. The results established in this paper include the previously known results for interval-valued convex functions on coordinates as a special case. For future directions, we can investigate H-H type inclusions for interval-valued coordinated preinvex functions using interval-valued fractional integrals on coordinates.

The organization of this paper is as follows: In Section 2, we present some necessary preliminaries. In Section 3, we define preinvex interval-valued functions on coordinates and investigate H-H type inclusions for coordinated preinvex interval-valued functions. Further, we present H-H type inclusions for the product of two interval-valued preinvex functions on coordinates. Some special cases of these results are also investigated in Section 3. In Section 4, we discuss the conclusions and future directions of this study.

2. Preliminaries

In this section, we recall some notations, basic definitions, and related results that are necessary for this paper.

Let $\mathbb{R}_I$, $\mathbb{R}_I^+$, $\mathbb{R}_I^-$ be the set of all closed intervals of $\mathbb{R}$, set of all positive closed intervals of $\mathbb{R}$, and set of all negative closed intervals of $\mathbb{R}$, respectively. If $\Lambda \in \mathbb{R}_I$, then interval Λ is defined by:

$$\Lambda = [\underline{\Lambda}, \overline{\Lambda}] = \{u \in \mathbb{R} : \underline{\Lambda} \leq u \leq \overline{\Lambda}\},\ \underline{\Lambda}, \overline{\Lambda} \in \mathbb{R}.$$

The interval $\Lambda = [\underline{\Lambda}, \overline{\Lambda}]$ is called degenerated if $\underline{\Lambda} = \overline{\Lambda}$; positive if $\underline{\Lambda} > 0$; and negative if $\overline{\Lambda} < 0$.

Let $\Lambda_1 = [\underline{\Lambda}_1, \overline{\Lambda}_1], \Lambda_2 = [\underline{\Lambda}_2, \overline{\Lambda}_2] \in \mathbb{R}_I$. We say $\Lambda_1 \subseteq \Lambda_2$ (or $\Lambda_2 \supseteq \Lambda_1$) if and only if $\underline{\Lambda}_2 \leq \underline{\Lambda}_1$ and $\overline{\Lambda}_1 \leq \overline{\Lambda}_2$.

The Hausdorff distance between $\Lambda_1 = [\underline{\Lambda}_1, \overline{\Lambda}_1]$ and $\Lambda_2 = [\underline{\Lambda}_2, \overline{\Lambda}_2]$ is defined as

$$d(\Lambda_1, \Lambda_2) = d([\underline{\Lambda}_1, \overline{\Lambda}_1], [\underline{\Lambda}_2, \overline{\Lambda}_2]) = max\{|\ \underline{\Lambda}_1 - \underline{\Lambda}_2\ |, |\ \overline{\Lambda}_1 - \overline{\Lambda}_2\ |\}.$$

For more properties and notations of intervals, we refer to [23,28].

Definition 1 ([23])**.** *A function Ω is called an interval-valued function on $[p,q]$ if it assigns a nonempty interval to each $u \in [p,q]$ and*

$$\Omega(u) = [\underline{\Omega}(u), \overline{\Omega}(u)],$$

where $\underline{\Omega}$ and $\overline{\Omega}$ are real-valued functions.

A partition P_1 of $[p,q]$ is a set of numbers $\{\omega_{i-1}, \nu_i, \omega_i\}_{i=1}^m$ such that

$$P_1 : p = \omega_0 < \omega_1 < \ldots < \omega_m = q$$

with $\omega_{i-1} \leq \nu_i \leq \omega_i$ for all $i = 1,2,3\ldots m$. Partition P_1 is said to be δ-fine if $\Delta\omega_i < \delta$ for all i, where $\Delta\omega_i = \omega_i - \omega_{i-1}$. Let the set of all δ-fine partitions of $[p,q]$ be denoted by $\mathcal{P}(\delta, [p,q])$. If $\{\omega_{i-1}, \nu_i, \omega_i\}_{i=1}^m$ is a δ-fine P_1 of $[p,q]$ and $\{\sigma_{j-1}, \mu_j, \sigma_j\}_{j=1}^n$ is a δ-fine P_2 of $[r,s]$, then the rectangles

$$\Delta_{i,j} = [\omega_{i-1}, \omega_i] \times [\sigma_{j-1}, \sigma_j]$$

partition rectangle $\Delta = [p,q] \times [r,s]$ with the points (ν_i, μ_j) are inside the rectangles $[\omega_{i-1}, \omega_i] \times [\sigma_{j-1}, \sigma_j]$. Furthermore, we denote the set of all δ-fine partitions of Δ with $P_1 \times P_2$ by $\mathcal{P}(\delta, \Delta)$, where $P_1 \in \mathcal{P}(\delta, [p,q])$ and $P_2 \in \mathcal{P}(\delta, [r,s])$. Let $\Delta A_{i,j}$ be the area of the rectangle $\Delta_{i,j}$. Choose an arbitrary (ν_i, μ_j) from each rectangle $\Delta_{i,j}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, and we get

$$S(\Omega, P, \delta, \Delta) = \sum_{i=1}^{m} \sum_{j=1}^{n} \Omega(\nu_i, \mu_j) \Delta A_{i,j},$$

where $\Omega : \Delta \to \mathbb{R}_I$. $S(\Omega, P, \delta, \Delta)$ denotes integral sum of Ω corresponding to the $P \in \mathcal{P}(\delta, \Delta)$.

Definition 2 ([26])**.** *A function $\Omega : [p,q] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[p,q]$ with (IR)-integral $I = (IR)\int_p^q \Omega(\lambda)d\lambda$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(S(\Omega, P, \delta, [p,q]), I) < \epsilon$$

for each $P \in \mathcal{P}(\delta, [p,q])$.

The collection of all (IR)-integrable functions on $[p,q]$ denoted by $IR_{([p,q])}$.

Definition 3 ([26])**.** *A function $\Omega : \Delta \to \mathbb{R}_I$ is called interval double integrable (ID-integrable) on Δ with (ID)-integral $I = (ID)\int\int_\Delta \Omega(u,v)dA$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(S(\Omega, P, \delta, \Delta), I) < \epsilon$$

for each $P \in \mathcal{P}(\delta, \Delta)$.

The collection of all (ID)-integrable functions on Δ denoted by $ID_{(\Delta)}$.

Theorem 1 ([28])**.** *Let $\Omega : [p,q] \to R_I$ be an interval-valued function such that $\Omega = [\underline{\Omega}, \overline{\Omega}]$. Then, ψ is (IR)-integrable on $[p,q]$ if and only if $\underline{\Omega}$ and $\overline{\Omega}$ are R-integrable on $[p,q]$ and*

$$(IR)\int_p^q \Omega(u)du = \left[(R)\int_p^q \underline{\Omega}(u)du, (R)\int_p^q \overline{\Omega}(u)du\right].$$

Theorem 2 ([26])**.** *Let $\Delta = [p,q] \times [r,s]$. If $\Omega : \Delta \to R_I$ be an interval-valued function such that $\Omega = [\underline{\Omega}, \overline{\Omega}]$ and $\Omega \in ID_{(\Delta)}$, then we have*

$$(ID)\int\int_\Delta \Omega(u,v)dA = (ID)\int_p^q (ID)\int_r^s \Omega(u,v)dvdu.$$

Definition 4 ([12]). *The set $X \subseteq \mathbb{R}^n$ is said to be invex with respect to vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, if*

$$v + \lambda\eta(u,v) \in X, \quad for\ all\ u,v \in X,\ \ \lambda \in [0,1].$$

Remark 1. *Every convex set is invex with respect to $\eta(u,v) = u - v$ but not conversely.*

Definition 5 ([12]). *The function Ω on the invex set X is said to be preinvex with respect to η, if*

$$\Omega(v + \lambda\eta(u,v)) \leq (1-\lambda)\Omega(v) + \lambda\Omega(u), \ \ for\ all\ u,v \in X,\ \ \lambda \in [0,1].$$

Remark 2. *Every convex function is preinvex with respect to $\eta(u,v) = u - v$ but not conversely.*

Condition C [10] Let $X \subseteq \mathbb{R}$ be an invex set with respect to $\eta(.,.)$. Then, function η satisfies Condition C if for any $\lambda \in [0,1]$ and any $u, v \in X$,

$$\eta(v, v + \lambda\eta(u,v)) = -\lambda\eta(u,v),$$

$$\eta(u, v + \lambda\eta(u,v)) = (1-\lambda)\eta(u,v).$$

For all $\lambda_1, \lambda_2 \in [0,1],\ u,v \in X$ and from Condition C, we have

$$\eta(v + \lambda_2\eta(u,v), v + \lambda_1\eta(u,v)) = (\lambda_2 - \lambda_1)\eta(u,v).$$

Theorem 3 ([13]). *Let $\Omega : [p, p+\eta(q,p)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers X^o (the interior of X) and $p,q \in X^o$ with $p < p + \eta(q,p)$. Then the following inequality holds:*

$$\Omega\left(\frac{2p + \eta(q,p)}{2}\right) \leq \frac{1}{\eta(q,p)} \int_p^{p+\eta(q,p)} \Omega(u)du \leq \frac{\Omega(p) + \Omega(q)}{2}.$$

Definition 6 ([33]). *If $X \subseteq \mathbb{R}$ is an invex set with respect to $\eta(.,.)$, $\Omega(u) = [\underline{\psi}(u), \overline{\psi}(u)]$ is an interval-valued function on X. Then Ω is preinvex interval-valued function on X with respect to $\eta(.,.)$ if*

$$\Omega(v + \lambda\eta(u,v)) \supseteq \lambda\Omega(u) + (1-\lambda)\Omega(v), \ \ for\ all\ \ u,v \in X,\ \ \lambda \in [0,1].$$

Let X_1 and X_2 be two nonempty subsets of $\mathbb{R}^n$, $\eta_1 : X_1 \times X_1 \to \mathbb{R}^n$ and $\eta_2 : X_2 \times X_2 \to \mathbb{R}^n$.

Definition 7 ([16]). *Let $(u,v) \in X_1 \times X_2$. The set $X_1 \times X_2$ is said to be invex at (u,v) with respect to η_1 and η_2, if for each $(w,z) \in X_1 \times X_2$ and λ_1,$\lambda_2 \in [0,1]$,*

$$(u + \lambda_1\eta_1(w,u), v + \lambda_2\eta_2(z,v)) \in X_1 \times X_2.$$

$X_1 \times X_2$ is said to be invex set with respect to η_1 and η_2 if $X_1 \times X_2$ is invex at each $(w,z) \in X_1 \times X_2$.

Theorem 4 ([33]). *Let $X \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : X \times X \to \mathbb{R}$ and $p,q \in X$ with $p < p + \eta(q,p)$. If $\Omega : [p, p+\eta(q,p)] \to \mathbb{R}_I^+$ is a preinvex interval-valued function such that $\Omega(\lambda) = [\underline{\Omega}(\lambda), \overline{\Omega}(\lambda)]$; $\Omega \in L[p, p+\eta(q,p)]$ and η satisfies Condition C and $\alpha > 0$, then*

$$\begin{aligned}\Omega\left(p + \frac{\eta(q,p)}{2}\right) &\supseteq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(q,p)}[J_{p^+}^\alpha \Omega(p+\eta(q,p)) + J_{(p+\eta(q,p))^-}^\alpha \Omega(p)] \\ &\supseteq \frac{\Omega(p) + \Omega(p+\eta(q,p))}{2} \supseteq \frac{\Omega(p)+\Omega(q)}{2}.\end{aligned}$$

Corollary 1. *If $\alpha = 1$, then Theorem 4 reduces to the following result:*

$$\Omega\left(p+\frac{\eta(q,p)}{2}\right) \supseteq \frac{1}{\eta(q,p)}\int_p^{p+\eta(q,p)} \Omega(\lambda)d\lambda \supseteq \frac{\Omega(p)+\Omega(p+\eta(q,p))}{2} \supseteq \frac{\Omega(p)+\Omega(q)}{2}.$$

Theorem 5 ([33]). *Let $X \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : X \times X \to \mathbb{R}$ and $p, q \in X$ with $p < p+\eta(q,p)$. If $\Omega, \Upsilon : [p, p+\eta(q,p)] \to \mathbb{R}_I^+$ is a preinvex interval-valued function such that $\Omega(\lambda) = [\underline{\Omega}(\lambda), \overline{\Omega}(\lambda)]$ and $\Upsilon(\lambda) = [\underline{\Upsilon}(\lambda), \overline{\Upsilon}(\lambda)]$; $\Omega, \Upsilon \in L[p, p+\eta(q,p)]$ and η satisfies Condition C and $\alpha > 0$, then*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\eta^\alpha(q,p)}[J^\alpha_{p^+}\Omega(p+\eta(q,p))\Upsilon(p+\eta(q,p)) + J^\alpha_{(p+\eta(q,p))^-}\Omega(p)\Upsilon(p)] \\ &\supseteq \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)F(p,p+\eta(q,p)) + \frac{\alpha}{(\alpha+1)(\alpha+2)}G(p,p+\eta(q,p))\end{aligned} \tag{1}$$

and

$$\begin{aligned}&2\Omega\left(p+\frac{1}{2}\eta(q,p)\right)\Upsilon\left(p+\frac{1}{2}\eta(q,p)\right) \\ &\supseteq \frac{\Gamma(\alpha+1)}{2\eta^\alpha(q,p)}[J^\alpha_{p^+}\Omega(p+\eta(q,p))\Upsilon(p+\eta(q,p)) + J^\alpha_{(p+\eta(q,p))^-}\Omega(p)\Upsilon(p)] \\ &+ \left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)G(p,p+\eta(q,p)) + \frac{\alpha}{(\alpha+1)(\alpha+2)}F(p,p+\eta(q,p)),\end{aligned} \tag{2}$$

where $F(p,p+\eta(q,p)) = \Omega(p)\Upsilon(p) + \Omega(p+\eta(q,p))\Upsilon(p+\eta(q,p))$ and $G(p,p+\eta(q,p)) = \Omega(p)\Upsilon(p+\eta(q,p)) + \Omega(p+\eta(q,p))\Upsilon(p)$.

Corollary 2. *If $\alpha = 1$, then (1) reduces to the following result:*

$$\frac{1}{\eta(q,p)}\int_p^{p+\eta(q,p)} \Omega(\lambda)\Upsilon(\lambda)d\lambda \supseteq \frac{1}{3}F(p,p+\eta(q,p)) + \frac{1}{6}G(p,p+\eta(q,p)).$$

Corollary 3. *If $\alpha = 1$, then (2) reduces to the following result:*

$$\begin{aligned}&2\Omega\left(p+\frac{1}{2}\eta(q,p)\right)\Upsilon\left(p+\frac{1}{2}\eta(q,p)\right) \\ &\supseteq \frac{1}{\eta(q,p)}\int_p^{p+\eta(q,p)} \Omega(\lambda)\Upsilon(\lambda)d\lambda + \frac{1}{3}G(p,p+\eta(q,p)) + \frac{1}{6}F(p,p+\eta(q,p)).\end{aligned}$$

3. Main Results

In this section, first, we give the definition of interval-valued coordinated preinvex function.

Definition 8. *Let $X_1 \times X_2$ be an invex set with respect to η_1 and η_2, $\Omega = [\underline{\Omega}, \overline{\Omega}]$ be an interval valued function defined on $X_1 \times X_2$. The function Ω is said to be interval-valued coordinated preinvex function with respect to η_1 and η_2 if the partial mappings $\Omega_v : X_1 \to \mathbb{R}_I^+$, $\Omega_v(w) = (w, v)$ and $\Omega_u : X_2 \to \mathbb{R}_I^+$, $\Omega_u(z) = (u, z)$ are interval-valued preinvex functions with respect to η_1 and η_2, respectively, for all $u \in X_1$ and $v \in X_2$.*

Remark 3. *From the definition of interval-valued coordinated preinvex functions, it follows that if* Ω *is an interval-valued coordinated preinvex function, then*

$$\begin{aligned}\Omega(u+\lambda_1\eta_1(w,u), v+\lambda_2\eta_2(z,v)) \supseteq\ & (1-\lambda_1)(1-\lambda_2)\Omega(u,v) + (1-\lambda_1)\lambda_2\Omega(u,z)\\ & + \lambda_1(1-\lambda_2)\Omega(w,v) + \lambda_1\lambda_2\Omega(w,z),\end{aligned}$$

for all $(u,v),(u,z),(w,v),(w,z) \in X_1 \times X_2$ *and* $\lambda_1, \lambda_2 \in [0,1]$.

If $\eta_1(w,u) = w-u$ and $\eta_2(z,v) = z-v$, then the definition of interval-valued coordinated preinvex function reduces to the definition of interval-valued coordinated convex function proposed by Zhao et al. [34].

Example 1. *An interval-valued function* $\Omega : [0,1] \times [\frac{1}{2},1] \to \mathbb{R}_I^+$ *defined as* $\Omega(u,v) = [u+v, (2-u)(2-v)]$ *is an interval-valued coordinated preinvex function with respect to* $\eta_1(w,u) = w-u-1$ *and* $\eta_2(z,v) = z-2v$ *for all* $u,w \in [0,1]$ *and* $v,z \in \left[\frac{1}{2},1\right]$.

Now, we establish H-H type inclusions for interval-valued preinvex functions on coordinates. In what follows, without any confusion, we will not include the symbol (R), (IR), or (ID) before the integral sign.

Theorem 6. *Let* $X_1 \times X_2$ *be an invex set with respect to* η_1 *and* η_2. *If* $\Omega : X_1 \times X_2 \to \mathbb{R}_I^+$ *is an interval-valued coordinated preinvex function with respect to* η_1 *and* η_2 *such that* $\Omega = [\underline{\Omega}, \overline{\Omega}]$ *and* $p < p+\eta_1(q,p)$, $r < r+\eta_2(s,r)$, *where* $p,q \in X_1$ *and* $r,s \in X_2$. *If* η_1, η_2 *satisfy Condition C, then we have*

$$\begin{aligned}\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r)\right) &\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu\\ &\supseteq \frac{1}{4}[\Omega(p,r)+\Omega(q,r)+\Omega(p,s)+\Omega(q,s)].\end{aligned}$$

Proof. Since Ω is an interval-valued preinvex function on coordinates with respect to η_1 and η_2, we have

$$\begin{aligned}\Omega(p+\lambda_1\eta_1(q,p), r+\lambda_2\eta_2(s,r)) \supseteq\ & (1-\lambda_1)(1-\lambda_2)\Omega(p,r) + (1-\lambda_1)\lambda_2\Omega(p,s)\\ & + \lambda_1(1-\lambda_2)\Omega(q,r) + \lambda_1\lambda_2\Omega(q,s). \end{aligned} \tag{3}$$

Integrating (3) with respect to (λ_1, λ_2) over $[0,1]\times[0,1]$, we get

$$\begin{aligned}&\int_0^1\int_0^1 \Omega(p+\lambda_1\eta_1(q,p), r+\lambda_2\eta_2(s,r))d\lambda_2 d\lambda_1\\ &\supseteq \int_0^1\int_0^1 (1-\lambda_1)(1-\lambda_2)\Omega(p,r)d\lambda_2 d\lambda_1 + \int_0^1\int_0^1 (1-\lambda_1)\lambda_2\Omega(p,s)d\lambda_2 d\lambda_1\\ &\quad + \int_0^1\int_0^1 \lambda_1(1-\lambda_2)\Omega(q,r)d\lambda_2 d\lambda_1 + \int_0^1\int_0^1 \lambda_1\lambda_2\Omega(q,s)d\lambda_2 d\lambda_1.\end{aligned}$$

This implies that

$$\frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \supseteq \frac{1}{4}[\Omega(p,r)+\Omega(p,s)+\Omega(q,r)+\Omega(q,s)]. \tag{4}$$

Using the definition of an interval-valued coordinated preinvex function and Condition C for η_1, η_2, we get

$$
\begin{aligned}
&\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r)\right) \\
&= \Omega(p+\lambda_1\eta_1(q,p)+\frac{1}{2}\eta_1(p+(1-\lambda_1)\eta_1(q,p), p+\lambda_1\eta_1(q,p)),\ r+\lambda_2\eta_2(s,r) \\
&\quad +\frac{1}{2}\eta_2(r+(1-\lambda_2)\eta_2(s,r), r+\lambda_2\eta_2(s,r))) \\
&\supseteq \frac{1}{4}[\Omega(p+\lambda_1\eta_1(q,p), r+\lambda_2\eta_2(s,r))+\Omega(p+\lambda_1\eta_1(q,p), r+(1-\lambda_2)\eta_2(s,r)) \\
&\quad +\Omega(p+(1-\lambda_1)\eta_1(q,p), r+\lambda_2\eta_2(s,r))+\Omega(p+(1-\lambda_1)\eta_1(q,p), r+(1-\lambda_2)\eta_2(s,r))]
\end{aligned} \tag{5}
$$

Thus, integrating (5) with respect to (λ_1, λ_2) over $[0,1]\times[0,1]$, we get

$$
\begin{aligned}
&\int_0^1\int_0^1 \Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r)\right)d\lambda_2 d\lambda_1 \\
&\supseteq \frac{1}{4}\int_0^1\int_0^1[\Omega(p+\lambda_1\eta_1(q,p), r+\lambda_2\eta_2(s,r))+\Omega(p+\lambda_1\eta_1(q,p), r+(1-\lambda_2)\eta_2(s,r)) \\
&\quad +\Omega(p+(1-\lambda_1)\eta_1(q,p), r+\lambda_2\eta_2(s,r))+\Omega(p+(1-\lambda_1)\eta_1(q,p), r+(1-\lambda_2)\eta_2(s,r))]d\lambda_2 d\lambda_1.
\end{aligned}
$$

This implies

$$
\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r)\right) \supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu. \tag{6}
$$

From (4) and (6), we get the desired result. □

Theorem 7. *Let $X_1\times X_2$ be an invex set with respect to η_1 and η_2. If $\Omega : [p, p+\eta_1(q,p)]\times[r, r+\eta_2(s,r)] \to \mathbb{R}_I^+$ is an interval-valued coordinated preinvex function with respect to η_1 and η_2 such that $\Omega = [\underline{\Omega}, \overline{\Omega}]$ and $p < p+\eta_1(q,p)$, $r < r+\eta_2(s,r)$, where $p, q \in X_1$ and $r, s \in X_2$. If η_1, η_2 satisfy Condition C, then we have*

$$
\begin{aligned}
&\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r))\right)du + \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv \\
&\supseteq \frac{2}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \\
&\supseteq \frac{1}{2}\left[\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)+\Omega(u, r+\eta_2(s,r)))du\right. \\
&\quad \left.+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)+\Omega(p+\eta_1(q,p), v))dv\right].
\end{aligned} \tag{7}
$$

Proof. Since Ω is an interval-valued preinvex function on coordinates $[p, p+\eta_1(q,p)]\times[r, r+\eta_2(s,r)]$, then $\Omega_u : [r, r+\eta_2(s,r)] \to \mathbb{R}_I^+$, $\Omega_u(v) = \Omega(u,v)$ is an interval-valued preinvex function on $[r, r+\eta_2(s,r)]$ for all $u \in [p, p+\eta_1(q,p)]$. From Corollary 1, we have

$$
\Omega_u\left(r+\frac{1}{2}\eta_2(s,r))\right) \supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega_u(v)dv \supseteq \frac{\Omega_u(r)+\Omega_u(r+\eta_2(s,r))}{2}.
$$

This implies

$$
\Omega\left(u, r+\frac{1}{2}\eta_2(s,r))\right) \supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dv \supseteq \frac{\Omega(u,r)+\Omega(u, r+\eta_2(s,r))}{2}. \tag{8}
$$

Integrating (8) over $[p, p+\eta_1(q,p)]$ with respect to u, then dividing by $\eta_1(q,p)$, we get

$$\begin{aligned}
&\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r)\right)du \\
&\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \\
&\supseteq \frac{1}{2\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)+\Omega(u,r+\eta_2(s,r)))du. \qquad (9)
\end{aligned}$$

Similarly, $\Omega_v : [p, p+\eta_1(p,q)] \to \mathbb{R}_I^+, \Omega_v(u) = \Omega(u,v)$ is interval-valued preinvex function on $[p, p+\eta_1(p,q)]$ for all $v \in [r, r+\eta_2(s,r)]$. Then, we have

$$\begin{aligned}
&\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv \\
&\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \\
&\supseteq \frac{1}{2\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)+\Omega(p+\eta_1(q,p),v))dv. \qquad (10)
\end{aligned}$$

By adding (9) and (10), we have

$$\begin{aligned}
&\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r)\right)du + \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv \\
&\supseteq \frac{2}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \\
&\supseteq \frac{1}{2}\left[\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)+\Omega(u,r+\eta_2(s,r)))du\right. \\
&\quad \left.+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)+\Omega(p+\eta_1(q,p),v))dv\right].
\end{aligned}$$

This completes the proof. □

Example 2. *Let* $[p, p+\eta_1(q,p)] = [\frac{1}{4}, \frac{1}{2}]$, $[r, r+\eta_2(s,r)] = [\frac{1}{4}, \frac{1}{2}]$ *and* $\eta_1(q,p) = q-2p$, $\eta_2(s,r) = s-2r$. *Let* $\Omega : [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{4}, \frac{1}{2}] \to \mathbb{R}_I^+$ *be defined by* $\Omega(u,v) = [uv, (1-u)(1-v)]$ $\forall\, u \in [\frac{1}{4}, \frac{1}{2}]$ *and* $v \in [\frac{1}{4}, \frac{1}{2}]$. *Then all assumptions of Theorem 7 are satisfied.*

Theorem 8. *Let* $X_1 \times X_2$ *be an invex set with respect to* η_1 *and* η_2. *If* $\Omega : [p, p+\eta_1(q,p)] \times [r, r+\eta_2(s,r)] \to \mathbb{R}_I^+$ *is an interval-valued coordinated preinvex function with respect to* η_1 *and* η_2 *such that* $\Omega = [\underline{\Omega}, \overline{\Omega}]$ *and* $p < p+\eta_1(q,p)$, $r < r+\eta_2(s,r)$, *where* $p, q \in X_1$ *and* $r, s \in X_2$. *If* η_1, η_2 *satisfy Condition C, then we have*

$$
\begin{aligned}
&\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r))\right) \\
&\supseteq \frac{1}{2}\left[\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r))\right)du+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv\right] \\
&\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)dvdu \\
&\supseteq \frac{1}{4}\left[\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)+\Omega(u,r+\eta_2(s,r)))du\right. \\
&\quad \left.+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)+\Omega(p+\eta_1(q,p),v))dv\right] \\
&\supseteq \frac{1}{4}[\Omega(p,r)+\Omega(p+\eta_1(q,p),r)+\Omega(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))] \\
&\supseteq \frac{1}{4}[\Omega(p,r)+\Omega(q,r)+\Omega(p,s)+\Omega(q,s)].
\end{aligned}
$$

Proof. Since Ω is an interval-valued preinvex function on coordinates $[p, p+\eta_1(q,p)] \times [r, r+\eta_2(s,r)]$, then from Corollary 1 we get

$$
\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r))\right) \supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r))\right)du, \tag{11}
$$

$$
\Omega\left(p+\frac{1}{2}\eta_1(q,p), r+\frac{1}{2}\eta_2(s,r))\right) \supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv. \tag{12}
$$

Adding (11) and (12), we have

$$
\begin{aligned}
&\Omega\left(p+\frac{1}{2}\eta_1(q,p),\ r+\frac{1}{2}\eta_2(s,r))\right) \\
&\supseteq \frac{1}{2}\left[\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega\left(u, r+\frac{1}{2}\eta_2(s,r))\right)du+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega\left(p+\frac{1}{2}\eta_1(q,p), v\right)dv\right].
\end{aligned} \tag{13}
$$

Again from Corollary 1, we get

$$
\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r)du \supseteq \frac{\Omega(p,r)+\Omega(p+\eta_1(q,p),r)}{2}, \tag{14}
$$

$$
\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\eta_2(s,r))du \supseteq \frac{\Omega(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))}{2}, \tag{15}
$$

$$
\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p,v)dv \supseteq \frac{\Omega(p,r)+\Omega(p,r+\eta_2(s,r))}{2}, \tag{16}
$$

$$
\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p+\eta_1(q,p),v)dv \supseteq \frac{\Omega(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))}{2}. \tag{17}
$$

Adding (14)–(17), we get

$$\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)+\Omega(u,r+\eta_2(s,r)))du$$
$$+\frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)+\Omega(p+\eta_1(q,p),v))dv$$
$$\supseteq \Omega(p,r)+\Omega(p+\eta_1(q,p),r)+\Omega(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r)). \quad (18)$$

By Corollary 1, we also have

$$\Omega(p,r)+\Omega(p+\eta_1(q,p),r)+\Omega(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))$$
$$\supseteq \Omega(p,r)+\Omega(q,r)+\Omega(p,s)+\Omega(q,s). \quad (19)$$

From (7), (13), (18), and (19), we get the desired result. □

Remark 4. *If we put $\eta_1(q,p)=q-p$ and $\eta_2(s,r)=s-r$ in Theorem 8, we obtain Theorem 7 of [34].*

Next, we prove H-H type inclusions for the product of two interval-valued coordinated preinvex functions.

Theorem 9. *Let $X_1\times X_2$ be an invex set with respect to η_1 and η_2. If $\Omega,\Upsilon:[p,p+\eta_1(q,p)]\times[r,r+\eta_2(s,r)]\to\mathbb{R}_I^+$ are interval-valued coordinated preinvex functions with respect to η_1 and η_2 such that $\Omega=[\underline{\Omega},\overline{\Omega}]$, $\Upsilon=[\underline{\Upsilon},\overline{\Upsilon}]$ and $p<p+\eta_1(q,p)$, $r<r+\eta_2(s,r)$, where $p,q\in X_1$ and $r,s\in X_2$. If η_1,η_2 satisfy Condition C, then*

$$\frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)\Upsilon(u,v)dvdu$$
$$\supseteq \frac{1}{9}N_1(p,q,r,s)+\frac{1}{18}N_2(p,q,r,s)+\frac{1}{18}N_3(p,q,r,s)+\frac{1}{36}N_4(p,q,r,s),$$

where
$N_1(p,q,r,s)=\Omega(p,r)\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r)+\Omega(p,r+\eta_2(s,r))$ $\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))$,

$N_2(p,q,r,s)=\Omega(p,r)\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r)\Upsilon(p,r)+\Omega(p,r+\eta_2(s,r))$ $\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))$,

$N_3(p,q,r,s)=\Omega(p,r)\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+$ $\Omega(p,r+\eta_2(s,r))\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)$,

$N_4(p,q,r,s)=\Omega(p,r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p,r+\eta_2(s,r))+$ $\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r)$.

Proof. Since Ω and Υ are interval-valued coordinated preinvex functions on $[p,p+\eta_1(q,p)]\times[r,r+\eta_2(s,r)]$, we have

$$\Omega_u(v):[r,r+\eta_2(s,r)]\to\mathbb{R}_I^+,\ \Omega_u(v)=\Omega(u,v)$$
and
$$\Upsilon_u(v):[r,r+\eta_2(s,r)]\to\mathbb{R}_I^+,\ \Upsilon_u(v)=\Upsilon(u,v)$$

are interval-valued preinvex functions on $[r, r+\eta_2(s,r)]$ for all $u \in [p, p+\eta_1(q,p)]$. Similarly,

$$\Omega_v(u) : [p, p+\eta_1(q,p)] \to \mathbb{R}_I^+,\ \Omega_v(u) = \Omega(u,v)$$
$$and$$
$$\Upsilon_v(u) : [p, p+\eta_1(q,p)] \to \mathbb{R}_I^+,\ \Upsilon_v(u) = \Upsilon(u,v)$$

are interval-valued preinvex functions on $[p, p+\eta_1(q,p)]$ for all $v \in [r, r+\eta_2(s,r)]$.

From Corollary 2, we get

$$\begin{aligned}
&\frac{1}{\eta_2(s,r)} \int_r^{r+\eta_2(s,r)} \Omega_u(v)\Upsilon_u(v)dv \\
&\supseteq \frac{1}{3}[\Omega_u(r)\Upsilon_u(r) + \Omega_u(r+\eta_2(s,r))\Upsilon_u(r+\eta_2(s,r))] + \frac{1}{6}[\Omega_u(r)\Upsilon_u(r+\eta_2(s,r)) + \Omega_u(r+\eta_2(s,r))\Upsilon_u(r)].
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{\eta_2(s,r)} \int_r^{r+\eta_2(s,r)} \Omega(u,v)\Upsilon(u,v)dv \\
&\supseteq \frac{1}{3}[\Omega(u,r)\Upsilon(u,r) + \Omega(u,r+\eta_2(s,r))\Upsilon(u,r+\eta_2(s,r))] \\
&\quad + \frac{1}{6}[\Omega(u,r)\Upsilon(u,r+\eta_2(s,r)) + \Omega(u,r+\eta_2(s,r))\Upsilon(u,r)].
\end{aligned} \tag{20}$$

Integrating (20) with respect to u over $[p, p+\eta_1(q,p)]$ and after then dividing by $\eta_1(q,p)$, we find

$$\begin{aligned}
&\frac{1}{\eta_1(q,p)\eta_2(s,r)} \int_p^{p+\eta_1(q,p)} \int_r^{r+\eta_2(s,r)} \Omega(u,v)\Upsilon(u,v)dvdu \\
&\supseteq \frac{1}{3\eta_1(q,p)} \int_p^{p+\eta_1(q,p)} [\Omega(u,r)\Upsilon(u,r) + \Omega(u,r+\eta_2(s,r))\Upsilon(u,r+\eta_2(s,r))]du \\
&\quad + \frac{1}{6\eta_1(q,p)} \int_p^{p+\eta_1(q,p)} [\Omega(u,r)\Upsilon(u,r+\eta_2(s,r)) + \Omega(u,r+\eta_2(s,r))\Upsilon(u,r)]du.
\end{aligned} \tag{21}$$

Again from Corollary 2, we have

$$\begin{aligned}
&\frac{1}{\eta_1(q,p)} \int_p^{p+\eta_1(q,p)} \Omega(u,r)\Upsilon(u,r)du \\
&\supseteq \frac{1}{3}[\Omega(p,r)\Upsilon(p,r) + \Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r] \\
&\quad + \frac{1}{6}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r) + \Omega(p+\eta_1(q,p),r)\Upsilon(p,r)],
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\frac{1}{\eta_1(q,p)} \int_p^{p+\eta_1(q,p)} \Omega(u,r+\eta_2(s,r))\Upsilon(u,r+\eta_2(s,r))du \\
&\supseteq \frac{1}{3}[\Omega(p,r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r)) + \Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))] \\
&\quad + \frac{1}{6}[\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r)) + \Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))],
\end{aligned} \tag{23}$$

$$\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r)\Upsilon(u,r+\eta_2(s,r))du$$
$$\supseteq \frac{1}{3}[\Omega(p,r))\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))]$$
$$+\frac{1}{6}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p,r+\eta_2(s,r))], \tag{24}$$

$$\frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\eta_2(s,r))\Upsilon(u,r)du$$
$$\supseteq \frac{1}{3}[\Omega(p,r+\eta_2(s,r))\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r]$$
$$+\frac{1}{6}[\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r)]. \tag{25}$$

Substituting (22)–(25) into (21), we obtain the desired result. Similarly, we can obtain the same result by using Corollary 2 for the product $\Omega_v(u)\Upsilon_v(u)$ on $[p,p+\eta_1(q,p)]$. □

Remark 5. *If we put $\eta_1(q,p) = q-p$ and $\eta_2(s,r) = s-r$ in Theorem 9, we obtain Theorem 8 of [34].*

Theorem 10. *Let $X_1 \times X_2$ be an invex set with respect to η_1 and η_2. If $\Omega,\Upsilon : [p,p+\eta_1(q,p)] \times [r,r+\eta_2(s,r)] \to \mathbb{R}_I^+$ are interval-valued coordinated preinvex functions with respect to η_1 and η_2 such that $\Omega = [\underline{\Omega},\overline{\Omega}]$, $\Upsilon = [\underline{\Upsilon},\overline{\Upsilon}]$ and $p < p+\eta_1(q,p)$, $r < r+\eta_2(s,r)$, where $p,q \in X_1$ and $r,s \in X_2$. If η_1,η_2 satisfy Condition C, then we have*

$$4\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)$$
$$\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)\Upsilon(u,v)dvdu$$
$$+\frac{5}{36}N_1(p,q,r,s)+\frac{7}{36}N_2(p,q,r,s)+\frac{7}{36}N_3(p,q,r,s)+\frac{2}{9}N_4(p,q,r,s),$$

where $N_1(p,q,r,s)$, $N_2(p,q,r,s)$, $N_3(p,q,r,s)$, and $N_4(p,q,r,s)$ are defined as previous.

Proof. Since Ω and Υ are interval-valued coordinated preinvex functions, therefore from Corollary 3, we have

$$2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)$$
$$\supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\frac{1}{2}\eta_2(s,r))\Upsilon(u,r+\frac{1}{2}\eta_2(s,r))du$$
$$+\frac{1}{6}\Big[\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r))$$
$$+\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Big]$$
$$+\frac{1}{3}\Big[\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))$$
$$+\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r))\Big] \tag{26}$$

and

$$\begin{aligned}
&2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\\
&\supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(r+\frac{1}{2}\eta_2(s,r),v)\Upsilon(p+\frac{1}{2}\eta_1(q,p),v)dv\\
&\quad+\frac{1}{6}\Big[\Omega(p+\frac{1}{2}\eta_1(q,p),r)\Upsilon(p+\frac{1}{2}\eta_1(q,p),r)\\
&\quad+\Omega(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Big]\\
&\quad+\frac{1}{3}\Big[\Omega(p+\frac{1}{2}\eta_1(q,p),r)\Upsilon(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\\
&\quad+\Omega(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\frac{1}{2}\eta_1(q,p),r)\Big]. \qquad (27)
\end{aligned}$$

Adding (26) and (27), then multiplying both sides of the resultant one by 2, we find

$$\begin{aligned}
&8\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\\
&\supseteq \frac{2}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\frac{1}{2}\eta_2(s,r))\Upsilon(u,r+\frac{1}{2}\eta_2(s,r))du\\
&\quad+\frac{2}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(r+\frac{1}{2}\eta_2(s,r),v)\Upsilon(p+\frac{1}{2}\eta_1(q,p),v)dv\\
&\quad+\frac{1}{6}\Big[2\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r))\\
&\quad+2\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\\
&\quad+2\Omega(p+\frac{1}{2}\eta_1(q,p),r)\Upsilon(p+\frac{1}{2}\eta_1(q,p),r)\\
&\quad+2\Omega(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Big]\\
&\quad+\frac{1}{3}\Big[2\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\\
&\quad+2\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r))\\
&\quad+2\Omega(p+\frac{1}{2}\eta_1(q,p),r)\Upsilon(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\\
&\quad+2\Omega(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\frac{1}{2}\eta_1(q,p),r)\Big]. \qquad (28)
\end{aligned}$$

Now, from Corollary 3, we have

$$\begin{aligned}
&2\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r))\\
&\supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p,v)\Upsilon(p,v)dv\\
&\quad+\frac{1}{6}[\Omega(p,r)\Upsilon(p,r)+\Omega(p,r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))]\\
&\quad+\frac{1}{3}[\Omega(p,r)\Upsilon(p,r+\eta_2(s,r))+\Omega(p,r+\eta_2(s,r))\Upsilon(p,r)], \qquad (29)
\end{aligned}$$

$$
\begin{aligned}
&2\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)) \\
&\supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p+\eta_1(q,p),v)\Upsilon(p+\eta_1(q,p),v)dv \\
&+\frac{1}{6}[\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))] \\
&+\frac{1}{3}[\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)],
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
&2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r\right) \\
&\supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r)\Upsilon(u,r)du \\
&+\frac{1}{6}[\Omega(p,r)\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r)] \\
&+\frac{1}{3}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r)\Upsilon(p,r)],
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r)\right) \\
&\supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\eta_2(s,r))\Upsilon(u,r+\eta_2(s,r))du \\
&+\frac{1}{6}[\Omega(p,r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))] \\
&+\frac{1}{3}[\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))],
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&2\Omega(p,r+\frac{1}{2}\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)) \\
&\supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p,v)\Upsilon(p+\eta_1(q,p),v)dv \\
&+\frac{1}{6}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r)+\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))] \\
&+\frac{1}{3}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)],
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&2\Omega(p+\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r))\Upsilon(p,r+\frac{1}{2}\eta_2(s,r)) \\
&\supseteq \frac{1}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p+\eta_1(q,p),v)\Upsilon(p,v)dv \\
&+\frac{1}{6}[\Omega(p+\eta_1(q,p),r)\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r+\eta_2(s,r))] \\
&+\frac{1}{3}[\Omega(p+\eta_1(q,p),r)\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r)],
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r)\right)\\
&\supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r)\Upsilon(u,r+\eta_2(s,r))du\\
&+\frac{1}{6}[\Omega(p,r)\Upsilon(p,r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))]\\
&+\frac{1}{3}[\Omega(p,r)\Upsilon(p+\eta_1(q,p),r+\eta_2(s,r))+\Omega(p+\eta_1(q,p),r)\Upsilon(p,r+\eta_2(s,r))],
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&2\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r\right)\\
&\supseteq \frac{1}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\eta_2(s,r))\Upsilon(u,r)du\\
&+\frac{1}{6}[\Omega(p,r+\eta_2(s,r))\Upsilon(p,r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)]\\
&+\frac{1}{3}[\Omega(p,r+\eta_2(s,r))\Upsilon(p+\eta_1(q,p),r)+\Omega(p+\eta_1(q,p),r+\eta_2(s,r))\Upsilon(p,r)].
\end{aligned}
\tag{36}
$$

Using (29)–(36) in (28), we get

$$
\begin{aligned}
&8\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\Upsilon\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\\
&\supseteq \frac{2}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\frac{1}{2}\eta_2(s,r))\Upsilon(u,r+\frac{1}{2}\eta_2(s,r))du\\
&\quad+\frac{2}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(r+\frac{1}{2}\eta_2(s,r),v)\Upsilon(p+\frac{1}{2}\eta_1(q,p),v)dv\\
&\quad+\frac{1}{6\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)\Upsilon(p,v)+\ \Omega(p+\eta_1(q,p),v)\Upsilon(p+\eta_1(q,p),v))dv\\
&\quad+\frac{1}{3\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)\Upsilon(p+\eta_1(q,p),v)+\Omega(p+\eta_1(q,p),v)\Upsilon(p,v))dv\\
&\quad+\frac{1}{6\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)\Upsilon(u,r)+\Omega(u,r+\eta_2(s,r))\Upsilon(u,r+\eta_2(s,r)))du\\
&\quad+\frac{1}{3\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)\Upsilon(u,r+\eta_2(s,r))+\Omega(u,r+\eta_2(s,r))\Upsilon(u,r))du\\
&\quad+\frac{1}{18}N_1(p,q,r,s)+\frac{1}{9}N_2(p,q,r,s)+\frac{1}{9}N_3(p,q,r,s)+\frac{2}{9}N_4(p,q,r,s).
\end{aligned}
\tag{37}
$$

Again from Corollary 3, we have

$$
\begin{aligned}
&\frac{2}{\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}\Omega(p+\frac{1}{2}\eta_1(q,p),v)\Upsilon(p+\frac{1}{2}\eta_1(q,p),v)dv\\
&\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)\Upsilon(u,v)dvdu\\
&+\frac{1}{6\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)\Upsilon(p,v)+\Omega(p+\eta_1(q,p),v)\Upsilon(p+\eta_1(q,p),v))dv\\
&+\frac{1}{3\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)\Upsilon(p+\eta_1(q,p),v)+\Omega(p+\eta_1(q,p),v)\Upsilon(p,v))dv,
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&\frac{2}{\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}\Omega(u,r+\frac{1}{2}\eta_2(s,r))Y(u,r+\frac{1}{2}\eta_2(s,r))du\\
&\supseteq \frac{1}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)Y(u,v)dvdu\\
&+\frac{1}{6\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)Y(u,r)+\Omega(u,r+\eta_2(s,r))Y(u,r+\eta_2(s,r)))du\\
&+\frac{1}{3\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)Y(u,r+\eta_2(s,r))+\Omega(u,r+\eta_2(s,r))Y(u,r))du.
\end{aligned}
\tag{39}
$$

Using (38) and (39) in (37), we get

$$
\begin{aligned}
&8\Omega\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)Y\left(p+\frac{1}{2}\eta_1(q,p),r+\frac{1}{2}\eta_2(s,r)\right)\\
&\supseteq \frac{2}{\eta_1(q,p)\eta_2(s,r)}\int_p^{p+\eta_1(q,p)}\int_r^{r+\eta_2(s,r)}\Omega(u,v)Y(u,v)dvdu\\
&+\frac{1}{3\eta_2(s,r)}\int_r^{r+\eta_2(s,r)}(\Omega(p,v)Y(p,v)+\Omega(p+\eta_1(q,p),v)Y(p+\eta_1(q,p),v)\\
&+2\Omega(p,v)Y(p+\eta_1(q,p),v)+2\Omega(p+\eta_1(q,p),v)Y(p,v))dv\\
&+\frac{1}{3\eta_1(q,p)}\int_p^{p+\eta_1(q,p)}(\Omega(u,r)Y(u,r)+\Omega(u,r+\eta_2(s,r))Y(u,r+\eta_2(s,r))\\
&+2\Omega(u,r)Y(u,r+\eta_2(s,r))+2\Omega(u,r+\eta_2(s,r))Y(u,r))du\\
&+\frac{1}{18}N_1(p,q,r,s)+\frac{1}{9}N_2(p,q,r,s)+\frac{1}{9}N_3(p,q,r,s)+\frac{2}{9}N_4(p,q,r,s).
\end{aligned}
\tag{40}
$$

Applying Corollary 3 for each integral in right side of (40), we obtain our desired result. □

Remark 6. *If we put $\eta_1(q,p)=q-p$ and $\eta_2(s,r)=s-r$ in Theorem 10, we obtain Theorem 9 of [34].*

4. Conclusions

In this article, we have introduced the concept of interval-valued preinvex functions on coordinates as a generalization of the convex interval-valued functions on coordinates. We have established H-H type inclusions for coordinated preinvex interval-valued functions. Moreover, some new H-H type inclusions for the product of two coordinated preinvex interval-valued functions are investigated. The results obtained in this paper may be extended for other kinds of interval-valued preinvex functions on the coordinates. In the future, we can investigate H-H type and H–H–Fejér type inclusions for interval-valued coordinated preinvex functions via interval-valued fractional integrals on coordinates. We hope that the ideas and results obtained in this article will encourage the readers towards further investigation.

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References

1. Kalsoom, H.; Latif, M.A.; Khan, Z.A.; Vivas-Cortez, M. Some New Hermite-Hadamard-Fejér fractional type inequalities for h-convex and harmonically h-Convex interval-valued Functions. *Mathematics* **2021**, *10*, 74. [CrossRef]
2. Sharma, N.; Bisht, J.; Mishra, S.K.; Hamdi, A. Some majorization integral inequalities for functions defined on rectangles via strong convexity. *J. Inequal. Spec. Funct.* **2019**, *10*, 21–34.
3. Khan, M.B.; Zaini, H.G.; Treanță, S.; Soliman, M.S.; Nonlaopon, K. Riemann–Liouville fractional integral inequalities for generalized preinvex functions of interval-valued settings based upon pseudo order relation. *Mathematics* **2022**, *10*, 204. [CrossRef]
4. Sharma, N.; Mishra, S.K.; Hamdi, A. A weighted version of Hermite-Hadamard type inequalities for strongly GA-convex functions. *Int. J. Adv. Appl. Sci.* **2020**, *7*, 113–118.
5. Zhao, D.; An, T.; Ye, G.; Torres, D.F.M. On Hermite-Hadamard type inequalities for harmonical h-convex interval-valued functions. *Math. Inequal. Appl.* **2020**, *23*, 95–105.
6. Sharma, N.; Bisht, J.; Mishra, S.K. Hermite-Hadamard type inequalities for functions whose derivatives are strongly $\eta-$convex via fractional integrals. In *Indo-French Seminar on Optimization, Variational Analysis and Applications*; Springer: Berlin, Germany, 2020; pp. 83–102.
7. Hanson, M.A. On sufficiency of the kuhn-tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [CrossRef]
8. Craven, B.D.; Glover, B.M. Invex functions and duality. *J. Aust. Math. Soc.* **1985**, *39*, 1–20. [CrossRef]
9. Ben-Israel, A.; Mond, B. What is invexity? *ANZIAM J.* **1986**, *28*, 1–9. [CrossRef]
10. Mohan, S.R.; Neogy, S.K. On invex sets and preinvex functions. *J. Math. Anal. Appl.* **1995**, *189*, 901–908. [CrossRef]
11. Weir, T.; Jeyakumar, V. A class of nonconvex functions and mathematical programming. *Bull. Aust. Math. Soc.* **1988**, *38*, 177–189. [CrossRef]
12. Weir, T.; Mond, B. Preinvex functions in multiple objective optimization. *J. Math. Anal. Appl.* **1988**, *136*, 29–38. [CrossRef]
13. Noor, M.A. Hermite-Hadamard integral inequalities for log-preinvex functions. *J. Math. Anal. Approx. Theory* **2007**, *2*, 126–131.
14. Dragomir, S.S. On the Hadamard's inequlality for convex functions on the coordinates in a rectangle from the plane. *Taiwan J. Math.* **2001**, 5, 775–788. [CrossRef]
15. Latif, M.A.; Dragomir, S.S. Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the cooordinates. *Facta Univ. Math. Inform.* **2013**, *28*, 257–270.
16. Matłoka, M. On some Hadamard type inequalities for (h_1, h_2)-preinvex functions on the coordinates. *J. Inequal. Appl.* **2013**, *2013*, 1–12. [CrossRef]
17. Matłoka, M. On Some new inequalities for differentiable (h_1, h_2)-preinvex functions on the coordinates. *Math. Stat.* **2014**, *2*, 6–14. [CrossRef]
18. Mehmood, S.; Zafar, F.; Yasmin, N. Hermite-Hadamard-Fejér type inequalities for preinvex functions using fractional integrals. *Mathematics* **2019**, *7*, 467. [CrossRef]
19. Noor, M.A.; Noor, K.I.; Rashid, S. Some new classes of preinvex functions and inequalities. *Mathematics* **2019**, *7*, 29. [CrossRef]
20. Rashid, S.; Latif, M.A.; Hammouch, Z.; Chu, Y.M. Fractional integral inequalities for strongly h-preinvex functions for a kth order differentiable functions. *Symmetry* **2019**, *11*, 1448. [CrossRef]
21. Sharma, N.; Mishra, S.K.; Hamdi, A. Hermite-Hadamard type inequality for ψ-Riemann-Liouville fractional integrals via preinvex functions. *Int. J. Nonlinear Anal. Appl.* **2022**, *13*, 3333–3345.
22. Moore, R.E. *Interval Analysis*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1966.
23. Moore, R.E. *Methods and Applications of Interval Analysis*; SIAM: Philadelphia, PA, USA, 1979.
24. Bhurjee, A.K.; Panda, G. Multi-objective interval fractional programming problems: An approach for obtaining efficient solutions. *Opsearch* **2015**, *52*, 156–167. [CrossRef]
25. Zhang, J.; Liu, S.; Li, L.; Feng, Q. The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function. *Optim. Lett.* **2014**, *8*, 607–631. [CrossRef]
26. Zhao, D.; An, T.; Ye, G.; Liu, W. Chebyshev type inequalities for interval-valued functions. *Fuzzy Sets Syst.* **2020**, *396*, 82–101. [CrossRef]
27. Guo, Y.; Ye, G.; Zhao, D.; Liu, W. gH-symmetrically derivative of interval-valued functions and applications in interval-valued optimization. *Symmetry* **2019**, *11*, 1203. [CrossRef]
28. Moore, R.E.; Kearfott, R.B.; Cloud, M.J. *Introduction to Interval Analysis*; SIAM: Philadelphia, PA, USA, 2009.
29. Rothwell, E.J.; Cloud, M.J. Automatic error analysis using intervals. *IEEE Trans. Educ.* **2011**, *55*, 9–15. [CrossRef]
30. Snyder, J.M. Interval analysis for computer graphics. In Proceedings of the 19th Annual Conference on Computer Graphics and Interactive Techniques, Chicago, IL, USA, 27–31 July 1992; pp. 121–130.

31. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft Comput.* **2015**, *19*, 3293–3300. [CrossRef]
32. Budak, H.; Tunç, T.; Sarikaya, M. Fractional Hermite-Hadamard type inequalities for interval-valued functions. *Proc. Amer. Math. Soc.* **2020**, *148*, 705–718. [CrossRef]
33. Sharma, N.; Singh, S.K.; Mishra, S.K.; Hamdi, A. Hermite-Hadamard type inequalities for interval-valued preinvex functions via Riemann–Liouville fractional integrals. *J. Inequal. Appl.* **2021**, *2021*, 98. [CrossRef]
34. Zhao, D.; Ali, M.A.; Murtaza, G.; Zhang, Z. On the Hermite-Hadamard inequalities for interval-valued coordinated convex functions. *Adv. Differ. Equ.* **2020**, *2020*, 1–14. [CrossRef]
35. Zhao, D.; Zhao, G.; Ye, G.; Liu, W.; Dragomir, S.S. On Hermite-Hadamard type inequalities for coordinated *h*-convex interval-valued functions. *Mathematics* **2021**, *9*, 2352. [CrossRef]
36. Budak, H.; Kara, H.; Ali, M.A.; Khan, S.; Chu, Y. Fractional Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions. *Open Math.* **2021**, *19*, 1081–1097. [CrossRef]
37. Kara, H.; Budak, H.; Ali, M.A.; Sarikaya, M.Z.; Chu, Y.M. Weighted Hermite–Hadamard type inclusions for products of coordinated convex interval-valued functions. *Adv. Differ. Equ.* **2021**, *2021*, 104. [CrossRef]
38. Kara, H.; Ali, M.A.; Budak, H. Hermite-Hadamard type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals. *Math. Methods Appl. Sci.* **2021**, *44*, 104–123. [CrossRef]
39. Lai, K.K.; Bisht, J.; Sharma, N.; Mishra, S.K. Hermite-Hadamard type fractional inclusions for interval-valued preinvex functions. *Mathematics* **2022**, *10*, 264. [CrossRef]
40. Shi, F.; Ye, G.; Zhao, D.; Liu, W. Some fractional Hermite-Hadamard type inequalities for interval-valued coordinated functions. *Adv. Differ. Equ.* **2021**, *2021*, 32. [CrossRef]
41. Tariboon, J.; Ali, M.A.; Budak, H.; Ntouyas, S.K. Hermite-Hadamard inclusions for coordinated interval-valued functions via post-quantum calculus. *Symmetry* **2021**, *13*, 1216. [CrossRef]
42. Du, T.; Zhou, T. On the fractional double integral inclusion relations having exponential kernels via interval-valued coordinated convex mappings. *Chaos Solitons Fractals* **2022**, *156*, 111846. [CrossRef]
43. Khan, M.B.; Santos-García, G.; Zaini, H.G.; Treanţă, S.; Soliman, M.S. Some new concepts related to integral operators and inequalities on coordinates in fuzzy fractional calculus. *Mathematics* **2022**, *10*, 534. [CrossRef]