

Article

Temporal Quantum Memory and Non-Locality of Two Trapped Ions under the Effect of the Intrinsic Decoherence: Entropic Uncertainty, Trace Norm Nonlocality and Entanglement

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Abstract: The engineering properties of trapped ions and their capacity to engender numerous quantum information resources determine many aspects of quantum information processing. We devise a setup of coherent and even coherent fields acting on two trapped ions to generate quantum memory, non-locality, and entanglement. Various effects, such as intrinsic decoherence, Lamb–Dicke regime, and dipole–dipole interaction are investigated. The inter-coupling of trapped ions, as well as the generation and dynamics of correlations between them, are analyzed. Using quantum memory assisted entropic uncertainty, trace-norm measurement induced non-locality, and concurrence, we find that the coherent and even coherent fields successfully generate non-local correlations in trapped-ions, with the latter being more resourceful for the dynamics and preservation of the non-local correlations. Furthermore, we observe that the entropic uncertainty and the trace norm induced non-locality present symmetrical dynamics. The dipole–dipole interaction improves correlation’s generation, robustness, and entropic uncertainty suppression.

Keywords: trapped ions; quantum memory; measurement-induced non-locality; entanglement



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1. Introduction

Engineering procedures for obtaining multi-qubit quantum states are critical in quantum information processing and computing [1,2]. Quantum computation [3], quantum simulations [4], communication [5], teleportation [6], dense-coding [7], efficient quantum dynamics [8] and quantum metrology [9] all require specific quantum states. Improved information transfer and multiple channel coupling are among the advantages of developing multi-qubit entangled and correlated states [10]. Many researchers have proposed various protocols and procedures to implement entangled and correlated states in recent years, such as in Refs. [11–16]. Entangling trapped-ion atoms is now considered one of the most efficient and reliable techniques [17,18].

The uncertainty principle [19] is a fundamental concept of quantum physics. This principle is one of the most notable illustrations of how quantum physics differs from classical physics. Robertson’s version of uncertainty relations [20], which extended Heisenberg’s finding to two arbitrary observables, is possibly the most well-known. Entropic uncertainty relations, according to Deutsch, are of great interest due to the lower limit of the Robertson’s uncertainty relation, which is conditional on the state of a quantum system [21].

Maassen and Uffink improved his uncertainty relation based on Kraus hypothesis [22]. The entropic uncertainty relation remained a valuable resource in many applications, including entanglement witness [23], quantum key distribution [24], quantum sampling [25], cryptographic security [26], quantum metrology [27], and quantum correlations probing [22]. In the current study, the entropic uncertainty relations will be used to study temporal quantum memory generation and evolution in two trapped-ion qubits.

Non-locality is one of the most intriguing aspects of quantum information resources [28]. It is a fundamental concept in quantum physics since, there is no classical counterpart. The violation of Bell inequalities is the basic criterion of quantum non-locality, demonstrating that the properties of one quantum system may be drastically affected by another [29]. Both Luo and Fu developed measurement-induced nonlocality based on von Neumann measurements, which differs from Bell inequalities violation [30].

Besides non-locality, entanglement is another non-local concept that boosts almost all quantum operations [31–33]. Even when the particles are separated by a huge distance, entangled states occur when a group of particles share physical properties in such a way that each particle's quantum state cannot be described independently of the other.

Entanglement is widely used to distinguish between classical and quantum physics since it is a fundamental property of non-classical systems.

Non-local operations where entanglement is a key resource include quantum teleportation [34,35], sensing [36], computing [37], communication [38], evolution of quantum systems [39,40] and information processing [41], to name a few.

Natural trapped-ion qubits are important tools for effective quantum information processing techniques because they are relatively easily controllable [42–45]. Various multi-qubit trapped ion models have been developed to address key computing difficulties [46].

After the quantum states have been entangled and correlated, the effective preservation of non-local correlations is the next critical step. Quantum correlations are vulnerable to decoherence effects, and as a result, quantum systems cannot be totally protected from decoherence. Even in closed quantum systems, decoherence occurs, such as the intrinsic decoherence [47–50]. In this work, we will explore the dynamics and preservation of quantum memory and non-local correlations in trapped ions under intrinsic decoherence. Beyond the Lamb–Dicke regime, the effect of dipole–dipole coupling strength will be considered in the system containing two trapped-ion qubits. The effects of intrinsic decoherence and the initial parameters of distinct initial coherent states on temporal quantum memory and non-locality will be investigated.

The paper is structured as follows: the considered two trapped-ions physical models are presented in the Section 2. The definitions of entropic uncertainty, measurement-induced non-locality, and entanglement concurrence, as well as their dynamical behavior, are presented in Sections 3 and 4, respectively. In Section 5, we discuss the relevance of the current study with the previous ones. In Section 6, we provide the conclusion of the current investigation.

2. The Physical Model

We explore two coupled two-level cold-ions beyond the Lamb–Dicke regime in this paper. Each cold-ion is considered as a qubit. In the resonant case with the resolved sideband limit and the resonant case (where the laser and the k -th vibrational sideband have the same frequency) [51,52], the two-ion Hamiltonian is:

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_{int}, \\ \hat{H}_0 &= \omega_A \hat{a}_A^\dagger \hat{a}_A + \omega_B \hat{a}_B^\dagger \hat{a}_B + \omega_0 (\hat{\sigma}_z^A + \hat{\sigma}_z^B), \\ \hat{H}_{int} &= \sum_{l=A,B} K_l E_l e^{(i\theta_l - |\mu_l|^2)} \sum_m \frac{(-i\mu_j)^{2m+k}}{(m+k)!m!} \hat{a}_l^{\dagger m} \hat{a}_j^m (\hat{a}_l^k |1_j\rangle \langle 0_j| + \hat{a}_l^{\dagger k} |0_j\rangle \langle 1_l|),\end{aligned}\quad (1)$$

θ_l depicts the initial applied laser phase of l -trap. $\hat{\sigma}_z^l$ designs the atomic flip operator between the upper $|1\rangle_l$ and lower $|0\rangle_l$ ($l = A, B$) state with the atomic transition frequency

ω_0 . The transition dipole moment is denoted by K_l . E_l is the amplitude of the applied laser field amplitude of the l -field. $\hat{a}_l$ and $\hat{a}_l^\dagger$ represent the lowering and rising operators of the center-of-mass vibrational modes with the frequency ω_l ($l = A, B$). μ_l is the atomic trapped l -ion's Lamb–Dicke parameter. Based on the approach outlined in [53] for designing N two-level cold-ions beyond the Lamb–Dicke limit, we investigate the case where the two-level cold-ions (two-qubit system) are close to one other and their total center of motion is the centre-of-mass vibration of the two cold-ions. As a result, each ion frequency's vibration ω_l ($l = A, B$) is replaced by the total centre of motion frequency ω .

For the resonant case, we take $\omega_l = \omega_0 = \omega$, $\mu_l = \mu$. The proposed study is inspired by the fact that the dipole–dipole coupling cannot be neglected while using the rotating-wave approximation [54]. Consequently, the interaction picture of the two-qubit Hamiltonian is:

$$\begin{aligned} \hat{H}_{int} = & \lambda \sum_{i=A,B} (|0_i\rangle\langle 1_i| \hat{a}^{\dagger k} F_\mu(\hat{a}^\dagger \hat{a}) + |1_i\rangle\langle 0_i| F_\mu^*(\hat{a}^\dagger \hat{a}) \hat{a}^k \\ & + D(|1_A 0_B\rangle\langle 0_A 1_B| + |0_A 1_B\rangle\langle 1_A 0_B|), \end{aligned} \quad (2)$$

where, we take $\lambda_j = K_j E_j e^{i\theta_j} e^{-|\mu|^2} = \lambda$ ($j = A, B$), D represents the dipole–dipole coupling. Using the associated Laguerre (k, μ) -polynomials $L_m^k(\mu^2)$ ($m = 0, 1, 2, \dots$), the diagonal elements of the nonlinear function operator $F_\mu(\hat{a}^\dagger \hat{a})$ [51] are given by:

$$F_\mu(m) = \frac{m!}{(m+k)!} L_m^k(\mu^2). \quad (3)$$

The dynamics of the two cold-ions interaction with the intrinsic decoherence are analytically explored by the Milburn model [47]. On sufficiently short time steps, the Milburn model assumes that the system does not evolve continuously under unitary evolution but rather as a stochastic sequence of identical unitary phase changes. The Milburn model is used to investigate the dynamics of numerous real-world systems, including polar molecules in pendular states [55] and two superconducting qubits [56]. In terms of the system density matrix $\hat{M}(t)$, the Milburn master equation is given by:

$$\frac{d\hat{M}(t)}{dt} = -i[\hat{H}_{int}, \hat{M}(t)] - \frac{\gamma}{2}[\hat{H}_{int}, [\hat{H}_{int}, \hat{M}(t)]], \quad (4)$$

where γ is the intrinsic decoherence parameter. This intrinsic decoherence has been proposed as a solution to decoherence problems in which quantum coherence is destroyed as the system progresses. Milburn developed an equation involving intrinsic decoherence, a term for quantum coherence loss that occurs without the system interacting with a reservoir and without energy decay.

We provide a specific analytical solution for Equation (4) when the two trapped ions are originally in a disentangled state to study the capacity of the two cold-ions interactions to form entangled two-qubit states, in particular the upper state $|1_A 1_B\rangle$. The nonclassical effects of the initial center-of-mass vibrational mode on the two ion-qubits dynamics is examined by considering the vibrational mode initially in two different coherent states. One of them is the coherent state and is given by:

$$|CS_\alpha\rangle = e^{-N/2} \sum_{m=1}^{\infty} \frac{N^{\frac{m}{2}}}{\sqrt{m!}} |m\rangle. \quad (5)$$

The initial intensity of the coherent field is designed by $N = |\alpha|^2$. Another initial state is the even coherent state, which is characterized by its high nonclassicality. It is given by:

$$|ECS_\alpha\rangle = \frac{e^{-N/2}}{\sqrt{2 + 2e^{-2N}}} \sum_{m=1}^{\infty} \frac{[1 + (-1)^m] N^{\frac{m}{2}}}{\sqrt{m!}} |m\rangle, \quad (6)$$

Therefore, the initial density matrix of the two cold-ions $M(0)$ is:

$$M(0) = \sum_{m,n=1}^{\infty} P_{mn} |m, 1_A 1_B\rangle \langle n, 1_A 1_B|, \quad (7)$$

where P_{mn} represents the photon number distribution of the coherent/even-coherent state.

Here, we use the eigenstates of the Hamiltonian given in Equation (2) to find an analytical solution of the Equation (4). In the basis of the two-level cold ions: $\{|S_1^m\rangle = |1_A 1_B, m\rangle, |S_2^m\rangle = |1_A 0_B, m+k\rangle, |S_3^m\rangle = |0_A 1_B, m+k\rangle, |S_4^m\rangle = |0_A 0_B, m+2k\rangle\}$ with $m = 0, 1, 2, \dots$, the eigenstates $|E_i\rangle$ are given by:

$$|U_j\rangle = \sum_{n=1}^4 X_{jn}^m |S_n^m\rangle, (j = 1-4) \quad (8)$$

where X_{jn}^m satisfies the eigenvalue-problem: $\hat{H}_{int}|U_j^m\rangle = E_j^m|U_j^m\rangle$ based on the eigenvalues:

$$\begin{aligned} E_1^m &= \omega(m+k), \\ E_2^m &= \omega(m+k) - D, \\ E_3^m &= \omega(m+k) + \frac{1}{2}D \\ &\quad - \lambda \left[\frac{D^2}{\lambda^2} + \frac{2(m+k)!F_{\mu}^2(m+k)}{m!} + \frac{2(m+2k)!F_{\mu}^2(m+2k)}{(m+k)!} \right]^{0.5}, \\ E_4^m &= \omega(m+k) + \frac{1}{2}D \\ &\quad + \lambda \left[\frac{D^2}{\lambda^2} + \frac{2(m+k)!F_{\mu}^2(m+k)}{m!} + \frac{2(m+2k)!F_{\mu}^2(m+2k)}{(m+k)!} \right]^{0.5}. \end{aligned}$$

Using the initial state of Equation (7) and the eigenvalues E_j^m ($j = 1-4$) and the eigenstates $|U_j^m\rangle$ of the Hamiltonian Equation (2), the time-dependent two trapped-ions density matrix is described by:

$$\begin{aligned} \hat{M}(t) &= \sum_{m,n=0} P_{mn} [Y_{11}|U_1^m\rangle \langle U_1^n| + Y_{31}|U_3^m\rangle \langle U_1^n| \\ &\quad + Y_{41}|U_4^m\rangle \langle U_1^n| + Y_{13}|U_1^m\rangle \langle U_3^n| + Y_{33}|U_3^m\rangle \langle U_3^n| \\ &\quad + Y_{43}|U_4^m\rangle \langle U_3^n| + Y_{14}|U_1^m\rangle \langle U_4^n| + Y_{34}|U_3^m\rangle \langle U_4^n| + Y_{44}|U_4^m\rangle \langle U_4^n|], \end{aligned} \quad (9)$$

where Y_{jk} are given by:

$$Y_{jk} = X_{j1}^m X_{k1}^m \Lambda_{mn}^{jk}(t) T_{mn}^{jk}(t),$$

The unitary evolution $\Lambda_{mn}^{jk}(t)$ and the intrinsic decoherence $T_{mn}^{jk}(t)$ terms are defined by:

$$\Lambda_{mn}^{jk}(t) = e^{-i(E_j^m - E_k^n)t}, \quad T_{mn}^{jk} = e^{-\frac{\gamma}{2}(E_j^m - E_k^n)^2 t}. \quad (10)$$

We aim to explore the dynamics of the correlations functions between the two trapped ions. We then need to find the two-ion reduced density matrix by considering the trace of the vibrational mode states from the system's density matrix $M(t)$. The time-dependent two-ion state is obtained by:

$$M^{AB}(t) = \text{tr}_F\{M(t)\}. \quad (11)$$

For the one-photon case $k = 1$, we can investigate the parameter effects of the Lamb-Dicke nonlinearity, the vibrational mode nonclassicality, the trapped-ion coupling and the decoherence on the non-local correlations between the two trapped ions.

3. Quantum Memory and Non-Localities Quantifiers

The entropic uncertainty, measurement-induced non-locality, and concurrence entanglement are the commonly employed quantum memory and non-locality measures.

3.1. Two Trapped-Ions Entropic Uncertainty

For incompatible observables P and Q , Bob's uncertainty regarding the two qubit (A and B) measurement outcome is verifying the following inequality [57]:

$$S(P|B) + S(Q|B) \geq S(A|B) + \log_2 \frac{1}{c}, \quad (12)$$

where $S(A|B) = S(\hat{R}_{AB}) - S(\hat{R}_B)$ represents the density $\hat{M}_{AB}$ operator's conditional von Neumann entropy with $S(\hat{M}) = -\text{tr}(\hat{M} \log_2 \hat{M})$ (for a density matrix $\hat{M}$). $S(X|B) = S(\hat{M}^{XB}) - S(\hat{M}_B)$, $X \in \{P, Q\}$ is the post measurement state $\hat{R}^{XB} = \sum_x (|\psi_x\rangle\langle\psi_x| \otimes \mathbf{I}) \hat{R}^{AB} (|\psi_x\rangle\langle\psi_x| \otimes \mathbf{I})$. Here, $\hat{R}_B = \text{tr}_A(\hat{R}^{AB})$ and $|\psi_x\rangle$ represent the eigenvectors of X and $\mathbf{I}$ is the identical operator.

The left UL and right UR entropic uncertainty sides of Equation (12) can be represented as follows:

$$L(t) = S(\hat{R}_{\sigma_x B}) + S(\hat{R}_{\sigma_z B}) - 2S(\hat{R}_B), \quad (13)$$

$$R(t) = S(\hat{R}_{AB}) - S(\hat{R}_B) + 1, \quad (14)$$

where $L(t)$ and $R(t)$ are respectively, the entropic uncertainty and its lower bound.

3.2. Trace Norm Measurement-Induced Nonlocality (Trace Norm-MIN)

For a two-qubit density matrix, ρ^{AB} , based on the Schatten p -norm [58], the p -MIN is defined as:

$$M_p(\rho^{AB}) = \max_{\chi \in \Pi^A} \|\rho^{AB} - \Pi^A(\rho^{AB})\|_p^p. \quad (15)$$

Here, we use 1-MIN and trace distance MIN, which is based on local invariant measurements. Therefore, it is based on the maximization of the trace distance between the pre-measurement and the post-measurement states.

The two trapped-ions trace-norm MIN is given by [59,60]:

$$M(t) = \begin{cases} \frac{\sqrt{K_+} + \sqrt{K_-}}{2\|\vec{r}\|_1}, & \text{If } \vec{r} \neq 0; \\ \max\{|t_{11}|, |t_{22}|, |t_{33}|\}, & \text{If } \vec{r} = 0. \end{cases} \quad (16)$$

where $K_{\pm} = \alpha \pm 2\sqrt{\beta}\|\vec{r}\|_1$, $\alpha = \|\mathbf{T}\|_2^2 \|r\|_1^2 - \sum_i t_{ii}^2 r_i^2$ and $\beta = \sum_{\langle ijk \rangle} r_i^2 t_{jj}^2 t_{kk}^2$, the summation of β runs over all the cyclic permutations of 1, 2, 3. $\vec{r}$ represents the local Bloch vectors, $t_{mn} = \text{tr}\{\rho^{AB}(\sigma_m \otimes \sigma_n)\}$ design the components of the correlation matrix $\mathbf{T} = [t_{mn}]$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

3.3. Concurrence Entanglement

Here, we investigate entanglement (CE) [61] between the two trapped-ions by using the concurrence function $C(t)$:

$$C(t) = \max\{0, R_1 - R_2 - R_3 - R_4\}, \quad (17)$$

with $R_i > R_{i+1}$, and R_i being the square roots of the M-matrix eigenvalues: $M = \rho^{AB}(t)(\sigma_3 \otimes \sigma_3)\rho^{AB*}(t)(\sigma_3 \otimes \sigma_3)$. The CE value $C(t) = 1$ corresponds to a maximally two trapped-ions state, and its zero-value corresponds to an unentangled state. Otherwise, the CE values correspond to partial two trapped-ions entanglement.

4. Dynamics of Correlations

This section presents the results of the time-dependent density matrix given in Equation (10) for two trapped ions linked with a laser beam in resonance. The two cold-ions are coupled with the field. The density matrix of the system is described by Equation (4), beyond the Lamb–Dicke regime and under intrinsic decoherence. Using Equations (13), (16) and (17), we study the generation and dynamics of temporal quantum memory, induced nonlocality, and entanglement in two trapped ions coupled to a laser field resonantly beyond the Lamb–Dicke regime. We study separately the two cases, i.e., under the absence and presence of intrinsic decoherence. In these cases we explore the properties of relevant fields' two trapped-ion qubits.

The effect of the laser l -field outside of the Lamb–Dicke regime under the absence of intrinsic decoherence and dipole–dipole interaction, is investigated in Figure 1. Our results show that the quantum memory functions $UL - 1$ and $UR - 1$ increase initially. The emergence of quantum memory, non-locality, and entanglement is examined when the initial coherent field intensity strength N is kept minimal. Before switching the interaction between the two-trapped ions and field, the functions $UL - 1$ and $UR - 1$ are both zero. When the interaction is turned on, the quantum memory assisted entropic functions grow. In other words, in addition to the increase of entropic uncertainty, quantum memory is becoming more prevalent. Similarly, the non-locality and entanglement functions, $M(t)$ and $C(t)$, arise and develop with time, ensuring that non-classical effects and entanglement are generated. It is worth noting that all the current measures appear to have similar qualitative dynamics initially. The dynamical behavior of the $UL - 1$ and $UR - 1$ functions becomes inverted to that of the $M(t)$ and $C(t)$ functions. It was seen that the minima of $UL - 1$ and $UR - 1$ at comparable intervals meets with the $M(t)$ and $C(t)$ functions' maxima, resulting in qualitatively opposite properties. This physically means that as the entropic uncertainty in the two-trapped ions coupled with a coherent state field increases, the entanglement and non-locality in the system decreases. In the resonant cases, the qualitative dynamical behavior of all functions in Figure 1 show a revival phenomenon. The values of quantum memory generation never reach a steady state, but rather fluctuate. The entanglement and non-locality functions show similar dynamics, indicating that entanglement and non-locality are continuously exchanged between the l -laser fields and the system. Furthermore, unlike the two-qubit system coupled with cavities in Ref. [62], the nonclassical correlations exhibit monotonic dynamical behavior, which presents opposite results to the current case. In the absence of intrinsic decoherence, the Lamb–Dicke parameter μ regulates the dynamical behavior, and as this parameter increases, the exchange between the two trapped-ions decreases. The exchange rate for $\mu = 0.5$ in Figure 1a is extremely high when compared to $\mu = 0.99$ in Figure 1c. The trapped-ion qubits coupled with the laser field are more resourceful in terms of non-locality and entanglement preservation for lower values of μ , as shown in Figure 1a. For increasing values of μ , however, the state in large time intervals becomes separable, as shown in Figure 1b,c. The revival character of $UL - 1$, $UR - 1$, $M(t)$, and $C(t)$ is enhanced by increasing the interaction time, however, with a slightly different patterns as can be seen in Figure 1a,b. The qualitative dynamical behavior of non-local correlations in the Lamb–Dicke regime has a different exchange rate than the current case [43].

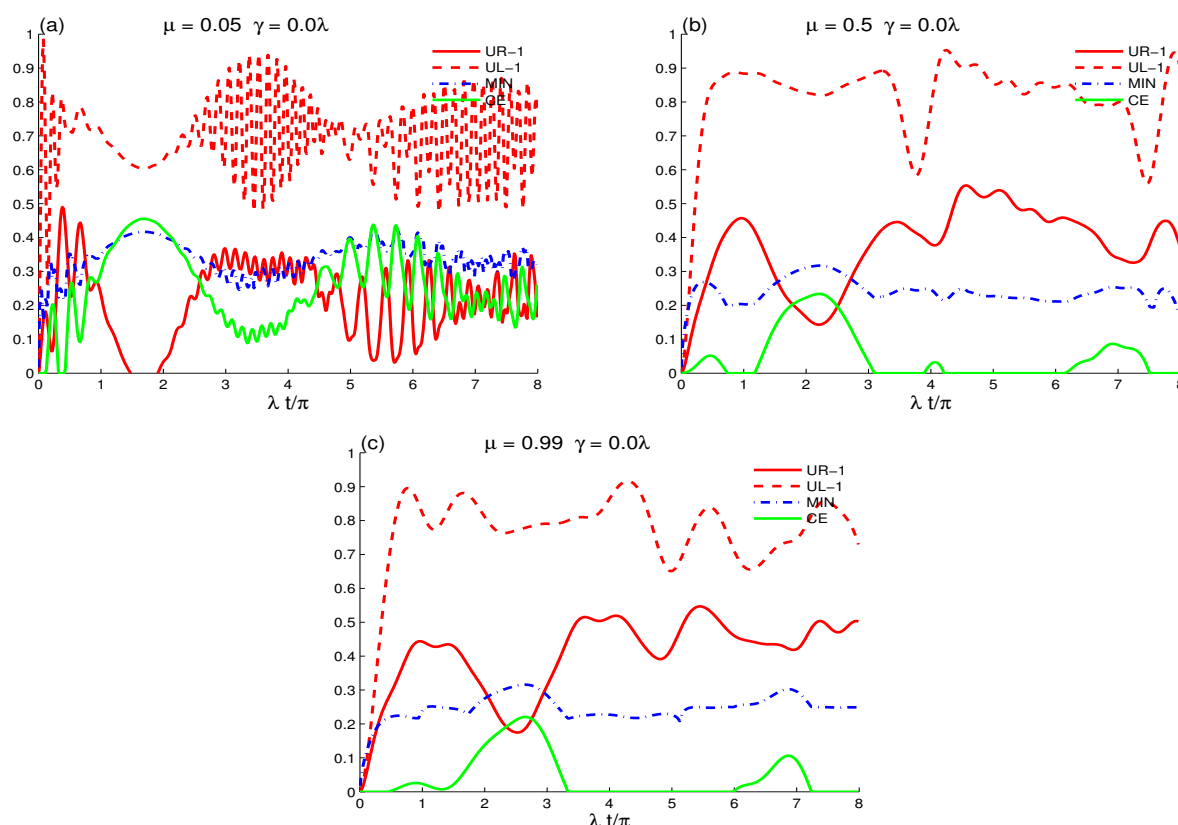


Figure 1. Temporal quantum memory and non-locality of the $UR - 1$, $UL - 1$, Trace norm-MIN and CE for $N = 0$ with different Lamb–Dicke nonlinearity values in the absence of the intrinsic decoherence ($\gamma = 0$) and dipole–dipole coupling ($D = 0$) when the vibrational mode state is initially in the coherent state.

The effects of initial even coherent state fields on the generated temporal quantum memory and non-locality is shown in Figure 2 when the centre-of-mass vibrational mode is in the even coherent state. The effects of varying the Lamb–Dicke parameter and the even coherent vibrational mode state on the time dynamics of entropic uncertainty, related lower bound, non-locality, and entanglement in two trapped-ion qubits coupled with laser field are investigated. We find that the left and right sides of the entropic relation are initially zero, but grow as interaction time increases, indicating that initially, the current configuration of trapped-ions and field is completely separable. The rapid revival of entropic relations implies that entropic uncertainty increases, but the system's order is restored because of the action of even coherent state fields. Compared to the coherent state field investigated in Figure 1, the current case appears to have a better recovery for entropic uncertainty regimes. As shown in Figure 1, the entropic uncertainty, entanglement, and non-locality functions for the even coherent state field experience fast revivals compared to the coherent state field. Furthermore, in the current case, the sudden decreasing of the $UL - 1$ and $UR - 1$ functions accommodate lower values than those observed in Figure 1, ensuring a quick decrease in the entropic uncertainty due to the even coherent state fields. In agreement, the maxima of the $M(t)$, and $C(t)$ functions grow, indicating an enhancement in entanglement and non-locality in the system. The generation and dynamics of entanglement and non-locality witnessed by $M(t)$ and $C(t)$ appear qualitatively similar and undergo rapid revivals in a similar pattern, suggesting the similar nature of the two phenomena. This implies that the state loses entanglement, however, as the entropic action of the field decreases, entanglement and non-locality are recovered. Besides, the exchange rate is sufficiently high for low μ values with respect to the normalized interaction time $\lambda t / \pi$ as shown in Figure 2a. For higher values of μ , the

system appears less entangled and non-local, with higher entropic uncertainty, as shown in Figure 2b. Because their quantitative values differ at comparable intervals, the current results also assess the different capacities of non-locality and entanglement to withstand against the intrinsic decoherence. This means that once the interaction is enabled, the state of the system oscillates non-periodically between locality and non-locality. We find that the current dynamical behavior of two trapped-ion qubits differs significantly from that of a three-qubit XY chain model with lower exchange rates, for example, see Ref. [63]. The upper bound $UL - 1$ appears to have a larger amplitude than the lower bound $UR - 1$ in the case of entropic relations, ensuring that they are quantitatively unequal.

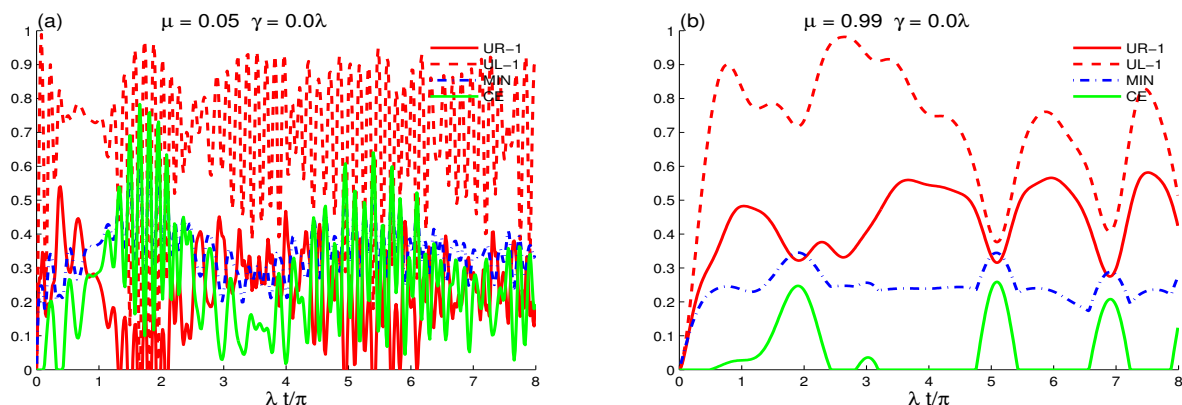


Figure 2. Temporal quantum memory and non-locality as in Figure 1a,c, but for the initial even super-position coherent state.

In the presence of intrinsic decoherence, the dynamics of the correlation's functions $UL - 1$, $UR - 1$, $M(t)$, and $C(t)$ for trapped-ion qubits coupled with coherent field are shown in Figure 3. The intrinsic decoherence parameter is set as $\gamma = 0.02\lambda$ in the existing case. The dynamical behavior of the correlations becomes completely changed when the intrinsic decoherence parameter is switched on. As the interaction between the system and the field is turned on, the entropic uncertainty rises quickly for low μ values. The functions $UL - 1$ and $UR - 1$ then gradually decline, but never reach zero, and take on a revival appearance. $UL - 1 = UR - 1 = 0$ at the initial time, which indicates that the two-trapped ions and field are initially uncorrelated, but as time evolves, the related functions increase, showing the generation of temporal quantum memory. In agreement, $M(t) = C(t) = 0$ initially, implying that the total configuration is comprized of the product states of two-trapped ions and the relative field. The $M(t)$ and $C(t)$ grow with time, indicating that the system is approaching high non-locality and entanglement. The initial generation of entanglement and non-locality in the current two-trapped ions is qualitatively similar to the trapped ions and XY-spin chains investigated in Refs. [43,63]. According to the time evolution of entropic relations, non-locality, and entanglement, the field's intrinsic decoherence constraints the system's revival character. The amount of induced entropic relations, as well as the corresponding stability of entanglement and non-locality, is affected by the Lamb–Dicke parameter, as illustrated in Figures 1 and 2. The lowest μ values produce less entropic uncertainty but more entanglement and non-locality generation, as shown in Figure 3a. For large values, such as those depicted in Figure 3b, entanglement, unlike non-locality, is greatly inhibited, eventually disappearing entirely at varied interaction intervals. For trapped ions, the current qualitative results are similar to those reported in [64], but with a different configuration setup. In Figure 3a, we notice that the $UR - 1$ functions reach zero in the interaction time range $1.5 < \lambda t/\pi < 1.9$, which means that the entropic uncertainty becomes minimum.

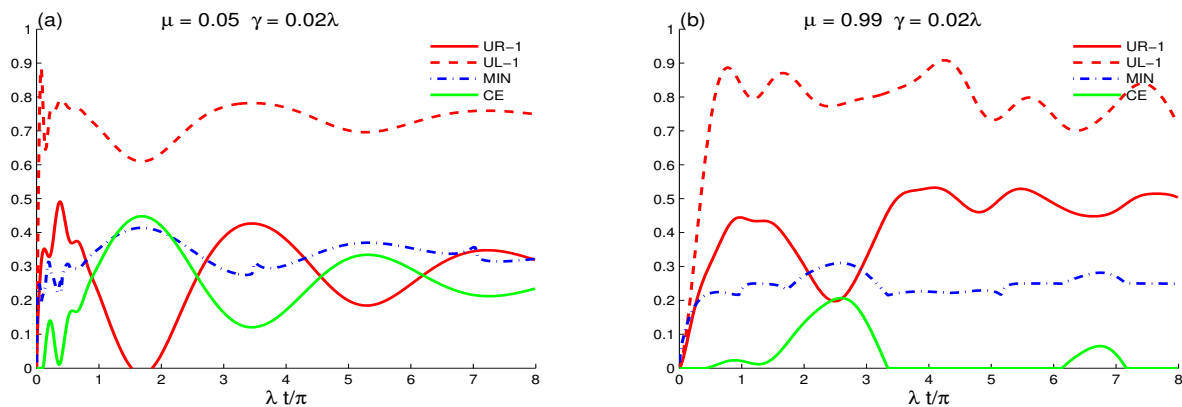


Figure 3. Temporal quantum memory and non-locality as in Figure 1a,c, but in the presence of the intrinsic decoherence.

In Figure 4, the dynamics of entropic, trace norm-*MIN*, and entanglement concurrence functions are discussed in the presence of two trapped-ion couplings. Here, we focus on the effect of the dipole–dipole interaction ($D = 10\lambda$) on trapped-ion coupling with coherent state fields (a) and even coherent state fields (b) in the absence of the intrinsic decoherence, i.e., $\gamma = 0.0\lambda$. The dynamical behaviors in Figure 4a,b differ from those in Figures 1a and 2a.

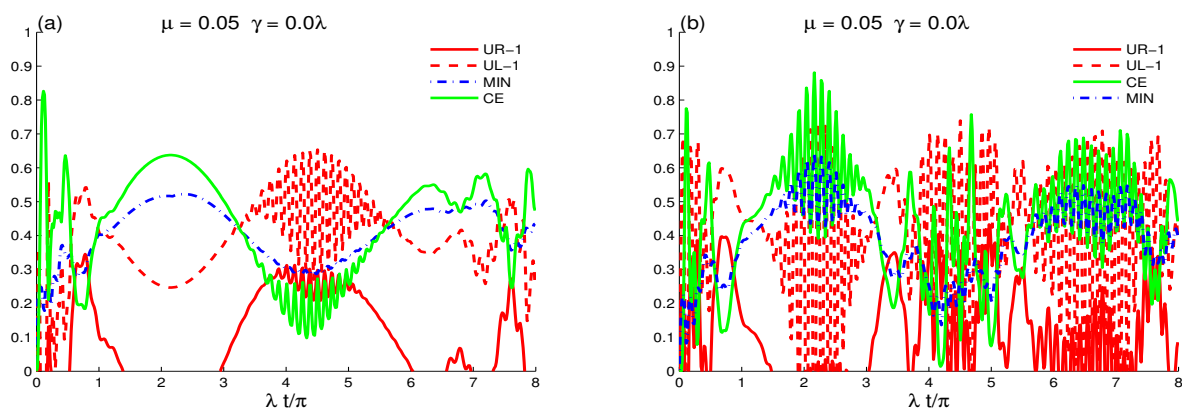


Figure 4. Non-classical correlation dynamics as in Figures 1a and 2a, but in the presence of the two trapped-ions coupling along with $D = 10\lambda$ and when the vibrational mode state is initially considered in coherent state (a) and when in even coherent state (b).

As a result, the trapped-ion coupling and the strength of the dipole–dipole interaction have a significant impact on the generation and preservation of quantum memory in two trapped ions. Because the initial values of all the functions are zero, there is no quantum memory, entanglement, or non-locality between the trapped ions. The entropic uncertainty, entanglement, and non-locality functions grow with interaction time as the coherent and even coherent state fields influence the trapped-ion coupling. It is worth mentioning that in the coherent state field, entanglement is more resilient when $0 \leq \frac{\lambda t}{\pi} \leq 0.5$. While the trapped-ion coupling's entanglement resilience interaction time varies $1.5 \leq \frac{\lambda t}{\pi} \leq 3$ in the even coherent state field. The sudden increase in entanglement robustness in the existing case is solely due to the dipole–dipole interaction. When compared to Figures 1–3, the current case's entropic relations remain suppressed. In the current cases, non-locality along with the entanglement in the trapped-ion coupling which is exposed to coherent and even coherent state fields remain more robust. The nonlocal correlations in two qubits have non-zero values when coupled with lossy cavities and classical fields, as seen in [62,65–68]. In comparison to the correlation functions in coherent state fields, trapped-ion coupling experiences more sudden rises and drops in even superposition coherent state fields,

generating results similar to Figures 2 and 3. Non-classical correlations perform better when associated fields are prepared in even superposition coherent states rather than in the standard coherent states. It is also worth noting that, for the even superposition coherent states, the correlations' unpredictability increases dramatically. This means that the two trapped-ion couplings and related centre-of-mass vibrational modes are quickly swapping their quantum information resources with the even coherent state field.

Figure 5 illustrates the robustness of the generated temporal quantum memory, non-locality and entanglement, in two trapped ions coupled with external coherent fields in the Lamb–Dicke regime limit $\mu \ll 1$ for the two trapped-ions coupling $D = 10\lambda$. It should be noted that, in this case, the effect of intrinsic decoherence on the generation and dynamics of the correlation functions is taken into account. The entropic uncertainty relations functions $UL - 1$ and $UR - 1$ are initially zero, showing that quantum memory does not exist initially. However, when the interaction between the system and coherent or even coherent states are activated, the entropic relations rapidly grow to their maximum values. The entropic uncertainty functions swing between their respective extrema over time. The random nature of the entropic relations, which strengthen with a higher revival rate at $\gamma = 0.0\lambda$ as shown in Figure 4, contrasts sharply with the current situation, which shows fewer revivals. The entanglement and non-locality functions, $M(t)$ and $C(t)$, are initially zero but grow with time. Despite this, the corresponding measures show fewer revivals than those seen in Figure 4 in the absence of intrinsic decoherence. When intrinsic decoherence is present, the difference in the dynamical patterns of quantum assisted memory entropic uncertainty relations, entanglement, and non-locality appears insignificant when compared to that observed in Figure 4 for the same set of parameters but with $\gamma = 0.0\lambda$. The amplitude of entropic relations, particularly the $UL - 1$ function, is significantly higher in the absence of dipole–dipole interaction, (see Figures 1–3). As soon as the dipole–dipole interaction becomes active, the relative amplitudes and maxima of the $UL - 1$ function become suppressed, as shown in Figures 4 and 5. This indicates that the dipole–dipole interaction strengthens coherent and even coherent fields, allowing the two-trapped ions to become more entangled and non-local.

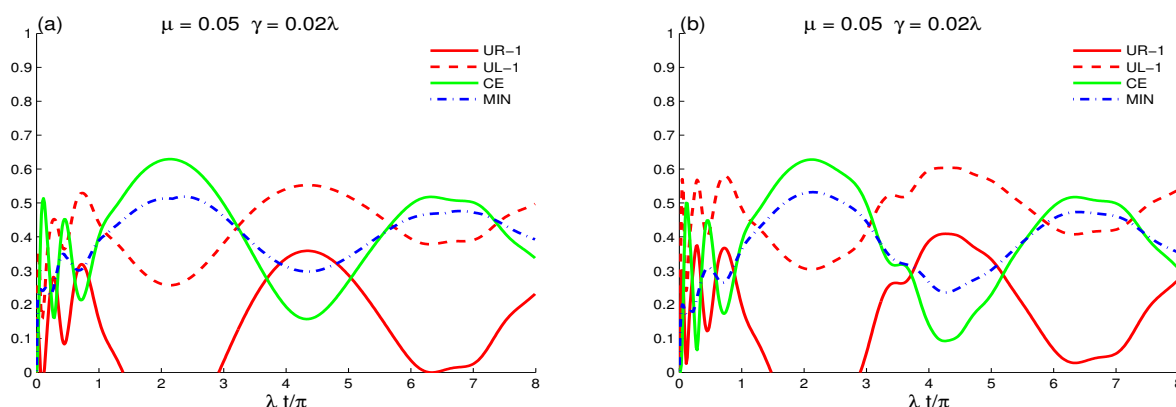


Figure 5. Quantum memory and non-locality dynamics as in Figure 4, but for the two trapped-ions coupling $D = 10\lambda$ and the decoherence $\gamma = 0.02\lambda$ when the vibrational mode state is initially considered in coherent state (a) and when in even coherent state (b).

The non-locality and entanglement functions grow over time as entropic uncertainty is suppressed. The relative maxima of the $UL - 1$ and $UR - 1$ functions coincide with the minima of the $M(t)$ and $C(t)$ functions, demonstrating that the increase in the entropic action of the fields corresponds to the decrease of the non-locality and entanglement and vice versa. The $UL - 1$ function's amplitude remains higher than that of the $UR - 1$ function, ensuring the inequality. Entanglement is more vulnerable to the intrinsic decoherence than the non-locality in the two trapped ions. From Figure 5, we note that the entropic uncertainty and the trace norm induced non-locality present symmetrical dynamics, which

confirms that the trace-norm MIN can be used as another indicator to the quantum memory assisted entropic uncertainty.

5. Comparison of Current and Previous Results

Two-qubit gate operations in trapped ions are typically based on exciting the ion motional sidebands with laser light, which results in a delayed process. An experimental technique to creating rapid entangling gate procedures involves carefully selected pulsed lasers, raising uneasy technological challenges. Late experiments, however, show that in order to design fast gates with higher speeds, an ultrafast entangling-gate source controlled by an optical frequency comb synthesizer appears to be an encouraging prospect [69]. Ref. [70] investigates another approach, which employs a new gate-optimizing concept that runs with substantially decreased power consumption and so accomplishes fast gate entangling operations. Other solutions for scaling multi-qubit operations are being considered, such as pulsed non-adiabatic gates based on coherent effects, or implementing techniques that extend repetitive indirect readout to qubits in a way that enables resilience to off-resonant photon scattering errors [71,72]. The paper of [73] focuses on the outcome of error mechanisms in laser-free one- and two-qubit gates in trapped ions and electrons. The effect of drive field inhomogeneities with respect to one-qubit gate fidelities is investigated. Moreover, the paper achieves in-depth study of two-qubit gate errors, including static frequency shifts, trap anharmonicities, field inhomogeneities, and heating. Another experimental approach demonstrates high-fidelity laser-free universal control of two trapped-ion qubits by creating maximally entangled states, based on using RF magnetic field gradients overlapped with microwave magnetic fields. The scheme shows robustness against multiple sources of decoherence, while its advantages are flexibility, adaptability, and scalability, as it can be practically employed for almost any trapped ion species [74]. Besides, the first demonstration of a qubit gate operation in a fault-tolerant control regime is achieved by Egan et al. in [75]. In this study, we have examined two-trapped ions coupled with coherent fields. It can be traced back to the configuration presented in Ref. [73], however, it has been affected by intrinsic decoherence. Furthermore, we assume the fields are beyond the Lamb–Dicke regime and in the presence of the dipole–dipole interaction. In addition, we investigate measurement-induced nonlocality, concurrence, the dynamics of temporal quantum memory, nonlocality, and entanglement in systems of two coupled trapped ions. To this we add the additional effects caused by intrinsic decoherence and the mode for initially distinct coherent states. Mihalcea et al. explored a similar configuration of two trapped ions in Ref. [76], which employs a model to analyze dynamical stability for systems of two coupled ions. The system under consideration here can be used to examine quantum dynamics for many-body systems composed of identical ions in 2D and 3D ion traps [77]. In the current context, and in light of the prior discoveries, we find that trapped ions are invaluable resources for quantum information processing, which can be linked back to the utility of trapped ions, as described by Haffner et al. in Refs. [78,79]. Similarly, Kaushal et al. proposed that trapped ions can be employed for a microstructured array of RF traps, which can be connected to the implementation of scalable quantum information processing nodes. Furthermore, employing trapped ions, the authors of Ref. [80] demonstrate procedures with ideal coherent controlled techniques that enable the realization of maximally dense universal parallelized quantum circuits. Thus, trapped ions are useful, not only for quantum information processing and computing but also for various different quantum mechanical protocols, such as for quantum error correction, sensing, and networks [81,82].

6. Conclusions

We have explored the generation and dynamics of quantum memory, non-locality, and entanglement in two trapped ions coupled to coherent or even coherent fields. The effects of the intrinsic decoherence, the Lamb–Dicke regime, and dipole–dipole interaction have been investigated. To analyze the correlations between the two trapped-ions, quantum

memory assisted entropic uncertainty, trace norm measurement-induced nonlocality, and concurrence are utilized. Note that under the Lamb–Dicke regime, the two trapped ions are viewed as two dipole-coupled qubits. Initially, the system is in coherent fields and the two trapped ions are fully uncorrelated, but owing to field interaction, non-local correlations arise over time. The properties of the coherent fields influence the dynamics of the correlations between trapped ions and fields. In the absence of intrinsic decoherence, the correlations show larger revivals. Correlations are suppressed when intrinsic decoherence occurs. In contrast, in even coherent vibrational fields, the dynamical patterns of the correlation functions have a higher rate of revival, ensuring better correlation preservation in two-trapped ions. The non-local correlations appear to be less preserved in the Lamb–Dicke regime as the relative parameter increases, along with higher induced entropic uncertainty. Unlike the Lamb–Dicke regime, the dipole–dipole interaction increases the non-locality and entanglement preservation while substantially reduces the entropic uncertainty. Thus, the robustness of the correlations between the two-trapped ions and coherent vibrational mode fields can be considerably enhanced by reducing the Lamb–Dicke and intrinsic decoherence parameters while rising the dipole–dipole interaction.

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