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Carlitz's Equations on Generalized Fibonacci Numbers

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Abstract: Carlitz solved some Diophantine equations on Fibonacci or Lucas numbers. We extend his results to the sequence of generalized Fibonacci and Lucas numbers. In this paper, we solve the Diophantine equations of the form $A_{n_1}\cdots A_{n_k} = B_{m_1}\cdots B_{m_r}C_{t_1}\cdots C_{t_s}$, where (A_n), (B_m), and (C_t) are generalized Fibonacci or Lucas numbers. Especially, we also find all solutions of symmetric Diophantine equations $U_{a_1}U_{a_2}\cdots U_{a_m} = U_{b_1}U_{b_2}\cdots U_{b_n}$, where $1 < a_1 \leq a_2 \leq \cdots \leq a_m$, and $1 < b_1 \leq b_2 \leq \cdots \leq b_n$.

Keywords: Fibonacci numbers; Lucas numbers; Diophantine equation

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1. Introduction

Let P, Q be nonzero coprime integers with $D = P^2 - 4Q \neq 0$. The sequences of generalized Fibonacci numbers and Lucas numbers, U_n and V_n satisfy the following recurrence relation:

$$U_0 = 0, \quad U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2} \quad (n \geq 2) \tag{1}$$

$$V_0 = 2, \quad V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2} \quad (n \geq 2) \tag{2}$$

Their close forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n. \tag{3}$$

where

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{P^2 - 4Q}}{2}$$

are roots of $x^2 - Px + Q = 0$.

For $m, n \in \mathbb{N}$. It is well known that these numbers have the following properties

- (a) $\gcd(U_m, U_n) = U_d$;
- (b) If $m \mid n$, then $U_m \mid U_n$;
- (c) If $U_m \mid U_n$ and $m > 2$, then $m \mid n$.

The generalized Fibonacci and Lucas numbers include many famous integer sequences such as Fibonacci numbers, Lucas numbers, Pell numbers, and Jacobsthal numbers. Their fascinated properties lead to abundant applications in totally surprising and unrelated fields (cf. [1–6]).

Consider equations:

$$U_n = U_m V_k, \tag{4}$$

$$U_n = U_m U_k, \tag{5}$$

$$U_n = V_m V_k, \tag{6}$$

$$V_n = U_m V_k, \tag{7}$$

$$V_n = V_m V_k, \tag{8}$$

$$V_n = U_m U_k, \tag{9}$$

where $n > m \geq k \geq 0$, $P > 1$, and $Q < -1$.

In 1964, L. Carlitz [7] solved the above equations for $P = 1$ and $Q = -1$, i.e., Fibonacci numbers and Lucas numbers. After half a century, M. Farrokhi D. G. [8] showed equation $F_n = kF_m$ has at most one solution (n, m) for $k > 1$. He gives the complete nontrivial solutions of the equation

$$F_m = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k}.$$

Moreover, he also gives the complete nontrivial solutions of the symmetric Diophantine equation

$$F_{m_1} F_{m_2} \cdots F_{m_k} = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_s}.$$

In 2011, as a byproduct of Lucas square classes, R. Keskin and B. Demirturk [9] rediscovered that $L_n = L_m L_r$ is impossible if m, $r > 1$. Two years later, R. Keskin and Z. Siar [10] proved that when $P > 1$ and $Q = \pm 1$, there is no generalized Lucas number V_n such that $V_n = V_m V_r$ for $m, r > 1$ as a byproduct of Lucas square classes. Lastly, they show that there is no generalized Fibonacci number U_n such that $U_n = U_m U_r$ for $Q = \pm 1$ and $1 < r < m$. With the help of Carmichael's primitive divisor theorem, P. Pongsriiam [11] solved equations:

$$F_m^a = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k},$$
$$F_m^a = L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k},$$
$$L_m^a = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k},$$
$$L_m^a = L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k},$$

where $a \geq 1, m \geq 0, k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$.

In 2017, P. Pongsriiam [12] considered the following Diophantine equations:

$$F_m = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} \pm 1,$$
$$F_m = L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} \pm 1,$$
$$L_m = F_{n_1} F_{n_2} F_{n_3} \cdots F_{n_k} \pm 1,$$
$$L_m = L_{n_1} L_{n_2} L_{n_3} \cdots L_{n_k} \pm 1,$$

where $m \geq 0, k \geq 1$, and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. There are various other types of equations involving generalized Fibonacci and Lucas numbers that many authors have also considered (cf. [13,14]).

Assume $P > 1$ and $Q < -1$. In this paper, we find Equation (4) holds if and only if $n = 2, m = k = 1$ or $n = 2m = 2k$. Equation (6) holds if and only if $n = 4, m = 2, k = 1$. Moreover, Equations (5) and (7)–(9) have no solution. Generally, we completely solved the Diophantine equations of the symmetric form

$$U_a = U_{b_1} U_{b_2} U_{b_3} \cdots U_{b_n}, \tag{10}$$

$$U_a = V_{b_1} V_{b_2} V_{b_3} \cdots V_{b_n}, \tag{11}$$

With the help of (10) and (11), we also find all solutions of symmetric Diophantine equations

$$U_{a_1} U_{a_2} \cdots U_{a_m} = U_{b_1} U_{b_2} \cdots U_{b_n}, \tag{12}$$

where $a > 1$, $1 < a_1 \leq a_2 \leq \cdots \leq a_m$, and $1 < b_1 \leq b_2 \leq \cdots \leq b_n$.

2. Preliminaries

In this section, we give some equalities and inequalities concerning generalized Fibonacci and Lucas numbers.

Lemma 1. *Let $n > 2$, $P \geq 1$, and $Q < -1$. Then*

$$(P-Q)U_{n-2} < U_n < (P-Q)U_{n-1},$$

$$(P-Q)V_{n-2} < V_n < (P-Q)V_{n-1}.$$

Proof. Using the recursive formula,

$$(P-Q)U_{n-2} < U_n = PU_{n-1} - QU_{n-2} < (P-Q)U_{n-1}.$$

It is easy to check that the above inequality holds. Proceed as in the proof of U_n. We have a similar result of V_n. □

Lemma 2. *Let $P \geq 1, Q < -1$. For $n \geq 2k > 0$.*

$$U_n > (-Q)^k U_{n-2k},$$

$$V_n > (-Q)^k V_{n-2k}.$$

Proof. Proceed by induction on k. It is easy to check that the above inequality holds when $k = 1$. Now, assume the inequality holds for $k = m$. By the induction method.

$$U_n > (-Q)^m U_{n-2m}.$$

Therefore, it follows that

$$U_n = PU_{n-1} - QU_{n-2} > -QU_{n-2} > (-Q)^{m+1} U_{n-2(m+1)}.$$

Proceed as in the proof of U_n. We have a similar result of V_n. □

Lemma 3. *Let n and k be positive integers. The following identities hold*

- *(a)* $U_m V_k = U_{m+k} + Q^k U_{m-k} \quad (m \geq k)$;
- *(b)* $U_m V_k = U_{m+k} - Q^m U_{k-m} \quad (k \geq m)$;
- *(c)* $V_m V_k = V_{m+k} + Q^k V_{m-k} \quad (m \geq k \geq 1)$;
- *(d)* $DU_m U_k = V_{m+k} + Q^k V_{m-k} \quad (m \geq k \geq 1)$.

Proof.

- (a) Using (3),

$$U_m V_k = \frac{\alpha^m - \beta^m}{\alpha - \beta} \times (\alpha^k + \beta^k) = \frac{\alpha^{m+k} - \beta^{m+k}}{\alpha - \beta} + (\alpha\beta)^k \frac{\alpha^{m-k} - \beta^{m-k}}{\alpha - \beta}.$$

- (b) It follows by the same method as in (a).
- (c) Using (3),

$$V_m V_k = (\alpha^m + \beta^m)\left(\alpha^k + \beta^k\right) = \left(\alpha^{m+k} + \beta^{m+k}\right) + (\alpha\beta)^k \left(\alpha^{m-k} + \beta^{m-k}\right).$$

- (d) Using (3),

$$DU_m U_k = \left(\alpha^{m+k} + \beta^{m+k}\right) - (\alpha\beta)^k \left(\alpha^{m-k} + \beta^{m-k}\right).$$

□

Corollary 1. $V_k = U_{k+1} - QU_{k-1}$.

Proof. It is easy to check Corollary 1 holds if one takes $m = 1$ in formula (b) of Lemma 3. □

Lemma 4. *Let $n \geq k+1 > 0$ and $Q < -1$. Then*

$$U_n = U_{k+1}U_{n-k} - QU_kU_{n-k-1}$$

and

$$V_n = U_{k+1}V_{n-k} - QU_kV_{n-k-1}.$$

Proof. Proceed by induction on k. It is easy to check that the above identities hold when $k = 1$. Now, assume the equation holds for integer k. By the induction method.

$$\begin{aligned} U_n &= U_{k+1}U_{n-k} - QU_kU_{n-k-1} \\ &= U_{k+1}(PU_{n-k-1} - QU_{n-k-2}) - QU_kU_{n-k-1} \\ &= U_{k+2}U_{n-(k+1)} - QU_{k+1}U_{n-(k+1)-1}. \end{aligned}$$

Proceed as in the proof of U_n. We have a similar result of V_n. □

Corollary 2. *For all $a, b, c, a_1, a_2, \ldots, a_n \in \mathbb{N}$. The following identities hold:*

- *(a)*
$$U_{a+b-1} = U_aU_b - QU_{a-1}U_{b-1}; \tag{13}$$
- *(b)*
$$U_{a+b-2} = \frac{1}{P}[U_aU_b - Q^2U_{a-2}U_{b-2}]; \tag{14}$$
- *(c)*
$$U_{a+b+c-3} = \frac{1}{P}[U_aU_bU_c - PQU_{a-1}U_{b-1}U_{c-1} + Q^3U_{a-2}U_{b-2}U_{c-2}]; \tag{15}$$
- *(d) If $n \geq 3$, $P > 1$, and $Q < -1$, then*
$$U_{a_1+\cdots+a_n-n} \geq \frac{1}{P}U_{a_1}U_{a_2}\cdots U_{a_n}. \tag{16}$$

Proof.

- (a) Formula (13) follows easily from Lemma 4, if one takes $k+1 = a$ and $n-k = b$.
- (b) Apply (13) to (14),

$$\begin{aligned} U_{a+b-2} &= U_{a-1}U_b - QU_{a-2}U_{b-1} \\ &= \frac{1}{P}[U_aU_b - QU_{a-2}(-U_b + PU_{b-1})] \\ &= \frac{1}{P}[U_aU_b - Q^2U_{a-2}U_{b-2}]. \end{aligned}$$

- (c) Combining Lemma 4 with (13) and (14),

$$\begin{aligned} U_{a+b+c-3} &= U_{a+b-1}U_{c-1} - QU_{a+b-2}U_{c-2} \\ &= \frac{1}{P}[U_aU_bU_c - PQU_{a-1}U_{b-1}U_{c-1} + Q^3U_{a-2}U_{b-2}U_{c-2}]. \end{aligned}$$

- (d) From (15), (16) holds. Then

$$\begin{aligned} U_{a_1+a_2+\cdots+a_{n+1}-(n+1)} &= U_{[a_1+a_2+\cdots+a_n-n]+a_{n+1}-1} \\ &= U_{a_1+a_2+\cdots a_n-n}U_{a_{n+1}} - QU_{a_1+a_2+\cdots+a_n-(n+1)}U_{a_{n+1}-1} \\ &\geq U_{a_1+a_2+\cdots+a_n-n}U_{a_{n+1}} \\ &\geq \frac{1}{P}U_{a_1}U_{a_2}\cdots U_{a_n}U_{a_{n+1}}. \end{aligned}$$

□

Lemma 5. *Assume $P > 2$, $Q < -1$, and $P^2 > -Q$. For all $a, b, c \in \mathbb{N}$. The following conditions hold*

- *(a)*
$$U_{a+b-2} < U_a U_b \quad (a + b \geq 2); \tag{17}$$
- *(b)*
$$U_{a+b+c-3} < U_a U_b U_c \quad (a + b + c \geq 3). \tag{18}$$

Proof.

- (a) It is easy to check that (17) holds for a,$b < 2$. Now, assume $a, b \geq 2$. By Lemma 2.
$$U_a U_b > -Q^2 U_{a-2} U_{b-2}.$$
Combine with (14),
$$PU_{a+b+c-2} < 2U_a U_b$$
implies $U_{a+b-2} < U_a U_b$.
- (b) Since
$$\begin{aligned} U_a U_b U_c &= (PU_{a-1} - QU_{a-2})(PU_{b-1} - QU_{b-2})(PU_{b-1} - QU_{b-2}) \\ &> P^3 U_{a-1} U_{b-1} U_{c-1} + Q^3 U_{a-2} U_{b-2} \\ &> -PQU_{a-1} U_{b-1} U_{c-1} + Q^3 U_{a-2} U_{b-2}. \end{aligned}$$
So
$$PU_{a+b+c-3} < 2U_a U_b U_c.$$
Since $P > 2$. It is easy to show that (18) holds.

□

Theorem 1 (Primitive divisor theorem of Carmichael [15]). *If α and β are real and $n \neq 1, 2, 6$, then U_n has a primitive divisor except when*
$$n = 12, \alpha + \beta = \pm 1, \alpha\beta = -1.$$

3. Main Theorems

Firstly, we begin this section by solving Equations (4)–(9) for $P > 1$ and $Q < -1$. Then, we solve (10)–(12) for $P > 2$, $Q < -1$, and $P^2 > -Q$.

Theorem 2. *Let $n > m \geq k$. Equation (4) holds if and only if $n = 2$, $m = k = 1$ and $n = 2m = 2k$.*

Proof. Equation (4) holds when $n = 2$ and $m = k = 1$. Thus, $m > 2$ and $k > 1$.

If $m = k$. Then $U_n = U_{2k}$ and $n = 2m = 2k$.

If $m - k = 1$ and k is even. Then $n \geq 2k + 2$. By Lemma 2.
$$\begin{aligned} U_{2k+1} + Q^k = U_n = U_{n-1} - QU_{n-2} &\geq U_{2k+1} - QU_{2k} \\ &> U_{2k+1} - QP(-Q)^{k-1} > U_{2k+1} + Q^k. \end{aligned}$$
If $m - k = 1$ and k is odd. Then $2k + 1 \geq n + 1$. By Lemma 2.
$$U_n - Q^k = U_{2k+1} = U_{2k} - QU_{2k-1} \geq U_n - QU_{2k-1} > U_n - Q^k.$$
If $k - m = 1$ and m is odd. Then $n \geq 2m + 2$. By Lemma 2.
$$\begin{aligned} U_{2m+1} - Q^m = U_n = U_{n-1} - QU_{n-2} &\geq U_{2m+1} - QU_{2m} \\ &> U_{2m+1} - Q(-Q)^{m-1} = U_{2m+1} - Q^m. \end{aligned}$$

If $k - m = 1$ and m is even. Then $2m + 1 \geq n + 1$. By Lemma 2.

$$U_n + Q^m = U_{2m+1} = U_{2m} - QU_{2m-1} \geq U_n - QU_{2m-1} > U_n + Q^m.$$

If $m - k > 1$ and k is even. Then $n \geq m + k + 1$. By Lemma 2.

$$\begin{aligned} U_{m+k} + Q^k U_{m-k} = U_n &= U_{n-1} - QU_{n-2} \geq U_{m+k} - QU_{m+k-1} \\ &> U_{m+k} + (-Q)^k U_{m-k+1} > U_{m+k} + Q^k U_{m-k}. \end{aligned}$$

If $m - k > 1$ and k is odd. Then $m + k \geq n + 1$. By Lemma 2.

$$\begin{aligned} U_n - Q^k U_{m-k} = U_{m+k} &= U_{m+k-1} - QU_{m+k-2} \geq U_n - QU_{m+k-1} \\ &> U_n + (-Q)^k U_{m-k+1} > U_n - Q^k U_{m-k}. \end{aligned}$$

If $k - m > 1$ and m is odd. Then $n \geq m + k + 1$. By Lemma 2.

$$\begin{aligned} U_{m+k} - Q^m U_{k-m} = U_n &= U_{n-1} - QU_{n-2} \geq U_{m+k} - QU_{m+k-1} \\ &> U_{m+k} + (-Q)^m U_{k-m+1} > U_{m+k} + (-Q)^m U_{k-m}. \end{aligned}$$

If $k - m > 1$ and m is even. Then $m + k \geq n + 1$. By Lemma 2.

$$\begin{aligned} U_n + Q^m U_{k-m} = U_{m+k} &= U_{m+k-1} - QU_{m+k-2} \geq U_n - QU_{m+k-2} \\ &> U_n + (-Q)^m U_{k-m}. \end{aligned}$$

□

Theorem 3. *Let $n > m \geq k$. Equation (5) possesses no solution.*

Proof. If $U_k = U_3$. By Lemma 4.

$$U_3 U_{n-2} < U_n = U_3 U_{n-2} - QU_{n-3} < U_3 U_{n-1}.$$

Then $U_{n-2} < U_m < U_{n-1}$.
If $U_k = U_4$. By Lemma 4.

$$U_4 U_{n-3} < U_n < U_4 U_{n-2}.$$

Then $U_{n-3} < U_m < U_{n-2}$.
It follows from induction and Lemma 4 that

$$U_{n-1} U_2 < U_n = U_3 U_{n-2} - QU_{n-3} < U_{n-1} U_3.$$

Then $U_2 < U_m < U_3$. □

Theorem 4. *Let $n > m \geq k$. Equation (6) holds if and only if $n = 4$, $m = 2$, and $k = 1$.*

Proof. If $n = 4$, $m = 2$, and $k = 1$. Equation (6) always holds. Suppose $m \geq k > 1$. By Lemma 3 (c) and Corollary 1. Equation (6) becomes

$$\begin{aligned} U_n &= U_{m+k+1} - QU_{m+k-1} + Q^k V_{m-k}. \\ &= U_{m+k+1} - QU_{m+k-1} + Q^k U_{m-k+1} - Q^{k+1} U_{m-k-1}. \end{aligned}$$

By Lemma 2

$$\begin{aligned} -QU_{m+k-1} &= -PQU_{m+k-2} + Q^2 U_{m+k-1} \\ &> -Q^{k+1} U_{m-k-1} + Q^k U_{m-k+1} = Q^k V_{m-k}. \end{aligned}$$

Then $n \geq m + k + 2$. By Lemma 2

$$PQ^2U_{m+k-3} > Q^kU_{m-k+1}$$

$$-Q^3U_{m+k-4} > -Q^{k+1}U_{m-k} > -Q^{k+1}U_{m-k-1}$$

$$\begin{aligned} U_{m+k+2} &= PU_{m+k+1} - PQU_{m+k-1} + PQ^2U_{m+k-3} - Q^3U_{m+k-4} \\ &> U_{m+k+1} - QU_{m+k-1} + Q^kU_{m-k+1} = Q^kV_{m-k} \end{aligned}$$

Then $n < m + k + 2$. It is a contradiction. □

Theorem 5. *Let $n > m \geq k$. Equation (7) possesses no solution.*

Proof. If $U_k = U_3$. By Lemma 4.

$$U_3V_{n-2} < V_n = U_3V_{n-2} - QV_{n-3} < U_3V_{n-1}.$$

Then $V_{n-2} < V_k < V_{n-1}$.
If $U_k = U_4$. By Lemma 4.

$$U_4U_{n-3} < U_n < U_4U_{n-2}.$$

Then $V_{n-3} < V_k < V_{n-2}$. Thus, we obtain the contradiction.
It follows from induction and Lemma 4 that

$$U_{n-1}V_2 < V_n < U_{n-1}V_3.$$

Then $V_2 < V_k < V_3$. Thus, no integer k makes Equation (7) hold. □

Theorem 6. *Let $n > m \geq k$. Equation (8) possesses no solution.*

Proof. Consider $V_n = V_{m+k} + Q^kV_{m-k}$ by Lemma 3 (c). If k is even. Then

$$n \geq m + k + 1.$$

$$V_{m+k+1} \leq V_{m+k} + Q^kV_{m-k} < PV_{m+k} + Q^kV_{m-k}$$

which is equal to

$$V_{m+k-1} < -Q^{k-1}V_{m-k}.$$

However,

$$V_{m+k-1} > (-Q)^{k-1}V_{m-k+1} > (-Q)^{k-1}V_{m-k},$$

a contradiction.
If k is odd. Then $n \leq m + k - 1$. By Lemma 2 and Lemma 3 (c).

$$V_n - Q^kV_{m-k} = V_{m+k} = PV_{m+k-1} - QV_{m+k-2} > V_n - QV_{m+k-2} > V_n - Q^kV_{m-k}.$$

□

Theorem 7. *Let $n > m \geq k$. Equation (9) possesses no solution.*

Proof. By Lemma 3 (d). We can transform Equation (9) into

$$DV_n = V_{m+k} - Q^kV_{m-k}. \tag{19}$$

By Lemma 2. We have $DV_n < 2V_{m+k}$. This implies that $n < m + k$.

Since

$$V_{m+k} = PV_{m+k-1} - QV_{m+k-2} = (P^3 - 2PQ)V_{m+k-3} + Q^2 - P^2QV_{m+k-4}. \quad (20)$$

We plug (20) back into (19).

$$(P^2 - 4Q)V_n = (P^3 - 2PQ)V_{m+k-3} + (Q^2 - P^2Q)V_{m+k-4} - Q^kV_{m-k}$$

By Lemma 2.

$$(Q^2 - P^2Q)V_{m+k-4} > (-Q)^2V_{m+k-4} > -Q^kV_{m-k}.$$

This implies

$$(P^2 - 4Q)V_n > (P^3 - 2PQ)V_{m+k-3}.$$

We see that $n > m + k - 3$. The proof falls into two conditions.
If $n = m + k - 1$. Then

$$(P^2 - 4Q)V_{m+k-1} = V_{m+k} - Q^kV_{m-k}.$$

It can be deduced that

$$-QV_{m+k-1} < (-Q)^kV_{m-k}.$$

However, we have

$$-QV_{m+k-1} > (-Q)^kV_{m+k+1}$$

by Lemma 2.
If $n = m + k - 2$. Then

$$(P^2 - 4Q)V_{m+k-1} = V_{ms+k} - Q^kV_{m-k},$$

which is equal to

$$-3QV_{m+k-2} = -PQV_{m+k-3} - Q^kV_{m-k}. \quad (21)$$

The left-hand side of (21) is equal to

$$-3QV_{m+k-2} = -3PQV_{m+k-3} + 3Q^2V_{m+k-4}.$$

Thus, we obtain

$$-3PQV_{m+k-3} + 3Q^2V_{m+k-4} = -PQV_{m+k-3} - Q^kV_{m-k}.$$

It is a contradiction by Lemma 2. Since

$$-3PQV_{m+k-3} > -PQV_{m+k-3}$$

$$3Q^2V_{m+k-4} > Q^2V_{m+k-4} > -Q^kV_{m-k}.$$

□

Theorem 8. *Let $n > m \geq k$. Equation (5) possesses no solution.*

Proof. If triple (a, b, c) is a solution of the equation. Then $b \mid c$ holds for $U_b \mid U_c$. Clearly, suppose $c = kb$ for $k \geq 2$.

$$U_aU_b = U_{kb} \geq U_{2b} = U_bU_{b+1} - QU_bU_{b-1} = PU_b^2 - 2QU_bU_{b-1} > PU_b^2 \geq U_aU_b,$$

which is impossible. □

Theorem 9. *Let a, b, c, d be natural numbers. Equation $U_a U_b = U_c U_d$ holds if and only if $U_a = U_c$ and $U_b = U_d$, $U_a = U_d$, and $U_b = U_c$.*

Proof. If $a < b, c, d \leq 3$. Then $U_b = U_c U_d$. Applying Theorem 8. We know that either $U_a = U_c = 1$ and $U_b = U_d$ or $U_a = U_d = 1$ and $U_b = U_c$. Next, we assume that $3 \leq a \leq b, c, d$. Apply (14),

$$\begin{aligned} U_{a+b-2} &= \frac{1}{P}[U_a U_b + Q^2 U_{a-2} U_{b-2}] \\ &< \frac{1}{P} U_a U_b \leq U_a U_b = U_c U_d \\ &= U_{c+d-1} + Q U_{c-2} U_{d-2} \\ &< U_{c+d-1}. \end{aligned}$$

Thus,

$$U_{a+b-2} < U_a U_b < U_{c+d-1}.$$

It is clear that $a + b \leq c + d$. The proof of $c + d \leq a + b$ follows in a similar manner. Thus, we obtain $a + b = c + d$. Repeatedly using (13),

$$\begin{aligned} & U_a U_b = U_c U_d \\ \Rightarrow \quad & U_{a-1} U_{b-1} = U_{c-1} U_{d-1} \\ & \vdots \\ \Rightarrow \quad & U_2 U_{b-a+2} = U_{c-a+2} U_{d-a+2} \\ \Rightarrow \quad & U_{b-a+2} = U_{c-a+2} U_{c-a+2}. \end{aligned}$$

By Theorem 8. We have $U_{c-a+2} = 1$ or $U_{c-a+2} = 1$, which implies that either $a = c$ and $b = d$ or $a = d$ and $b = c$. □

Theorem 10. *Let a, b, c, d, and e be natural numbers. Equation $U_a U_b U_c = U_d U_e$ has no solution with $3 \leq a, b, c, d, e$.*

Proof. If $a, b, c, d, e \in \mathbb{N}$. We assume that $(a, b, c; d, e)$ is a solution of the equation $U_a U_b U_c = U_d U_e$. Suppose $3 \leq a, b, c, d, e$. By (15),

$$\begin{aligned} U_{a+b+c-2} &= U_a U_{b+c-1} - Q U_{a-1} U_{b+c-2} \\ &= U_a U_b U_c - Q U_a U_{b-1} U_{c-1} - Q U_{a-1} U_{b+c-2} \\ &> U_a U_b U_c. \end{aligned}$$

By (17),

$$U_{d+e-2} < U_d U_e = U_a U_b U_c < U_{a+b+c-2}.$$

By (18),

$$U_{a+b+c-3} < U_a U_b U_c = U_d U_e < U_{d+e-1},$$

which implies that $a + b + c - 3 = d + e - 2$.

$$\begin{aligned} U_{d+e-6} &< U_{d-2} U_{e-2} = U_{d+e-5} + Q U_{d-3} U_{e-2} \\ &= U_{a+b+c-6} + Q U_{d-3} U_{e-2} \\ &< U_{a+b+c-6} = \frac{1}{P} U_{a+b+c-5} + \frac{Q}{P} U_{a+b+c-7} \\ &< \frac{1}{P} U_{a+b+c-5} \\ &< U_{a+b+c-8}. \end{aligned}$$

Then, we have $d + e - 6 < a + b + c - 8$, which is impossible. □

Let $P \geq 1$, $Q < 0$, and $\gcd(P, Q) = 1$. Next, we solve Equations (10)–(12) using the primitive divisor theorem of Carmichael.

Theorem 11. *The only nontrivial solutions of Equation (10) with $a > 1$, $b_i > 1$. and $n \geq 2$ are*

$$(2; 2 \cdots 2), (6; 3^3, 2 \cdots 2), (6; 3, 2 \cdots 2), (12; 6, 4^2, 3, 2 \cdots 2), (12; 4^2, 3^4, 2 \cdots 2)$$

Here, nontrivial solution means that $n \geq 2$, $b_i \leq 1$ for all $i = 1, \ldots, n$, and $a > 1$.

Proof. If $a = 12$, $P = 1$, and $Q = -1$. Then, we obtain

$$U_{12} = 144, \quad U_6 = 8, \quad U_4 = 3, \quad U_3 = 2.$$

Note that

$$144 = 2^4 3^2.$$

Then

$$U_{12} = U_6 U_4^2 U_3 U_2^k.$$
$$U_{12} = U_4^2 U_3^4 U_2^k.$$

If $m \geq 7$ or $m = 3, 4, 5$. By Theorem 1, there exists an odd primitive prime divisor p of U_a. We see p does not divide any generalized Fibonacci numbers U_k with index less than a. Next, consider $m = 2, 6$. If $m = 2$. Then

$$U_2 = U_2 U_2 \cdots U_2.$$

holds if and only if $U_2 = P = 1$.

If $a = 6$. Assume Equation (10) has a solution $(n_1, n_2, \cdots, n_s)$ for $6 > n_1 \geq n_2 \cdots \geq n_s$, $s \geq 2$, and $n_s > 1$. Obviously, $n_i \neq 5$. If $n_1 = 4$. Since $\gcd(U_6, U_4) = U_2$. It follows that $U_4 = U_2$.

Then

$$U_4 - U_2 = P(P^2 - 2Q - 1) = 0.$$

Because $P \geq 1$ and $Q < 0$, Equation (10) has no solution. If $n_1 = n_2 = 3$. Since $U_2 = P$.

$$U_3 = P^2 - Q | P^2 - 3Q.$$

Then

$$U_3 = P^2 - Q | -2Q.$$

It follows that $P^2 = -Q$, $P^2 - 3Q = 2U_3$ and

$$U_6 = 2U_3^2 U_2.$$

If $U_3 = P^2 - Q = -2Q = 2$. We obtain $P = 1$ $Q = -1$ and

$$U_6 = U_3^3 U_2^k.$$

If $U_2 = P = 2$ and $Q = -4$. It is contradict to $\gcd(P, Q) = 1$.

If $n_1 = 3$ and $n_2 = 2$. We have

$$U_6 = U_3 U_2^{l+1}.$$

and

$$U_2^l = P^l = P^2 - 3Q. \quad (l > 2)$$

However, it is contradict to $\gcd(P, Q) = 1$ if $P \neq 1$ and $P \neq 3$. If $P = 1$.

$$P^2 - 3Q = 1,$$

which has no solution with $P \geq 1$ and $Q < 0$. If $P = 3$.

$$81 - 36Q + 3Q^2 = 3^{l+2} - 3^l Q,$$

which has a solution with $P \geq 1$ and $Q < -6$.

If $n_1 = 2$. Then $U_2 = P$ divides

$$\frac{U_6}{U_2} = P^4 - 4P^2Q + 3Q^2.$$

Observe that $\gcd(U_6, U_2) = U_2 = P$, which is also contradict to P not divides

$$\frac{U_6}{U_2} = P^4 - 4P^2Q + 3Q^2.$$

□

Theorem 12. *Equation (11) has a finite nontrivial solution. Here, nontrivial solution means that $n \geq 2$, $b_i \leq 1$ for all $i = 1, \ldots, n$ and $a > 1$.*

Proof.

Case 1 a is odd and $a \geq 6$. By Theorem 1. There exists an odd primitive divisor p of U_a. Therefore, p does not divide any generalized Lucas number with index less than a. Since $p|U_a$, we see that U_a is not a product of generalized Lucas numbers.

Case 2 $a = 2^l m$, $l \geq 1, m \geq 5$ and m is odd. By Theorem 1. There exists an odd primitive divisor p of U_m. Moreover, p does not divide any generalized Lucas number with index less than a. Since $p|U_m$ and $U_m|U_a$. We obtain U_a is not a product of generalized Lucas numbers with index less than a.

Case 3 $a = 3 \cdot 2^l$

$$U_{3 \cdot 2^l} = V_{3 \cdot 2^{l-1}} V_{3 \cdot 2^{l-2}} \cdots V_6 V_3 U_3 \tag{22}$$

Since

$$U_3 = P^2 - Q > P = V_1$$

.

Equation (11) holds if and only if

$$U_3 = P^2 - Q = V_0 = 2.$$

Which contradict to $b_i \geq 2$.

Case 4 $a = 2^l$

$$U_{2^l} = V_{2^{l-1}} V_{2^{l-2}} \cdots V_2 V_1 (l \geq 2). \tag{23}$$

We show that the representation of U_{2^l} is unique for $l \geq 0$. It is easy to check that

$$U_2 = V_1 U_1.$$

Consider the equation

$$U_{2^l} = V_{a_1} V_{a_2} \cdots V_{a_k} = V_{2^{l-1}} V_{2^{l-2}} \cdots V_2 V_1. \tag{24}$$

where $l \geq 2$, $a^1 \geq a^2 \cdots \geq a^k$. By the identity $U_{2m} = U_m V_m$. Transform (24) into

$$U_{2^{l-1}} U_{2a_1} V_{a_2} \cdots V_{a_k} = U_{a_1} U_{2^l} V_{2^{l-2}} \cdots V_2 V_1. \tag{25}$$

If $2^l > 2a_1$. By Theorem 1, there exists a prime p dividing 2^l but p does not divide any term on the left-hand side of Equation (25). It is a contradiction. Similarly, $2a_1 > 2^l$ leads to a contradiction. Therefore, $2a_1 = 2^l$. Equation (24) is reduced to

$$V_{a_2} \cdots V_{a^k} = V_{2^{l-2}} V_{2^{l-2}} \cdots V_2 V_1.$$

Repeat the same process, we obtain

$$a_2 = 2^{l-2}, \quad a_3 = 2^{l-3}, \cdots.$$

Equation (24) is reduced to

$$V_2 V_1 = V_{a_k} \cdots V_{a_i}.$$

It is obvious that $a_k = 1, a_i = 2$. $\square$

By Theorem 11, Equation (12) has a nontrivial solution if and only if $P = 1$ and $Q = -1$.

Theorem 13. *The only nontrivial solutions of Equation* (12) *with* $1 < a_1 \le a_2 \le \cdots \le a_m$, $1 < b_1 \le b_2 \le \cdots \le b_n$ *are*

$$\begin{array}{rl}
(3,\ldots,3;6,\ldots,6) & , \quad m = 3n \\
(\overbrace{3,\ldots,3}^{a},\overbrace{6,\ldots,6}^{b},4,\ldots,4;12,\ldots,12) & , \quad a + 3b = 4n \\
(\overbrace{3,\ldots,3}^{a},4,\ldots,4;\overbrace{6,\ldots,6}^{b},12,\ldots,12) & , \quad a = 3b + 4n \\
(\overbrace{6,\ldots,6}^{a},4,\ldots,4;\overbrace{3,\ldots,3}^{b},12,\ldots,12) & , \quad 3a = b + 4n \\
(3,\ldots,3,4,\ldots,4;\overbrace{12,\ldots,12}^{a}\overbrace{6,\ldots,6}^{b}) & , \quad 3b + 6a = m
\end{array}$$

Here, nontrivial solution means that $a_i, b_j > 1$ *and* $a_i \neq b_j$ *for all* $i = 1,\ldots,m$ *and* $j = 1,\ldots,n$.

4. Conclusions

In this paper, we mainly solve some Diophantine equations of the form $A_{n_1} \cdots A_{n_k} = B_{m_1} \cdots B_{m_r} C_{t_1} \cdots C_{t_s}$, where (A_n), (B_m), and (C_t) are generalized Fibonacci or Lucas numbers. Our theorems show that no generalized Fibonacci numbers can be expressed as the product of generalized Fibonacci or Lucas numbers except the trivial cases. In general, two different products of generalized Fibonacci numbers are not equal except the trivial cases.

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