symmetry

MDPI

Article

Some Fuzzy Riemann–Liouville Fractional Integral Inequalities for Preinvex Fuzzy Interval-Valued Functions

Muhammad Bilal Khan [1], Hatim Ghazi Zaini [2], Jorge E. Macías-Díaz [3,4,*], Savin Treanţă [5,*] and Mohamed S. Soliman [6]

1 Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan; bilal42742@gmail.com
2 Department of Computer Science, College of Computers and Information Technology, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; h.zaini@tu.edu.sa
3 Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Avenida, Universidad 940, Ciudad Universitaria, Aguascalientes 20131, Mexico
4 Department of Mathematics, School of Digital Technologies, Tallinn University, Narva Rd. 25, 10120 Tallinn, Estonia
5 Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania
6 Department of Electrical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; soliman@tu.edu.sa
* Correspondence: jemacias@correo.uaa.mx (J.E.M.-D.); savin.treanta@upb.ro (S.T.)

Abstract: The main objective of this study is to introduce new versions of fractional integral inequalities in fuzzy fractional calculus utilizing the introduced preinvexity. Due to the behavior of its definition, the idea of preinvexity plays a significant role in the subject of inequalities. The concepts of preinvexity and symmetry have a tight connection thanks to the significant correlation that has developed between both in recent years. In this study, we attain the Hermite-Hadamard ($H{\cdot}H$) and Hermite-Hadamard-Fejér ($H{\cdot}H$ Fejér) type inequalities for preinvex fuzzy-interval-valued functions (preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$) via Condition C and fuzzy Riemann–Liouville fractional integrals. Furthermore, we establish some refinements of fuzzy fractional $H{\cdot}H$ type inequality. There are also some specific examples of the reported results for various preinvex functions deduced. To support the newly introduced ideal, we have provided some nontrivial and logical examples. The results presented in this research are a significant improvement over earlier results. This paper's awe-inspiring notions and formidable tools may energize and revitalize future research on this worthwhile and fascinating topic.

Keywords: preinvex fuzzy interval-valued function; fuzzy fractional integral operator; Hermite-Hadamard type inequality; Hermite-Hadamard Fejér type inequality

Citation: Khan, M.B.; Zaini, H.G.; Macías-Díaz, J.E.; Treanţă, S.; Soliman, M.S. Some Fuzzy Riemann–Liouville Fractional Integral Inequalities for Preinvex Fuzzy Interval-Valued Functions. *Symmetry* **2022**, *14*, 313. https://doi.org/10.3390/sym14020313

Academic Editor: Alexander Zaslavski

Received: 4 January 2022
Accepted: 21 January 2022
Published: 3 February 2022

1. Introduction

Convex function theory has a wide range of potential applications in a variety of unique and fascinating disciplines of study. Furthermore, this theory is useful in a variety of fields, including physics, information theory, coding theory, engineering, optimization, and inequality theory. This theory is currently making a significant contribution to the extensions and improvements of a wide range of mathematical and practical fields. Many authors analyzed, celebrated, and executed their work on the concept of convexity, and used fruitful methodologies and novel ideas to extend its many variations in helpful ways. In the literature, several new families of classical convex functions have been proposed. The references [1–5] are provided for the benefit of the readers. Many authors and scientists have always attempted to contribute to the theory of inequality by producing high-quality work. Integral inequalities on convex functions, both derivative and integration, have likewise been a hot and engaging area of study in recent years. The theory of inequalities has significant applications in the field of applied analysis, such as geometric function theory, impulsive diffusion equations, coding theory, numerical analysis, and fractional calculus, to name a few. Sun [6] and co-workers [7] recently used the local fractional integral operator to generalize the Hermite-Hadamard condition for harmonically convex and s-preinvex functions. The references [8–13] are provided for the benefit of the readers.

Several writers have recently proposed novel inequalities for various types of convexities, preinvexities, statistical theory, and other topics. Several discussions show a tight connection between inequality theory and convex functions. Hanson examined the invex function in the context of bi-function $\varphi(.,.)$ for the first time in 1981 (see [14]). Following Hanson's work, Ben-Israel and Mond attempted to delve deeper into linked invexity, introducing the concepts of invex sets and preinvex functions for the first time (see [15]). Under certain conditions, the preinvex and invex functions in the form of differentiability are comparable, according to Mohan and Neogy [16]. Antczak [17] discovered and analyzed the features of preinvex functions for the first time in 2005.

Note that fuzzy mappings ($F{\cdot}Ms$) are fuzzy-interval-valued functions. On the other hand, the concept of convex $F{\cdot}Ms$ from $\mathbb{R}^n$ to the set of fuzzy numbers was introduced by Nanda and Kar [18], Syau [19], and Furukawa [20]. They also explored Lipschitz continuity of fuzzy valued mappings and created other types of convex $F{\cdot}Ms$, such as logarithmic convex $F{\cdot}Ms$ and quasi-convex $F{\cdot}Ms$. Based on Goetschel and Voxman's concept of ordering [21], Yan and Xu [22] introduced the conceptions of epigraphs and convexity of $F{\cdot}Ms$, as well as the properties of convex $F{\cdot}Ms$ and quasi-convex $F{\cdot}Ms$. Khan et al. [23–26] extended the class of convex $F{\cdot}Ms$ and defined h-convex and (h_1, h_2)-convex $F{\cdot}I{\cdot}V{\cdot}Fs$ using fuzzy partial order relation. Moreover, they introduced $H{\cdot}H$, $H{\cdot}H$ Fejér, $H{\cdot}H$ fractional, $H{\cdot}H$ fractional Fejér for h-convex and (h_1, h_2)-convex $F{\cdot}I{\cdot}V{\cdot}Fs$ via fuzzy Riemannian and fuzzy Riemann–Liouville fractional integrals. Noor [27] proposed and investigated the notion of fuzzy preinvex mapping on the invex set. He also showed how to express the fuzzy optimality conditions of differentiable preinvex fuzzy mappings using variational inequalities. Recently Khan et al. [28] generalized the concept of preinvex fuzzy mappings in terms of (h_1, h_2)-preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$. Moreover, they established relation between $H{\cdot}H$ inequalities and (h_1, h_2)-preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ by using fuzzy Riemannian integrals. Recently Khan et al. [29–33] proposed the concepts of strongly preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$, higher strongly preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$, generalized strongly preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ and characterized their optimality conditions by introducing different variational like inequalities. Moreover, they proposed $H{\cdot}H$ inequalities for strongly preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ by utilizing fuzzy Riemannian.

At one step forward, Khan et al. introduced new classes of convex and generalized convex $F{\cdot}I{\cdot}V{\cdot}Fs$, and derived new $H{\cdot}H$ type inequalities for log-s-convex $F{\cdot}I{\cdot}V{\cdot}Fs$ in the second sense [34], log-h-convex $F{\cdot}I{\cdot}V{\cdot}Fs$ [35] and the references therein. We refer to the readers for further analysis of literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex $F{\cdot}Ms$, see [36–56] and the references therein.

The goal of this study is to complete the fuzzy Riemann–Liouville fractional integrals for $F{\cdot}I{\cdot}V{\cdot}Fs$ and use these integrals to get the $H{\cdot}H$ inequalities. These integrals are also used to derive $H{\cdot}H$ type inequalities for preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$.

2. Preliminaries

Let $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\eta \in \mathcal{K}_C$ be defined by

$$\eta = [\eta_*, \eta^*] = \{\omega \in \mathbb{R} | \eta_* \le \omega \le \eta^*\}, \quad (\eta_*, \eta^* \in \mathbb{R})$$

if $\eta_* = \eta^*$ then, η is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\eta_* \ge 0$, then $[\eta_*, \eta^*]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\eta_*, \eta^*] : [\eta_*, \eta^*] \in \mathcal{K}_C \text{ and } \eta_* \ge 0\}$.

Let $\varsigma \in \mathbb{R}$ and $\varsigma\eta$ be defined by

$$\varsigma{\cdot}\eta = \begin{cases} [\varsigma\eta_*, \varsigma\eta^*] & \text{if } \varsigma \ge 0, \\ [\varsigma\eta^*, \varsigma\eta_*] & \text{if } \varsigma < 0. \end{cases} \tag{1}$$

Then the Minkowski difference $\xi - \eta$, addition $\eta + \xi$ and $\eta \times \xi$ for $\eta, \xi \in \mathcal{K}_C$ are defined by

$$\begin{aligned} [\xi_*,\ \xi^*] - [\eta_*,\ \eta^*] &= [\xi_* - \eta_*,\ \xi^* - \eta^*], \\ [\xi_*,\ \xi^*] + [\eta_*,\ \eta^*] &= [\xi_* + \eta_*,\ \xi^* + \eta^*], \end{aligned} \tag{2}$$

and

$$[\xi_*,\ \xi^*] \times [\eta_*,\ \eta^*] = [min\{\xi_*\eta_*,\ \xi^*\eta_*,\ \xi_*\eta^*,\ \xi^*\eta^*\},\ max\{\xi_*\eta_*,\ \xi^*\eta_*,\ \xi_*\eta^*,\ \xi^*\eta^*\}]$$

The inclusion "$\subseteq$" means that

$$\xi \subseteq \eta \text{ if and only if, } [\xi_*,\ \xi^*] \subseteq [\eta_*,\ \eta^*], \text{ if and only if } \eta_* \leq \xi_*,\ \xi^* \leq \eta^* \tag{3}$$

Remark 2.1. [38] The relation " $\leq_I$ " defined on $\mathcal{K}_C$ by

$$[\nabla_*,\ \nabla^*] \leq_I [\eta_*,\ \eta^*] \text{ if and only if } \nabla_* \leq \eta_*, \nabla^* \leq \eta^*, \tag{4}$$

for all $[\nabla_*,\ \nabla^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$, it is an order relation. For given $[\nabla_*,\ \nabla^*],\ [\eta_*,\ \eta^*] \in \mathcal{K}_C$, we say that $[\nabla_*,\ \nabla^*] \leq_I [\eta_*,\ \eta^*]$ if and only if $\nabla_* \leq \eta_*,\ \nabla^* \leq \eta^*$ or $\nabla_* \leq \eta_*,\ \nabla^* < \eta^*$.

A fuzzy subset A of $\mathbb{R}$ is characterize by a mapping $\zeta : \mathbb{R} \to [0,1]$ called the membership function, for each fuzzy set and $\theta \in (0,\ 1]$, then θ-level sets of ζ is denoted and defined as follows $\zeta_\theta = \{u \in \mathbb{R} |\ \zeta(u) \geq \theta\}$. If $\theta = 0$, then $supp(\zeta) = \{\omega \in \mathbb{R} \mid \zeta(\omega) \rangle 0\}$ is called support of ζ. By $[\zeta]^0$ we define the closure of $supp(\psi)$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\zeta \in \mathbb{F}(\mathbb{R})$ denote the family of all nonempty sets. $\zeta \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then we define the following:

(1) ζ is said to be normal if there exists $\omega \in \mathbb{R}$ and $\zeta(\omega) = 1$;
(2) ζ is said to be upper semi continuous on $\mathbb{R}$ if for given $\omega \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\zeta(\omega) - \zeta(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\omega - y| < \delta$;
(3) ζ is said to be fuzzy convex if ζ_θ is convex for every $\theta \in [0,1]$;
(4) ζ is compactly supported if $supp(\zeta)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties (1), (2), (3) and (4). We denote by $\mathbb{F}_0$ the family of all intervals.

Let $\zeta \in \mathbb{F}_0$ be a fuzzy-interval, if and only if, θ-levels $[\zeta]^\theta$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\zeta]^\theta = [\zeta_*(\theta),\ \zeta^*(\theta)],$$

where

$$\zeta_*(\theta) = inf\{\omega \in \mathbb{R} |\ \zeta(\omega) \geq \theta\}, \zeta^*(\theta) = sup\{\omega \in \mathbb{R} |\ \zeta(\omega) \geq \theta\}. \tag{5}$$

Proposition 2.2. [47] If $\zeta, \eta \in \mathbb{F}_0$ then relation " $\preccurlyeq$ " defined on $\mathbb{F}_0$ by

$$\zeta \preccurlyeq \eta \text{ if and only if, } [\zeta]^\theta \leq_I [\eta]^\theta, \text{ for all } \theta \in [0,1], \tag{6}$$

this relation is known as partial order relation.

For $\zeta, \eta \in \mathbb{F}_0$ and $\varsigma \in \mathbb{R}$, the sum $\zeta \tilde{+} \eta$, product $\zeta \tilde{\times} \eta$, scalar product $\varsigma.\zeta$ and sum with scalar are defined by:

Then, for all $\theta \in [0,\ 1]$, we have

$$[\zeta \tilde{+} \eta]^\theta = [\zeta]^\theta + [\eta]^\theta, \tag{7}$$

$$[\zeta \tilde{\times} \eta]^\theta = [\zeta]^\theta \times [\eta]^\theta, \tag{8}$$

$$[\varsigma.\zeta]^\theta = \varsigma.[\zeta]^\theta. \tag{9}$$

$$\left[\varsigma \tilde{+} \zeta\right]^{\theta} = \varsigma + [\zeta]^{\theta}. \tag{10}$$

For $\psi \in \mathbb{F}_0$ such that $\zeta = \eta \tilde{+} \psi$, then by this result we have existence of Hukuhara difference of ζ and η, and we say that ψ is the H-difference of ζ and η, and denoted by $\zeta \tilde{-} \eta$. If H-difference exists, then

$$(\psi)^{*}(\theta) = \left(\zeta \tilde{-} \eta\right)^{*}(\theta) = \zeta^{*}(\theta) - \eta^{*}(\theta),\ (\psi)_{*}(\theta) = \left(\zeta \tilde{-} \eta\right)_{*}(\theta) = \zeta_{*}(\theta) - \eta_{*}(\theta)$$

Definition 2.3. [36] A fuzzy map $\Psi : [u,\ v] \subset \mathbb{R} \to \mathbb{F}_0$ is called $F{\cdot}I{\cdot}V{\cdot}F$. For each $\theta \in [0,\ 1]$, whose θ-levels define the family of $I{\cdot}V{\cdot}F$ $\Psi_\theta : [u,\ v] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\theta(\omega) = [\Psi_*(\omega,\theta),\ \Psi^*(\omega,\theta)]$ for all $\omega \in [u,\ v]$. Here, for each $\theta \in [0,\ 1]$, the left and right real valued functions $\Psi_*(\omega,\theta),\ \Psi^*(\omega,\theta) : [u,\ v] \to \mathbb{R}$ are also called lower and upper functions of Ψ.

Remark 2.4. If $\Psi : [u,\ v] \subset \mathbb{R} \to \mathbb{F}_0$ is a $F{\cdot}I{\cdot}V{\cdot}F$, then $\Psi(\omega)$ is called continuous function at $\omega \in [u,\ v]$, if for each $\theta \in [0,\ 1]$, both left and right real valued functions $\Psi_*(\omega,\theta)$ and $\Psi^*(\omega,\theta)$ are continuous at $\omega \in [u,\ v]$.

The following FI Riemann–Liouville fractional integral operators were introduced by Allahviranloo et al. [40]:

Definition 2.5. Let $\beta > 0$ and $L([\mu,\ \upsilon],\mathbb{F}_0)$ be the collection of all Lebesgue measurable $F{\cdot}I{\cdot}V{\cdot}Fs$ on $[\mu, \upsilon]$. Then the fuzzy left and right Riemann–Liouville fractional integral of $\Psi \in L([\mu,\ \upsilon],\mathbb{F}_0)$ with order $\beta > 0$ are defined by

$$\mathcal{I}_{\mu^+}^{\beta}\ \Psi(\omega) = \frac{1}{\Gamma(\beta)}\int_{\mu}^{\omega}(\omega-\varsigma)^{\beta-1}\Psi(\varsigma)d\varsigma,\ \ (\omega > \mu) \tag{11}$$

and

$$\mathcal{I}_{\upsilon^-}^{\beta}\ \Psi(\omega) = \frac{1}{\Gamma(\beta)}\int_{\omega}^{\upsilon}(\varsigma-\omega)^{\beta-1}\Psi(\varsigma)d\varsigma,\ (\omega < \upsilon), \tag{12}$$

respectively, where $\Gamma(\omega) = \int_0^\infty \varsigma^{\omega-1}e^{-\varsigma}d\varsigma$ is the Euler gamma function. The fuzzy left and right Riemann–Liouville fractional integral ω based on left and right end point functions can be defined, that is

$$\begin{aligned}\left[\mathcal{I}_{\mu^+}^{\beta}\ \Psi(\omega)\right]^{\theta} &= \tfrac{1}{\Gamma(\beta)}\int_{\mu}^{\omega}(\omega-\varsigma)^{\beta-1}\Psi_\theta(\varsigma)d\varsigma \\ &= \tfrac{1}{\Gamma(\beta)}\int_{\mu}^{\omega}(\omega-\varsigma)^{\beta-1}[\Psi_*(\varsigma,\ \theta),\Psi^*(\varsigma,\ \theta)]d\varsigma,\ (\omega > \mu)\end{aligned} \tag{13}$$

where

$$\mathcal{I}_{\mu^+}^{\beta}\ \Psi_*(\omega,\ \theta) = \frac{1}{\Gamma(\beta)}\int_{\mu}^{\omega}(\omega-\varsigma)^{\beta-1}\Psi_*(\varsigma,\ \theta)d\varsigma,\ \ (\omega > \mu), \tag{14}$$

and

$$\mathcal{I}_{\mu^+}^{\beta}\ \Psi^*(\omega,\ \theta) = \frac{1}{\Gamma(\beta)}\int_{\mu}^{\omega}(\omega-\varsigma)^{\beta-1}\Psi^*(\varsigma,\ \theta)d\varsigma,\ (\omega > \mu), \tag{15}$$

Similarly, the left and right end point functions can be used to define the right Riemann–Liouville fractional integral Ψ of ω.

Definition 2.6. [18]. The $F{\cdot}I{\cdot}V{\cdot}F$ $\Psi : [u,\ v] \to \mathbb{F}_0$ is called convex $F{\cdot}I{\cdot}V{\cdot}F$ on $[u,\ v]$ if

$$\Psi(\varsigma\omega + (1-\varsigma)y) \preccurlyeq \varsigma\Psi(\omega)\tilde{+}(1-\varsigma)\Psi(y), \tag{16}$$

for all $\omega,\ y \in\ [u,\ \nu]\ ,\ \varsigma \in\ [0,\ 1]\ $, where for all $\Psi(\omega)\ \succcurlyeq \widetilde{0}$ for all $\omega \in\ [u,\ \nu]\ $. If (16) is reversed, then Ψ is called concave $F{\cdot}I{\cdot}V{\cdot}F$ on $[u,\ \nu]$. Ψ is affine if and only if, it is both convex and concave $F{\cdot}I{\cdot}V{\cdot}F$.

Definition 2.7. [27]. The $F{\cdot}I{\cdot}V{\cdot}F$ $\Psi:\ [u,\ \nu]\ \to \mathbb{F}_0$ is called preinvex $F{\cdot}I{\cdot}V{\cdot}F$ on invex interval $[u,\ \nu]$ if

$$\Psi(\omega +\ (1-\varsigma)\varphi(\omega,y))\ \preccurlyeq \varsigma\Psi(\omega)\widetilde{+}(1-\varsigma)\Psi(y)\ , \tag{17}$$

for all $\omega,\ y \in\ [u,\ \nu]\ ,\ \varsigma \in\ [0,\ 1]\ $, where $\Psi(\omega)\ \succcurlyeq \widetilde{0}$ for all $\omega \in\ [u,\ \nu]$ and $\varphi:\ [u,\ \nu]\ \times [u,\ \nu] \to \mathbb{R}$. If (17) is reversed then, Ψ is called preconcave $F{\cdot}I{\cdot}V{\cdot}F$ on $[u,\ \nu]$. Ψ is affine if and only if, it is both preinvex and preconcave $F{\cdot}I{\cdot}V{\cdot}F$.

We need the following assumption regarding the function $\varphi:\ [u,\ \nu]\ \times\ [u,\ \nu]\ \to \mathbb{R}$, which plays an important role in upcoming main results.

Condition C. [16]

$$\varphi(y,\omega+\tau\varphi(y,\omega))\ =\ (1-\tau)\varphi(y,\omega)\ ,$$
$$\varphi(\omega,\omega+\tau\varphi(y,\omega))\ = -\tau\varphi(y,\omega)$$

Note that $\forall\ \omega,\ y \in\ [u,\ \nu]$ and $\varsigma \in\ [0,\ 1]$, then from Condition C we have

$$\varphi(\omega+\tau_2\varphi(y,\omega)\ ,\omega+\tau_1\varphi(y,\omega))\ =\ (\tau_2-\tau_1)\varphi(y,\omega)$$

Clearly for $\tau = 0$, we have $\xi(y,\omega) = 0$ if and only if $y = \omega$, for all $\omega,\ y \in\ [u,\ \nu]$. For the application of Condition C, see [27–33].

Theorem 2.8. [28] *Let $[u,\ \nu]$ be an invex set with resoect to bifunvtion φ and $\Psi:\ [u,\ \nu]\ \to \mathbb{F}_C(\mathbb{R})$ be a $F{\cdot}I{\cdot}V{\cdot}F$ with $\Psi(\omega)\ \succcurlyeq \widetilde{0}$, whose θ-levels define the family of $I{\cdot}V{\cdot}Fs$ $\Psi_\theta:\ [u,\ \nu]\ \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by*

$$\Psi_\theta(\omega)\ =\ [\Psi_*(\omega,\theta)\ ,\ \Psi^*(\omega,\theta)]\ ,\ \forall\ \omega \in\ [u,\ \nu] \tag{18}$$

for all $\omega \in\ [u,\ \nu]$ and for all $\theta \in\ [0,\ 1]$. Then, Ψ is preinvex F·I·V·F *on $[u,\ \nu]\ $, if and only if, for all $\theta \in\ [0,\ 1]\ $, $\Psi_*(\omega,\ \theta)$ and $\Psi^*(\omega,\ \theta)$ both are preinvex functions.*

Remark 2.9. If $\varphi(\omega,y)\ = \omega-y$, then we obtain inequality (16).

If $\Psi_*(\omega,\theta)\ = \Psi^*(\omega,\theta)$ with $\theta = 1$, then from (17), we obtain the definition of classical preinvex function, see [16].

If $\Psi_*(\omega,\theta)\ = \Psi^*(\omega,\theta)$ with $\varphi(\omega,y)\ = \omega-y$ and $\theta = 1$, then from (17), we obtain the definition of classical convex function.

3. Fuzzy-Interval Fractional Hermite-Hadamard Inequalities

The major goal of this section is to build a new version of fractional $H{\cdot}H$ and $H{\cdot}H$ Fejér type inequality in the mode of preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$, which is a classical studied topic. We also study some related inequalities. In what follows, we denote by $L([u,\ u+\varphi(\nu,u)]\ ,\mathbb{F}_0)$ the family of Lebesgue measureable $F{\cdot}I{\cdot}V{\cdot}Fs$.

Theorem 3.1. *Let $\Psi:\ [u,\ u+\varphi(\nu,u)]\ \to \mathbb{F}_0$ be a preinvex $F{\cdot}I{\cdot}V{\cdot}F$ on $[u,\ u+\varphi(\nu,u)]\ $, whose θ-levels define the family of $I{\cdot}V{\cdot}Fs$ $\Psi_\theta:\ [u,\ u+\varphi(\nu,u)]\ \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by $\Psi_\theta(\omega) = [\Psi_*(\omega,\theta)\ ,\ \Psi^*(\omega,\theta)]$ for all $\omega \in\ [u,\ u+\varphi(\nu,u)]$ and for all $\theta \in\ [0,\ 1]$. If φ satisfies Condition C and $\Psi \in L([u,\ u+\varphi(\nu,u)]\ ,\mathbb{F}_0)$, then*

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right)\ \preccurlyeq \frac{\Gamma(\beta+1)}{2(\varphi(\nu,u))^\beta}\left[\mathcal{I}^\beta_{u^+}\ \Psi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^\beta_{u+\varphi(\nu,u)^-}\ \Psi(u)\right]\ \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2}\ \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2} \tag{19}$$

If $\Psi(\omega)$ is preconcave $F{\cdot}I{\cdot}V{\cdot}F$ then

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \succcurlyeq \frac{\Gamma(\beta+1)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^{+}}^{\beta}\ \Psi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^{-}}^{\beta}\ \Psi(u)\right] \succcurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2} \succcurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2} \quad (20)$$

Proof. Let $\Psi:\ [u,\ u+\varphi(\nu,u)]\ \to \mathbb{F}_0$ be a preinvex $F{\cdot}I{\cdot}V{\cdot}F$. If Condition C holds then, by hypothesis, we have that

$$2\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \Psi(u+\ (1-\varsigma)\varphi(\nu,u))\widetilde{+}\Psi(u+\varsigma\varphi(\nu,u))$$

Therefore, for every $\theta\,[0,1]$, we have

$$2\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \le \Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ +\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ ,$$
$$2\Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \le \Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ +\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\theta)\ .$$

Multiplying both sides by $\varsigma^{\beta-1}$ and integrating the obtained result with respect to ς over $(0,1)$, we have

$$2\int_0^1 \varsigma^{\beta-1}\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)d\varsigma$$
$$\le \int_0^1 \varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\theta)d\varsigma+\int_0^1 \varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)d\varsigma,$$
$$2\int_0^1 \varsigma^{\beta-1}\Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\theta\right)d\varsigma$$
$$\le \int_0^1 \varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)d\varsigma+\int_0^1 \varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\theta)d\varsigma.$$

Let $\omega=u+\ (1-\varsigma)\varphi(\nu,u)$ and $y=u+\varsigma\varphi(\nu,u)$. Then we have

$$\tfrac{2}{\beta}\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \le \tfrac{1}{(\varphi(\nu,u))^{\beta}}\int\limits_u^{u+\varphi(\nu,u)}(u+\varphi(\nu,u)\ -y)^{\beta-1}\Psi_*(y,\theta)dy$$
$$+\tfrac{1}{(\varphi(\nu,u))^{\beta}}\int\limits_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi_*(\omega,\theta)d\omega$$
$$\tfrac{2}{\beta}\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \le \tfrac{1}{(\varphi(\nu,u))^{\beta}}\int\limits_u^{u+\varphi(\nu,u)}(u+\varphi(\nu,u)\ -y)^{\beta-1}\Psi^*(y,\theta)dy$$
$$+\tfrac{1}{(\varphi(\nu,u))^{\beta}}\int\limits_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi^*(\omega,\theta)d\omega,$$
$$\le \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^{+}}^{\beta}\ \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^{-}}^{\beta}\ \Psi_*(u,\ \theta)\right]$$
$$\le \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^{+}}^{\beta}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^{-}}^{\beta}\ \Psi^*(u,\ \theta)\right],$$

That is

$$\tfrac{2}{\beta}\left[\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right),\ \Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\right]$$

$$\le_I \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^{+}}^{\beta}\ \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^{-}}^{\beta}\ \Psi_*(u,\ \theta)\ ,\ \mathcal{I}_{u^{+}}^{\beta}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{\nu^{-}}^{\beta}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\right]$$

Thus,

$$\frac{2}{\beta}\,\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\,\Psi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\,\Psi(u)\right] \tag{21}$$

In a similar way as above, we have

$$\frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\,\Psi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\,\Psi(u)\right] \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2} \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}. \tag{22}$$

Combining (21) and (22), we have

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{\Gamma(\beta+1)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\,\Psi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\,\Psi(u)\right] \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2} \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}$$

Hence, the required result. □

Remark 3.2. From Theorem 3.1 we clearly see that

If $\varphi(\omega,y) = \omega - y$, then from Theorem 3.1, we get following result in fuzzy fractional calculus, see [23].

$$\Psi\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{\Gamma(\beta+1)}{2(\nu-u)^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\,\Psi(\nu)\widetilde{+}\mathcal{I}^{\beta}_{\nu^-}\,\Psi(u)\right] \preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}$$

Let $\beta = 1$. Then Theorem 3.1 reduces to the result for preinvex $F{\cdot}I{\cdot}V{\cdot}F$ given in [28]:

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{1}{\varphi(\nu,u)}\int_u^{u+\varphi(\nu,u)}\Psi(\omega)d\omega \preccurlyeq \frac{\Psi(u)\ \widetilde{+}\ \Psi(\nu)}{2}.$$

Let $\beta = 1$ and $\varphi(\omega,y) = \omega - y$. Then Theorem 3.1 reduces to the result for convex $F{\cdot}I{\cdot}V{\cdot}F$ given in [26]:

$$\Psi\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\int_u^{\nu}\Psi(\omega)d\omega \preccurlyeq \frac{\Psi(u)\ \widetilde{+}\ \Psi(\nu)}{2}$$

Let $\beta = 1 = \theta$ and $\Psi_*(\omega,\ \theta) = \Psi^*(\omega,\ \theta)$ with $\varphi(\omega,y) = \omega - y$. Then from Theorem 3.1 we obtain classical $H{\cdot}H$ Fejér type inequality.

Example 3.3. Let $\beta = \frac{1}{2}$, $\omega \in [2, 2+\varphi(3,2)]$, and the $F{\cdot}I{\cdot}V{\cdot}F$ $\Psi : [u,\ u+\varphi(\nu,u)] = [2, 2+\varphi(3,2)] \to \mathbb{F}_0$, defined by

$$\Psi(\omega)(\theta) = \begin{cases} \frac{\theta}{2-\omega^{\frac{1}{2}}} & \theta \in \left[0,\ 2-\omega^{\frac{1}{2}}\right] \\ \frac{2\left(2-\omega^{\frac{1}{2}}\right)-\theta}{2-\omega^{\frac{1}{2}}} & \theta \in \left(2-\omega^{\frac{1}{2}},\ 2\left(2-\omega^{\frac{1}{2}}\right)\right] \\ 0 & \textit{otherwise}, \end{cases}$$

Then, for each $\theta \in [0,\ 1]$, we have $\Psi_\theta(\omega) = \left[\theta\left(2-\omega^{\frac{1}{2}}\right),\ (2-\theta)\left(2-\omega^{\frac{1}{2}}\right)\right]$. Since left and right end point functions $\Psi_*(\omega,\theta) = \theta\left(2-\omega^{\frac{1}{2}}\right)$, $\Psi^*(\omega,\ \theta) = (2-\theta)\left(2-\omega^{\frac{1}{2}}\right)$, are preinvex functions with respect to $\varphi(\nu,u) = \nu - u$, for each $\theta \in [0,1]$, then $\Psi(\omega)$ is preinvex $F{\cdot}I{\cdot}V{\cdot}F$. We clearly see that $\Psi \in L([u,\ u+\varphi(\nu,u)], \mathbb{F}_0)$ and

$$\Psi_*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right) = \Psi_*\left(\frac{5}{2},\ \theta\right) = \theta\frac{4-\sqrt{10}}{2}$$

$$\Psi^*\left(\frac{2u+\varphi(\nu,u)}{2},\,\theta\right) = \Psi^*\left(\frac{5}{2},\,\theta\right) = (2-\theta)\frac{4-\sqrt{10}}{2}$$

$$\frac{\Psi_*(u,\theta)\ +\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)}{2} = \theta\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)$$

$$\frac{\Psi^*(u,\theta)\ +\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)}{2} = (2-\theta)\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)$$

Note that

$$\frac{\Gamma(\beta+1)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\ \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\ \Psi_*(u,\theta)\right]$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{2+\varphi(3,2)}(3-\omega)^{\frac{-1}{2}}\ .\ \theta\left(2-\omega^{\frac{1}{2}}\right)d\omega$$

$$+ \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{2+\varphi(3,2)}(\omega-2)^{\frac{-1}{2}}\ .\ \theta\left(2-\omega^{\frac{1}{2}}\right)d\omega$$

$$= \frac{1}{4}\theta\left[\frac{7393}{10,000}+\frac{9501}{10,000}\right]$$

$$= \theta\frac{8447}{20,000}$$

$$\frac{\Gamma(\beta+1)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^+}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\ \Psi^*(u,\theta)\right]$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{2+\varphi(3,2)}(3-\omega)^{\frac{-1}{2}}\ .\ (2-\theta)\left(2-\omega^{\frac{1}{2}}\right)d\omega$$

$$+ \frac{\Gamma\left(\frac{3}{2}\right)}{2}\frac{1}{\sqrt{\pi}}\int_{2}^{2+\varphi(3,2)}(\omega-2)^{\frac{-1}{2}}\ .\ (2-\theta)\left(2-\omega^{\frac{1}{2}}\right)d\omega$$

$$= \frac{1}{4}(2-\theta)\left[\frac{7393}{10,000}+\frac{9501}{10,000}\right]$$

$$= (2-\theta)\frac{8447}{20,000}$$

Therefore

$$\left[\theta\frac{4-\sqrt{10}}{2},\ (2-\theta)\frac{4-\sqrt{10}}{2}\right] \leq_I \left[\theta\frac{8447}{20,000},\ (2-\theta)\frac{8447}{20,000}\right] \leq_I \left[\theta\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right),\ (2-\theta)\left(\frac{4-\sqrt{2}+\sqrt{3}}{2}\right)\right]$$

and Theorem 3.1 is verified.

It is well known fact that $H{\cdot}H$ Fejér type inequality is a generalization of $H{\cdot}H$ type inequality. In Theorem 3.4 and Theorem 3.5, we obtain second and first fuzzy fractional $H{\cdot}H$ Fejér type inequalities for introduced preinvex $F{\cdot}I{\cdot}V{\cdot}F$.

Theorem 3.4. *Let* $\Psi : [u,\ u+\varphi(\nu,u)] \to \mathbb{F}_0$ *be a preinvex* $F{\cdot}I{\cdot}V{\cdot}F$ *with* $u < \nu$, *whose* θ-*levels define the family of* $I{\cdot}V{\cdot}F$ $\Psi_\theta : [u,\ u+\varphi(\nu,u)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\Psi_\theta(\omega) = [\Psi_*(\omega,\theta), \Psi^*(\omega,\theta)]$ *for all* $\omega \in [u,\ u+\varphi(\nu,u)]$ *and for all* $\theta \in [0,\ 1]$. *Let* $\Psi \in$

$L([u,\ u+\varphi(\nu,u)],\mathbb{F}_0)$ and $\Omega:\ [u,\ u+\varphi(\nu,u)]\ \to\mathbb{R},\ \Omega(\omega)\ \geq 0$, symmetric with respect to $\frac{2u+\varphi(\nu,u)}{2}$. If φ satisfies Condition C, then

$$\left[\mathcal{I}^{\beta}_{u^+}\ \Psi\Omega(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\ \Psi\Omega(u)\right]$$
$$\preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2}\left[\mathcal{I}^{\beta}_{u^+}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\ \Omega(u)\right] \tag{23}$$
$$\preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}\left[\mathcal{I}^{\beta}_{u^+}\ \Omega(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^-}\ \Omega(u)\right]$$

If Ψ is preconcave $F{\cdot}I{\cdot}V{\cdot}F$, then inequality (23) is reversed.

Proof. Let Ψ be a preinvex $F{\cdot}I{\cdot}V{\cdot}F$ and $\varsigma^{\beta-1}\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ \geq 0$. Then, for each $\theta\in\ [0,\ 1]$, we have

$$\begin{gathered}\varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\\ \leq\varsigma^{\beta-1}(\varsigma\Psi_*(u,\ \theta)\ +\ (1-\varsigma)\Psi_*(u+\varphi(\nu,u)\ ,\ \theta))\Omega(u+\ (1-\varsigma)\varphi(u+\varphi(\nu,u)\ ,u))\\ \varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\\ \leq\varsigma^{\beta-1}(\varsigma\Psi^*(u,\ \theta)\ +\ (1-\varsigma)\Psi^*(u+\varphi(\nu,u)\ ,\ \theta))\Omega(u+\ (1-\varsigma)\varphi(u+\varphi(\nu,u)\ ,u)).\end{gathered} \tag{24}$$

and

$$\begin{gathered}\varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))\\ \leq\varsigma^{\beta-1}((1-\varsigma)\Psi_*(u,\ \theta)\ +\varsigma\Psi_*(u+\varphi(\nu,u)\ ,\ \theta))\Omega(u+\varsigma\varphi(\nu,u))\\ \varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))\\ \leq\varsigma^{\beta-1}((1-\varsigma)\Psi^*(u,\ \theta)\ +\varsigma\Psi^*(u+\varphi(\nu,u)\ ,\ \theta))\Omega(u+\varsigma\varphi(\nu,u))\end{gathered} \tag{25}$$

After adding (24) and (25), and integrating over $[0,\ 1]$, we get

$$\begin{aligned}&\int_0^1\varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))d\varsigma\\ &\quad+\int_0^1\varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\\ &\leq\int_0^1\left[\begin{array}{c}\varsigma^{\beta-1}\Psi_*(u,\ \theta)\{\varsigma\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ +\ (1-\varsigma)\Omega(u+\varsigma\varphi(\nu,u))\}\\ +\varsigma^{\beta-1}\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\{(1-\varsigma)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ +\varsigma\Omega(u+\varsigma\varphi(\nu,u))\}\end{array}\right]d\varsigma,\\ &\int_0^1\varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\\ &\quad+\int_0^1\varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))d\varsigma\\ &\leq\int_0^1\left[\begin{array}{c}\varsigma^{\beta-1}\Psi^*(u,\ \theta)\{\varsigma\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ +\ (1-\varsigma)\Omega(u+\varsigma\varphi(\nu,u))\}\\ +\varsigma^{\beta-1}\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\{(1-\varsigma)\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ +\varsigma\Omega(u+\varsigma\varphi(\nu,u))\}\end{array}\right]d\varsigma,\\ &=\Psi_*(u,\ \theta)\int_0^1\ \varsigma^{\beta-1}\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ d\varsigma+\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\int_0^1\ \varsigma^{\beta-1}\Omega(u+\varsigma\varphi(\nu,u))\ d\varsigma,\\ &=\Psi^*(u,\ \theta)\int_0^1\ \varsigma^{\beta-1}\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ d\varsigma+\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\int_0^1\ \varsigma^{\beta-1}\Omega(u+\varsigma\varphi(\nu,u))\ d\varsigma.\end{aligned}$$

Since Ω is symmetric, then

$$= [\Psi_*(u,\ \theta)\ + \Psi_*(u + \varphi(\nu, u)\ ,\ \theta)] \int_0^1\ \varsigma^{\beta-1}\Omega(u + \varsigma\varphi(\nu, u))\ d\varsigma$$

$$= [\Psi^*(u,\ \theta)\ + \Psi^*(u + \varphi(\nu, u)\ ,\ \theta)] \int_0^1\ \varsigma^{\beta-1}\Omega(u + \varsigma\varphi(\nu, u))\ d\varsigma.$$

$$= \frac{\Psi_*(u,\ \theta)\ + \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right], \tag{26}$$

$$= \frac{\Psi^*(u,\ \theta)\ + \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right].$$

Since

$$\begin{aligned}
&\int_0^1 \varsigma^{\beta-1}\Psi_*(u +\ (1-\varsigma)\varphi(\nu, u)\ ,\ \theta)\Omega(u + \varsigma\varphi(\nu, u))d\varsigma \\
&\quad + \int_0^1 \varsigma^{\beta-1}\Psi_*(u + \varsigma\varphi(\nu, u)\ ,\ \theta)\Omega(u + \varsigma\varphi(\nu, u))d\varsigma \\
&= \frac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)} (\omega-u)^{\beta-1}\Psi_*(2u + \varphi(\nu, u)\ -\omega, \theta)\Omega(\omega)d\omega \\
&\quad + \frac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)} (\omega-u)^{\beta-1}\Psi_*(\omega, \theta)\Omega(\omega)d\omega \\
&= \frac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)} (\omega-u)^{\beta-1}\Psi_*(\omega, \theta)\Omega(2u + \varphi(\nu, u)\ -\omega)d\omega \\
&\quad + \frac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)} (\omega-u)^{\beta-1}\Psi_*(\omega, \theta)\Omega(\omega)d\omega \\
&= \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Psi_*\Omega(\nu)\ + \mathcal{I}_{\nu^-}^\beta\ \Psi_*\Omega(u)\right], \\
&\int_0^1 \varsigma^{\beta-1}\Psi^*(u +\ (1-\varsigma)\varphi(\nu, u)\ ,\ \theta)\Omega(u + \varsigma\varphi(\nu, u))d\varsigma \\
&\quad + \int_0^1 \varsigma^{\beta-1}\Psi^*(u + \varsigma\varphi(\nu, u)\ ,\ \theta)\Omega(u + \varsigma\varphi(\nu, u))d\varsigma \\
&= \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Psi^*\Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Psi^*\Omega(u)\right].
\end{aligned} \tag{27}$$

Then from (26), we have

$$\begin{aligned}
&\frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Psi_*\Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Psi_*\Omega(u)\right] \\
&\leq \frac{\Psi_*(u,\ \theta)\ + \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right] \\
&\leq \frac{\Psi_*(u,\ \theta)\ + \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right], \\
&\frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Psi^*\Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Psi^*\Omega(u)\right] \\
&\leq \frac{\Psi^*(u,\ \theta)\ + \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right] \\
&\leq \frac{\Psi^*(u,\ \theta)\ + \Psi^*(u+\varphi(\nu)\ ,\ \theta)}{2} \frac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta} \left[\mathcal{I}_{u^+}^\beta\ \Omega(u + \varphi(\nu, u))\ + \mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Omega(u)\right],
\end{aligned}$$

that is

$$\frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*\Omega(u)\right],\ \mathcal{I}_{u^+}^{\beta}\ \Psi^*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*\Omega(u)\right]$$

$$\leq_I \frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\frac{\Psi_*(u,\theta)+\Psi_*(u+\varphi(\nu,u),\theta)}{2},\ \frac{\Psi^*(u,\theta)+\Psi^*(u+\varphi(\nu,u),\theta)}{2}\right]\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]$$

$$\leq_I \frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\frac{\Psi_*(u,\theta)+\Psi_*(\nu,\theta)}{2},\ \frac{\Psi^*(u,\theta)+\Psi^*(\nu,\theta)}{2}\right]\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]$$

hence

$$\left[\mathcal{I}_{u^+}^{\beta}\ \Psi\Omega(u+\varphi(\nu,u))\ \widetilde{+}\,\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi\Omega(u)\right]$$

$$\preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]$$

$$\preccurlyeq \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\widetilde{+}\,\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]$$

□

Theorem 3.5. *Let* $\Psi:\ [u,\ u+\varphi(\nu,u)]\ \to \mathbb{F}_0$ *be a preinvex* $F{\cdot}I{\cdot}V{\cdot}F$ *with* $u<\nu$, *whose* θ-*levels define the family of* $I{\cdot}V{\cdot}Fs$ $\Psi_\theta:\ [u,\ u+\varphi(\nu,u)]\ \subset\mathbb{R}\to\mathcal{K}_C{}^{+}$ *are given by* $\Psi_\theta(\omega)=[\Psi_*(\omega,\theta),\ \Psi^*(\omega,\theta)]$ *for all* $\omega\in\ [u,\ u+\varphi(\nu,u)]$ *and for all* $\theta\in\ [0,\ 1]$. *If* $\Psi\in L\ ([u,\ u+\varphi(\nu,u)],\mathbb{F}_0)$ *and* $\Omega:\ [u,\ u+\varphi(\nu,u)]\ \to\mathbb{R}$, $\Omega(\omega)\ \geq 0$, *symmetric with respect to* $\frac{2u+\varphi(\nu,u)}{2}$. *If* φ *satisfies Condition C and then*

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]$$
$$\preccurlyeq\ \left[\mathcal{I}_{u^+}^{\beta}\ \Psi\Omega(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi\Omega(u)\right]. \tag{28}$$

If Ψ is preconcave $F{\cdot}I{\cdot}V{\cdot}F$, then inequality (28) is reversed.

Proof. Since Ψ is a preinvex $F{\cdot}I{\cdot}V{\cdot}F$, then for $\theta\in\ [0,\ 1]$, we have

$$\Psi_*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \leq\frac{1}{2}(\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u),\ \theta)\ +\Psi_*(u+\varsigma\varphi(\nu,u),\ \theta))$$
$$\Psi^*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\ \leq\frac{1}{2}(\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u),\ \theta)\ +\Psi^*(u+\varsigma\varphi(\nu,u),\ \theta)), \tag{29}$$

Since $\Omega(u+\ (1-\varsigma)\varphi(\nu,u))\ =\ \Omega(u+\varsigma\varphi(\nu,u))$, then by multiplying (29) by $\varsigma^{\beta-1}\Omega(u+\varsigma\varphi(\nu,u))$ and integrate it with respect to ς over $[0,\ 1]$, we obtain

$$\Psi_*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\int_0^1\varsigma^{\beta-1}\Omega(u+\varsigma\varphi(\nu,u))d\varsigma$$
$$\leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u),\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\\ +\int_0^1\varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u),\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\end{array}\right),$$
$$\Psi^*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\int_0^1\Omega(u+\varsigma\varphi(\nu,u))d\varsigma$$
$$\leq\frac{1}{2}\left(\begin{array}{c}\int_0^1\varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u),\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\\ +\int_0^1\varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u),\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma\end{array}\right). \tag{30}$$

Let $\omega = u + \varsigma\varphi(\nu, u)$. Then we have

$$\begin{aligned}
&\int_0^1 \varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma \\
&\quad + \int_0^1 \varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma \\
&= \tfrac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi_*(2u+\varphi(\nu,u)\ -\omega,\theta)\Omega(\omega)d\omega \\
&\quad + \tfrac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi_*(\omega,\theta)\Omega(\omega)d\omega \\
&= \tfrac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi_*(\omega,\theta)\Omega(2u+\varphi(\nu,u)\ -\omega)d\omega \\
&\quad + \tfrac{1}{(\varphi(\nu,u))^\beta}\ \int_u^{u+\varphi(\nu,u)}(\omega-u)^{\beta-1}\Psi_*(\omega,\theta)\Omega(\omega)d\omega \\
&= \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*\Omega(u)\right], \\
&\int_0^1 \varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma \\
&\quad + \int_0^1 \varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Omega(u+\varsigma\varphi(\nu,u))d\varsigma \\
&= \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi^*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*\Omega(u)\right].
\end{aligned} \tag{31}$$

Then from (31), we have

$$\begin{aligned}
&\tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right] \\
&\quad \le \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\ \left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*\Omega(u)\right] \\
&\tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right] \\
&\quad \le \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\ \left[\mathcal{I}_{u^+}^{\beta}\ \Psi^*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*\Omega(u)\right],
\end{aligned}$$

from which, we have

$$\begin{aligned}
&\tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\left[\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right),\ \ \Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\ \theta\right)\right]\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right] \\
&\le\ \ {}_I\tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\left[\ \mathcal{I}_{u^+}^{\beta}\ \Psi_*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*\Omega(u)\ ,\ \mathcal{I}_{u^+}^{\beta}\ \Psi^*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*\Omega(u)\right],
\end{aligned}$$

that is

$$\begin{aligned}
&\tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\Psi\left(\tfrac{2u+\varphi(\nu,u)}{2}\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right] \\
&\quad \preccurlyeq \tfrac{\Gamma(\beta)}{(\varphi(\nu,u))^\beta}\ \left[\mathcal{I}_{u^+}^{\beta}\ \Psi\Omega(u+\varphi(\nu,u))\tilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi\Omega(u)\right]
\end{aligned}$$

This completes the proof. □

Example 3.6. We consider the $F{\cdot}I{\cdot}V{\cdot}F$ $\Psi :\ [0,\ 2]\ \to \mathbb{F}_0$ defined by,

$$\Psi(\omega)(\sigma)\ =\ \begin{cases} \frac{\sigma}{2-\sqrt{\omega}}, & \sigma \in\ \left[0,\ 2-\sqrt{\omega}\right], \\ \frac{2\left(2-\sqrt{\omega}\right)\ -\sigma}{2-\sqrt{\omega}}, & \sigma \in\ \left(2-\sqrt{\omega},\ 2\left(2-\sqrt{\omega}\right)\right], \\ 0, & otherwise. \end{cases}$$

Then, for each $\theta \in\ [0,\ 1]$, we have $\Psi_\theta(\omega)\ =\ \left[\theta\left(2-\sqrt{\omega}\right)\ ,\ (2-\theta)\left(2-\sqrt{\omega}\right)\right]$. Since end point functions $\Psi_*(\omega,\theta)$, $\Psi^*(\omega,\theta)$ are preinvex functions with respect to $\varphi(\nu,u)\ =\ \nu-u$ for each $\theta \in\ [0,1]$, then $\Psi(\omega)$ is preinvex $F{\cdot}I{\cdot}V{\cdot}F$. If

$$\Omega(\omega)\ =\ \begin{cases} \sqrt{\omega}, & \sigma \in\ [0,1]\ , \\ \sqrt{2-\omega}, & \sigma \in\ (1,\ 2]\ , \end{cases}$$

then $\Omega(2-\omega)\ \ =\Omega(\omega)\ \ \geq\ 0$, for all $\omega\ \in\ \ [0,\ 2]$. Since $\Psi_*(\omega,\theta)\ \ =\theta\left(2-\sqrt{\omega}\right)$ and $\Psi^*(\omega,\theta)\ =\ (2-\theta)\left(2-\sqrt{\omega}\right)$. If $\beta = \frac{1}{2}$, then we compute the following:

$$\left[\mathcal{I}_{u^+}^{\beta}\ \Psi\Omega(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi\Omega(u)\right]\ \preccurlyeq\ \frac{\Psi(u)\widetilde{+}\Psi(u+\varphi(\nu,u))}{2}\left[\begin{array}{c}\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u)) \\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\end{array}\right]$$

$$\preccurlyeq\ \frac{\Psi(u)\widetilde{+}\Psi(\nu)}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right] \tag{32}$$

$$\frac{\Psi_*(u)\ +\Psi_*(u+\varphi(\nu,u))}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\ \frac{\pi}{\sqrt{2}}\theta\left(\frac{4-\sqrt{2}}{2}\right)$$

$$\frac{\Psi^*(u)\widetilde{+}\Psi^*(u+\varphi(\nu,u))}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\ \frac{\pi}{\sqrt{2}}(2-\theta)\left(\frac{4-\sqrt{2}}{2}\right),$$

$$\frac{\Psi_*(u)\ +\Psi_*(\nu)}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\ \frac{\pi}{\sqrt{2}}\theta\left(\frac{4-\sqrt{2}}{2}\right)$$

$$\frac{\Psi^*(u)\widetilde{+}\Psi^*(\nu)}{2}\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\ \frac{\pi}{\sqrt{2}}(2-\theta)\left(\frac{4-\sqrt{2}}{2}\right), \tag{33}$$

$$\left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*\Omega(u)\right]\ =\ \frac{1}{\sqrt{\pi}}\theta\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),$$

$$\left[\mathcal{I}_{u^+}^{\beta}\ \Psi^*\Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*\Omega(u)\right]\ =\ \frac{1}{\sqrt{\pi}}(2-\theta)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right). \tag{34}$$

From (32), (33) and (34), we have

$$\frac{1}{\sqrt{\pi}}\left[\theta\left(2\pi+\frac{4-8\sqrt{2}}{3}\right),\ (2-\theta)\left(2\pi+\frac{4-8\sqrt{2}}{3}\right)\right]\quad \leq_I\ \frac{\pi}{\sqrt{2}}\left[\theta\left(\frac{4-\sqrt{2}}{2}\right),\ (2-\theta)\left(\frac{4-\sqrt{2}}{2}\right)\right]$$

$$=\ \frac{\pi}{\sqrt{2}}\left[\theta\left(\frac{4-\sqrt{2}}{2}\right),\ (2-\theta)\left(\frac{4-\sqrt{2}}{2}\right)\right]$$

for each $\theta \in\ [0,\ 1]$. Hence, Theorem 10 is verified.

For Theorem 11, we have

$$\Psi_*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\theta\sqrt{\pi},$$

$$\Psi^*\left(\frac{2u+\varphi(\nu,u)}{2},\ \theta\right)\left[\mathcal{I}_{u^+}^{\beta}\ \Omega(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Omega(u)\right]\ =\ (2-\theta)\sqrt{\pi}. \tag{35}$$

From (34) and (35), we have $\sqrt{\pi}[\theta,\ (2-\theta)] \leq_I \frac{1}{\sqrt{\pi}}[\theta(2\pi + \frac{4-8\sqrt{2}}{3}), (2-\theta)(2\pi + \frac{4-8\sqrt{2}}{3})]$, for each $\theta \in\ [0,\ 1]$.

Remark 3.7. If $\Omega(\omega)\ = 1$. Then from Theorem 3.4 and Theorem 3.5, we get Theorem 3.1.

Let $\beta = 1$. Then we obtain following $H{\cdot}H$ Fejér type inequality for preinvex $F{\cdot}I{\cdot}V{\cdot}F$, see [28].

$$\Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right) \preccurlyeq \frac{1}{\int_u^{u+\varphi(\nu,u)}\Omega(\omega)d\omega}\ (FR)\int_u^{u+\varphi(\nu,u)}\Psi(\omega)\Omega(\omega)d\omega \preccurlyeq \frac{\Psi(u)\ +\Psi(\nu)}{2}$$

If $\Psi_*(\omega,\ \theta)\ = \Psi^*(\omega,\ \theta)$ with $\varphi(\omega,y)\ = \omega - y$ and $\Omega(\omega)\ = \beta = 1 = \theta$. Then from Theorem 3.4 and Theorem 3.5, we get the classical $H{\cdot}H$ inequality.

If $\Psi_*(\omega,\ \theta)\ = \Psi^*(\omega,\ \theta)$ with $\varphi(\omega,y)\ = \omega - y$ and $\beta = 1$, then from Theorem 3.4 and Theorem 3.5, we obtain the classical $H{\cdot}H$ Fejér inequality, see [46].

From Theorems 3.8 and 3.9, now we get several fuzzy-interval fractional integral inequalities linked to fuzzy-interval fractional $H{\cdot}H$ type inequality for the product of preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$.

Theorem 3.8. *Let $\Psi,\Phi\ :\ [u,\ u+\varphi(\nu,u)]\ \to \mathbb{F}_0$ be two preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ on $[u,\ u+\varphi(\nu,u)]$, whose θ-levels $\Psi_\theta,\ \Phi_\theta :\ [u,\ u+\varphi(\nu,u)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are defined by $\Psi_\theta(\omega) = [\Psi_*(\omega,\theta)\ ,\ \Psi^*(\omega,\theta)]$ and $\Phi_\theta(\omega) = [\Phi_*(\omega,\theta)\ ,\ \Phi^*(\omega,\theta)]$ for all $\omega \in\ [u,\ u+\varphi(\nu,u)]$ and for all $\theta \in\ [0,\ 1]$. If $\Psi\widetilde{\times}\Phi \in\ L([u,\ u+\varphi(\nu,u)]\ ,\mathbb{F}_0)$ and φ satisfies Condition C, then*

$$\frac{\Gamma(\beta)}{2(\varphi(\nu,u))^\beta}\left[\mathcal{I}_{u^+}^\beta\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^\beta\ \Psi(u)\widetilde{\times}\Phi(u)\right]$$
$$\preccurlyeq\ \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta\ (u,u+\varphi(\nu,u))\widetilde{+}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla\ (u,u+\varphi(\nu,u))$$

where $\Delta\ (u,u+\varphi(\nu,u))\ = \Psi(u)\widetilde{\times}\Phi(u)\ \widetilde{+}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))$, $\nabla\ (u,u+\varphi(\nu,u))\ = \Psi(u)\widetilde{\times}\Phi(u+\varphi(\nu,u))\ \widetilde{+}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u)$, and $\Delta_\theta(u,u+\varphi(\nu,u)) = [\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,\ \Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$ and $\nabla_\theta(u,\nu) = [\nabla_*((u,u+\varphi(\nu,u)),\theta), \nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$.

Proof. Since $\Psi,\ \Phi$ both are preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ and Condition C holds φ for then, for each $\theta \in\ [0,\ 1]$ we have

$$\begin{aligned}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ &= \Psi_*(u+\varphi(\nu,u)\ +\varsigma\varphi(u,u+\varphi(\nu,u))\ ,\ \theta)\\ &\leq \varsigma\Psi_*(u,\theta)\ +\ (1-\varsigma)\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\\ \Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ &= \Psi^*(u+\varphi(\nu,u)\ +\varsigma\varphi(u,u+\varphi(\nu,u))\ ,\ \theta)\\ &\leq \varsigma\Psi^*(u,\theta)\ +\ (1-\varsigma)\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ .\end{aligned}$$

and

$$\begin{aligned}\Phi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ &= \Phi_*(u+\varphi(\nu,u)\ +\varsigma\varphi(u,u+\varphi(\nu,u))\ ,\ \theta)\\ &\leq \varsigma\Phi_*(u,\theta)\ +\ (1-\varsigma)\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\\ \Phi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ &= \Phi^*(u+\varphi(\nu,u)\ +\varsigma\varphi(u,u+\varphi(\nu,u))\ ,\ \theta)\\ &\leq \varsigma\Phi^*(u,\theta)\ +\ (1-\varsigma)\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\ .\end{aligned}$$

From the definition of preinvex $F{\cdot}I{\cdot}V{\cdot}Fs$ it follows that $\widetilde{0} \preccurlyeq \Psi(\omega)$ and $\widetilde{0} \preccurlyeq \Phi(\omega)$, so

$$\begin{aligned}
&\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&\leq\ (\varsigma\Psi_*(u,\theta)\ +\ (1-\varsigma)\Psi_*(u+\varphi(\nu,u)\ ,\ \theta))(\varsigma\Phi_*(u,\theta)\ +\ (1-\varsigma)\Phi_*(u+\varphi(\nu,u)\ ,\ \theta))\\
&=\varsigma^2\Psi_*(u,\theta)\ \times\Phi_*(u,\theta)\ +(1-\varsigma)^2\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma(1-\varsigma)\Psi_*(u,\theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\varsigma(1-\varsigma)\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u,\theta)\\
&\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&\leq\ (\varsigma\Psi^*(u,\theta)\ +\ (1-\varsigma)\Psi^*(u+\varphi(\nu,u)\ ,\ \theta))(\varsigma\Phi^*(u,\theta)\ +\ (1-\varsigma)\Phi^*(u+\varphi(\nu,u)\ ,\ \theta))\\
&=\varsigma^2\Psi^*(u,\theta)\ \times\Phi^*(u,\theta)\ +(1-\varsigma)^2\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma(1-\varsigma)\Psi^*(u,\theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\varsigma(1-\varsigma)\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u,\theta)\ ,
\end{aligned}\tag{36}$$

Analogously, we have

$$\begin{aligned}
&\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\Phi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\\
&\leq(1-\varsigma)^2\Psi_*(u,\theta)\ \times\Phi_*(u,\theta)\ +\varsigma^2\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma(1-\varsigma)\Psi_*(u,\theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\varsigma(1-\varsigma)\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u,\theta)\\
&\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\\
&\leq(1-\varsigma)^2\Psi^*(u,\theta)\ \times\Phi^*(u,\theta)\ +\varsigma^2\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma(1-\varsigma)\Psi^*(u,\theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\varsigma(1-\varsigma)\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u,\theta)\ .
\end{aligned}\tag{37}$$

Adding (36) and (37), we have

$$\begin{aligned}
&\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&+\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\\
&\leq\ \left[\varsigma^2+(1-\varsigma)^2\right][\Psi_*(u,\theta)\ \times\Phi_*(u,\theta)\ +\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)]\\
&+2\varsigma(1-\varsigma)[\Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u,\theta)\ +\Psi_*(u,\theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)]\\
&\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&+\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\\
&\leq\ \left[\varsigma^2+(1-\varsigma)^2\right][\Psi^*(u,\theta)\ \times\Phi^*(u,\theta)\ +\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)]\\
&+2\varsigma(1-\varsigma)[\Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u,\theta)\ +\Psi^*(u,\theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)]\ .
\end{aligned}\tag{38}$$

Taking multiplication of (38) by $\varsigma^{\beta-1}$ and integrating the obtained result with respect to ς over (0,1), we have

$$\begin{aligned}
&\int_0^1\varsigma^{\beta-1}\Psi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma^{\beta-1}\Psi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)d\varsigma\\
&\leq\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)\int_0^1\varsigma^{\beta-1}\left[\varsigma^2+(1-\varsigma)^2\right]d\varsigma\\
&+2\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)\int_0^1\varsigma^{\beta-1}\varsigma(1-\varsigma)d\varsigma\\
&\int_0^1\varsigma^{\beta-1}\Psi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\ (1-\varsigma)\varphi(\nu,u)\ ,\ \theta)\\
&+\varsigma^{\beta-1}\Psi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varsigma\varphi(\nu,u)\ ,\ \theta)d\varsigma\\
&\leq\Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)\int_0^1\varsigma^{\beta-1}\left[\varsigma^2+(1-\varsigma)^2\right]d\varsigma\\
&+2\nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)\int_0^1\varsigma^{\beta-1}\varsigma(1-\varsigma)d\varsigma.
\end{aligned}$$

It follows that,

$$\frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*(u,\theta)\ \times\Phi_*(u,\theta)\right]$$
$$\leq \frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)$$
$$\frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*(u,\theta)\ \times\Phi^*(u,\theta)\right]$$
$$\leq\ \frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,$$

that is

$$\frac{\Gamma(\beta)}{(\varphi(\nu,u))^{\beta}}[\mathcal{I}_{u^+}^{\beta}\ \Psi_*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi_*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*(u,\theta)\ \times\Phi_*(u,\theta)\ ,$$
$$\mathcal{I}_{u^+}^{\beta}\ \Psi^*(u+\varphi(\nu,u)\ ,\ \theta)\ \times\Phi^*(u+\varphi(\nu,u)\ ,\ \theta)\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*(u,\theta)\ \times\Phi^*(u,\ \theta)]$$
$$\leq_I\ \frac{2}{\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)[\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,\ \Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$$
$$+\frac{2}{\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)[\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,\ \nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$$

Thus,

$$\frac{\Gamma(\beta)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\Psi(u)\widetilde{\times}\Phi(u)\right]$$
$$\preccurlyeq\ \left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \Delta\ (u,u+\varphi(\nu,u))\widetilde{+}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \nabla\ (u,u+\varphi(\nu,u))$$

and the theorem has been established. □

Theorem 3.9. *Let $\Psi,\Phi\ :\ [u,\ u+\varphi(\nu,u)]\ \to\mathbb{F}_0$ be two preinvex F·I·V·Fs, whose θ-levels define the family of I·V·Fs $\Psi_\theta,\ \Phi_\theta:\ [u,\ u+\varphi(\nu,u)]\ \subset\mathbb{R}\to\mathcal{K}_C{}^{+}$ are given by $\Psi_\theta(\omega)\ =\ [\Psi_*(\omega,\theta)\ ,\ \Psi^*(\omega,\theta)]$ and $\Phi_\theta(\omega)\ =\ [\Phi_*(\omega,\theta)\ ,\ \Phi^*(\omega,\theta)]$ for all $\omega\in\ [u,\ u+\varphi(\nu,u)]$ and for all $\theta\in\ [0,\ 1]$. If $\Psi\widetilde{\times}\Phi\in L([u,\ u+\varphi(\nu,u)]\ ,\mathbb{F}_0)$ and φ satisfies Condition C, then*

$$\frac{1}{\beta}\ \Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right)\widetilde{\times}\Phi\left(\frac{2u+\varphi(\nu,u)}{2}\right)$$
$$\preccurlyeq\ \frac{\Gamma(\beta+1)}{4(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi(u)\widetilde{\times}\Phi(u)\right]$$
$$\widetilde{+}\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \nabla\ (u,u+\varphi(\nu,u))\widetilde{+}\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \Delta\ (u,u+\varphi(\nu,u))\ .$$

where $\Delta\ (u,u+\varphi(\nu,u))\ =\Psi(u)\widetilde{\times}\Phi(u)\ \widetilde{+}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))$, $\nabla\ (u,\nu)\ =\ \Psi(u)\widetilde{\times}\Phi(u+\varphi(\nu,u))\ \widetilde{+}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u)$, and $\Delta_\theta(u,u+\varphi(\nu,u))\ =\ [\Delta_*((u,u+\varphi(u+\varphi(\nu,u)))\ ,\ \theta)\ ,\ \Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$ and $\nabla_\theta(u,u+\varphi(\nu,u))\ =\ [\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,\ \nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)]$.

Proof. Consider $\Psi,\Phi\ :\ [u,\ u+\varphi(\nu,u)]\ \to\mathbb{F}_0$ are preinvex *F·I·V·Fs*. Then by hypothesis, for each $\theta\in\ [0,\ 1]$, we have

$$
\begin{aligned}
&\Psi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\theta\right)\times\Phi_*\left(\tfrac{2u+\varphi(\nu,u)}{2},\theta\right)\\
&\Psi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\theta\right)\times\Phi^*\left(\tfrac{2u+\varphi(\nu,u)}{2},\theta\right)\\
&\le\frac{1}{4}\left[\begin{array}{c}\Psi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi_*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}\Psi_*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi_*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi_*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\le\frac{1}{4}\left[\begin{array}{c}\Psi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi^*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}\Psi^*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi^*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi^*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right],\\
&\le\frac{1}{4}\left[\begin{array}{c}\Psi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi_*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi_*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}(\varsigma\Psi_*(u,\theta)+(1-\varsigma)\Psi_*(u+\varphi(\nu,u),\theta))\\\times((1-\varsigma)\Phi_*(u,\theta)+\varsigma\Phi_*(u+\varphi(\nu,u),\theta))\\+((1-\varsigma)\Psi_*(u,\theta)+\varsigma\Psi_*(u+\varphi(\nu,u),\theta))\\\times(\varsigma\Phi_*(u,\theta)+(1-\varsigma)\Phi_*(u+\varphi(\nu,u),\theta))\end{array}\right]\\
&\le\frac{1}{4}\left[\begin{array}{c}\Psi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi^*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi^*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}(\varsigma\Psi^*(u,\theta)+(1-\varsigma)\Psi^*(u+\varphi(\nu,u),\theta))\\\times((1-\varsigma)\Phi^*(u,\theta)+\varsigma\Phi^*(u+\varphi(\nu,u),\theta))\\+((1-\varsigma)\Psi^*(u,\theta)+\varsigma\Psi^*(u+\varphi(\nu,u),\theta))\\\times(\varsigma\Phi^*(u,\theta)+(1-\varsigma)\Phi^*(u+\varphi(\nu,u),\theta))\end{array}\right],\\
&=\frac{1}{4}\left[\begin{array}{c}\Psi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi_*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi_*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi_*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}\left\{\varsigma^2+(1-\varsigma)^2\right\}\nabla_*((u,u+\varphi(\nu,u)),\theta)\\+\{\varsigma(1-\varsigma)+(1-\varsigma)\varsigma\}\Delta_*((u,u+\varphi(\nu,u)),\theta)\end{array}\right]\\
&=\frac{1}{4}\left[\begin{array}{c}\Psi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\times\Phi^*(u+(1-\varsigma)\varphi(\nu,u),\theta)\\+\Psi^*(u+\varsigma\varphi(\nu,u),\theta)\times\Phi^*(u+\varsigma\varphi(\nu,u),\theta)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}\left\{\varsigma^2+(1-\varsigma)^2\right\}\nabla^*((u,u+\varphi(\nu,u)),\theta)\\+\{\varsigma(1-\varsigma)+(1-\varsigma)\varsigma\}\Delta^*((u,u+\varphi(\nu,u)),\theta)\end{array}\right].
\end{aligned}
\tag{39}
$$

Taking multiplication of (39) with $\varsigma^{\beta-1}$ and integrating over $(0,\ 1)$, we get

$$\frac{1}{\beta}\ \Psi_*\left(\frac{2u+\varphi(\nu,u)}{2},\theta\right)\ \times\Phi_*\left(\frac{2u+\varphi(\nu,u)}{2},\theta\right)$$

$$\leq \frac{1}{4(\varphi(\nu,u))^{\beta}}\left[\begin{array}{c}\int_u^{u+\varphi(\nu,u)}(u+\varphi(\nu,u)\ -\omega)^{\beta-1}\Psi_*(\omega,\theta)\ \times\Phi_*(\omega,\theta)d\omega\\ +\int_u^{u+\varphi(\nu,u)}(y-u)^{\beta-1}\Psi_*(y,\theta)\ \times\Phi_*(y,\theta)dy\end{array}\right]$$

$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)$$

$$=\frac{\Gamma(\beta+1)}{4(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi_*(u+\varphi(\nu,u))\ \times\Phi_*(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi_*(u)\ \times\Phi_*(u)\right]$$

$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta_*((u,u+\varphi(\nu,u))\ ,\ \theta)$$

$$\frac{1}{\beta}\ \Psi^*\left(\frac{2u+\varphi(\nu,u)}{2},\theta\right)\ \times\Phi^*\left(\frac{2u+\varphi(\nu,u)}{2},\theta\right)$$

$$\leq \frac{1}{4(\varphi(\nu,u))^{\beta}}\left[\begin{array}{c}\int_u^{u+\varphi(\nu,u)}(u+\varphi(\nu,u)\ -\omega)^{\beta-1}\Psi^*(\omega,\theta)\ \times\Phi^*(\omega,\theta)d\omega\\ +\int_u^{u+\varphi(\nu,u)}(y-u)^{\beta-1}\Psi^*(y,\theta)\ \times\Phi^*(y,\theta)dy\end{array}\right]$$

$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)$$

$$=\frac{\Gamma(\beta+1)}{4(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi^*(u+\varphi(\nu,u))\ \times\Phi^*(u+\varphi(\nu,u))\ +\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi^*(u)\ \times\Phi^*(u)\right]$$

$$+\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla^*((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta^*((u,u+\varphi(\nu,u))\ ,\ \theta)\ ,$$

that is

$$\frac{1}{\beta}\ \Psi\left(\frac{2u+\varphi(\nu,u)}{2}\right)\widetilde{\times}\Phi\left(\frac{2u+\varphi(\nu,u)}{2}\right)$$

$$\preccurlyeq\frac{\Gamma(\beta+1)}{4(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}_{u^+}^{\beta}\ \Psi(u+\varphi(\nu,u))\widetilde{\times}\Phi(u+\varphi(\nu,u))\widetilde{+}\mathcal{I}_{u+\varphi(\nu,u)^-}^{\beta}\ \Psi(u)\widetilde{\times}\Phi(u)\right]$$

$$\widetilde{+}\frac{1}{2\beta}\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \nabla\ (u,u+\varphi(\nu,u))\widetilde{+}\frac{1}{2\beta}\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\ \Delta\ (u,u+\varphi(\nu,u))$$

Hence, the required result. □

Example 3.10. Let $[u,u+\varphi(\nu,u)]\ =\ [0,\varphi(2,0)]$, $\beta=\frac{1}{2}$, $\Psi(\omega)\ =\ [\omega,2\omega]$, and $\Phi(\omega)=[\omega,3\omega]$.

$$\Psi(\omega)(\theta)\ =\ \begin{cases}\frac{\theta}{\omega} & \theta\in\ [0,\ \omega]\\ \frac{2\omega-\theta}{\omega} & \theta\in\ (\omega,\ 2\omega]\\ 0 & otherwise,\end{cases}$$

$$\Phi(\omega)(\theta)\ =\ \begin{cases}\frac{\theta}{2\omega} & \theta\in\ [0,\ 2\omega]\\ \frac{4\omega-\theta}{2\omega} & \theta\in\ (2\omega,\ 4\omega]\\ 0 & otherwise.\end{cases}$$

Then, for each $\theta\in\ [0,\ 1]$, we have $\Psi_\theta(\omega)\ =\ [\theta\omega,\ (2-\theta)\omega]$ and $\Phi_\theta(\omega)=[2\theta\omega,2(2-\theta)\omega]$. Since left and right end point functions $\Psi_*(\omega,\theta)\ =\theta\omega$, $\Psi^*(\omega,\ \theta)\ =(2-\theta)\omega$, $\Phi_*(\omega,\theta)\ =2\theta\omega$ and $\Phi^*(\omega,\ \theta)\ =2(2-\theta)\omega$ are preinvex functions with respect to $\varphi(\nu,u)\ =\nu-u$ and for each $\theta\in\ [0,1]$, then $\Psi(\omega)$ and $\Phi(\omega)$ both are preinvex $F{\cdot}I{\cdot}V{\cdot}F$. We clearly see that $\Psi(\omega)\widetilde{\times}\Phi(\omega)\ \in L([u,\ u+\varphi(\nu,u)]\ ,\mathbb{F}_0)$ and

$$\frac{\Gamma(1+\beta)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^{+}}\ \Psi_{*}(u+\varphi(\nu,u))\ \times\Phi_{*}(u+\varphi(\nu,u))\ +\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^{-}}\ \Psi_{*}(u)\ \times\Phi_{*}(u)\right]$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\varphi(2,0)}(2-\omega)^{\frac{-1}{2}}\ \left(2\theta^{2}\omega^{2}\right)d\omega+\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\varphi(2,0)}(\omega)^{\frac{-1}{2}}\ \left(2\theta^{2}\omega^{2}\right)d\omega\approx 2.9332\theta^{2},$$

$$\frac{\Gamma(1+\beta)}{2(\varphi(\nu,u))^{\beta}}\left[\mathcal{I}^{\beta}_{u^{+}}\ \Psi^{*}(u+\varphi(\nu,u))\ \times\Phi^{*}(u+\varphi(\nu,u))\ +\mathcal{I}^{\beta}_{u+\varphi(\nu,u)^{-}}\ \Psi^{*}(u)\ \times\Phi^{*}(u)\right]$$

$$=\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\varphi(2,0)}(2-\omega)^{\frac{-1}{2}}\ .2(2-\theta)^{2}\omega^{2}d\omega+\frac{\Gamma\left(\frac{3}{2}\right)}{2\sqrt{2}}\frac{1}{\sqrt{\pi}}\int\limits_{0}^{\varphi(2,0)}(\omega)^{\frac{-1}{2}}\ .2(2-\theta)^{2}\omega^{2}d\omega$$

$$\approx 2.9332(2-\theta)^{2},$$

Note that

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta_{*}(u,u+\varphi(\nu,u))\ =\ [\Psi_{*}(u)\ \times\Phi_{*}(u)\ +\Psi_{*}(u+\varphi(\nu,u))\ \times\Phi_{*}(u+\varphi(\nu,u))]$$

$$=\frac{11}{30}.8\theta^{2},$$

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta^{*}(u,u+\varphi(\nu,u))$$

$$=\ [\Psi^{*}(u)\ \times\Phi^{*}(u)\ +\Psi^{*}(u+\varphi(\nu,u))\ \times\Phi^{*}(u+\varphi(\nu,u))]\ =\frac{11}{30}.8(2-\theta)^{2},$$

$$\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_{*}(u,u+\varphi(\nu,u))\ =\ [\Psi_{*}(u)\ \times\Phi_{*}(u+\varphi(\nu,u))\ +\Psi_{*}(u+\varphi(\nu,u))\ \times\Phi_{*}(u)]$$

$$=\frac{2}{15}(0)\ ,\left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_{*}(u,u+\varphi(\nu,u))\ =\ [\Psi^{*}(u)\ \times\Phi^{*}(u+\varphi(\nu,u))\ +\Psi^{*}(u+\varphi(\nu,u))\ \times\Phi^{*}(u)]$$

$$=\frac{2}{15}(0)\ .$$

Therefore, we have

$$\left(\frac{1}{2}-\frac{\beta}{(\beta+1)(\beta+2)}\right)\Delta_{\theta}((u,u+\varphi(\nu,u))\ ,\ \theta)\ +\ \left(\frac{\beta}{(\beta+1)(\beta+2)}\right)\nabla_{\theta}((u,u+\varphi(\nu,u))\ ,\ \theta)$$

$$=\frac{11}{30}\left[8\theta^{2},8(2-\theta)^{2}\right]\ +\frac{2}{15}[0,0]\ \approx\ \left[2.9332\theta^{2},2.9332(2-\theta)^{2}\right].$$

It follows that

$$\left[2.9332\theta^{2},2.9332(2-\theta)^{2}\right]\leq_{I}\left[2.9332\theta^{2},2.9332(2-\theta)^{2}\right],$$

and Theorem 3.7. has been demonstrated.

4. Conclusions and Future Plan

In this article, we established relation between integral inequalities and preinvex *F·I·V·Fs* using fuzzy Riemann–Liouville fractional integrals and Condition C. We addressed *H·H* type inequalities and *H·H* Fejér type inequalities for introduced preinvex *F·I·V·F*. Moreover, some related fuzzy fractional inequalities were also obtained. We gave useful examples to verify the validity of presented results. In future, we will try to explore this concept for generalized preinvex *F·I·V·Fs* and using fuzzy Riemann–Liouville fractional integrals, we will try to get new inequalities for preinvex *F·I·V·Fs.* We believe that the implications and methodologies presented in this article will energize and encourage scholars to pursue a more intriguing follow-up in this field. Finally, we think that our findings may be applied to other fractional calculus models having Mittag-Liffler functions in their kernels, such as Atangana-Baleanue and Prabhakar fractional operators. This consideration has been kept as an open problem for academics interested in this topic. Researchers that are interested might follow the steps outlined in references [52,53].

Author Contributions: Conceptualization, M.B.K.; methodology, M.B.K.; validation, S.T., M.S.S. and H.G.Z.; formal analysis, J.E.M.-D.; investigation, M.S.S.; resources, S.T.; data curation, H.G.Z.; writing—original draft preparation, M.B.K., J.E.M.-D. and H.G.Z.; writing—review and editing, M.B.K. and S.T.; visualization, H.G.Z.; supervision, M.B.K. and M.S.S.; project administration, M.B.K.; funding acquisition, J.E.M.-D., M.S.S. and H.G.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research. This work was funded by Taif University Researchers Supporting Project number (TURSP-2020/345), Taif University, Taif, Saudi Arabia and this work was also supported by Consejo Nacional de Ciencia y Tecnología, Grant No. A1-S-45928.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Xi, B.-Y.; Qi, F. Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means. *J. Funct. Spaces Appl.* **2012**, *2012*, 980438. [CrossRef]
2. Niculescu, C.P.; Persson, L.E. *Convex Functions and Their Applications*; Springer: Berlin/Heidelberg, Germany, 2006.
3. Ozcan, S.; Iscan, I. Some new Hermite-Hadamard type integral inequalities for s-convex functions and theirs applications. *J. Inequal. Appl.* **2019**, *1*, 201. [CrossRef]
4. Sahoo, S.K.; Ahmad, H.; Tariq, M.; Kodamasingh, B.; Aydi, H.; De la Sen, M. Hermite-Hadamard type inequalities involving k-fractional operator for (h, m)-convex functions. *Symmetry* **2021**, *13*, 1686. [CrossRef]
5. Tariq, M.; Sahoo, S.K.; Nasir, J.; Aydi, H.; Alsamir, H. Some Ostrowski type inequalities via n-polynomial exponentially s-convex functions and their applications. *AIMS Math.* **2021**, *6*, 13272–13290. [CrossRef]
6. Sun, W.; Xu, R. Some new Hermite-Hadamard type inequalities for generalized harmonically convex functions involving local fractional integrals. *AIMS Math.* **2021**, *6*, 10679–10695. [CrossRef]
7. Sun, W. Hermite-Hadamard type local fractional integral inequalities for generalized s-preinvex functions and their generalization. *Fractals* **2021**, *29*, 2150098. [CrossRef]
8. Khan, M.A.; Chu, Y.M.; Khan, T.U.; Khan, J. Some new inequalities of Hermite-Hadamard type for s-convex functions with applications. *Open Math.* **2017**, *15*, 1414–1430. [CrossRef]
9. Du, T.; Awan, M.U.; Kashuri, A.; Zhao, S. Some *k*-fractional extensions of the trapezium inequalities through generalized relative semi-(*m*,*h*)-preinvexity. *Appl. Anal.* **2021**, *100*, 642–662. [CrossRef]
10. Butt, S.I.; Tariq, M.; Aslam, A.; Ahmad, H.; Nofel, T.A. Hermite-Hadamard type inequalities via generalized harmonic exponential convexity and applications. *J. Funct. Space.* **2021**, *2021*, 5533491. [CrossRef]
11. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Nasir, J.; Aydi, H.A.; Mukheimer, A. New Ostrowski-Type Fractional Integral Inequalities via Generalized Exponential-Type Convex Functions and Applications. *Symmetry* **2021**, *13*, 1429. [CrossRef]
12. Tariq, M.; Nasir, J.; Sahoo, S.K.; Mallah, A.A. A note on some Ostrowski type inequalities via Generalized Exponentially Convexity. *J. Math. Anal. Model.* **2021**, *2*, 1–15. [CrossRef]
13. Tariq, M.; Sahoo, S.K.; Nasir, J.; Awan, S.K. Some Ostrowski Type Integral Inequalities using Hypergeometric Functions. *J. Fract. Calc. Nonlinear Syst.* **2021**, *2*, 24–41. [CrossRef]
14. Hanson, M.A. On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl.* **1981**, *80*, 545–550. [CrossRef]
15. Ben-Isreal, A.; Mond, B. What is invexity? *Anziam. J.* **1986**, *28*, 1–9. [CrossRef]
16. Mohan, S.R.; Neogy, S.K. On Invex Sets and Preinvex Functions. *J. Math. Anal. Appl.* **1995**, *189*, 901–908. [CrossRef]
17. Antczak, T. Mean value in invexity analysis. *Nonlinear Anal. Theory Methods Appl.* **2005**, *60*, 1473–1484. [CrossRef]
18. Nanda, S.; Kar, K. Convex fuzzy mappings. *Fuzzy Sets Syst.* **1992**, *48*, 129–132. [CrossRef]
19. Syau, Y.-R. On convex and concave fuzzy mappings. *Fuzzy Sets Syst.* **1999**, *103*, 163–168. [CrossRef]
20. Furukawa, N. Convexity and local Lipschitz continuity of fuzzy-valued mappings. *Fuzzy Sets Syst.* **1998**, *93*, 113–119. [CrossRef]
21. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [CrossRef]
22. Yan, H.; Xu, J. A class of convex fuzzy mappings. *Fuzzy Sets Syst.* **2002**, *129*, 47–56. [CrossRef]

23. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Hamed, Y.S. New Hermite–Hadamard Inequalities in Fuzzy-Interval Fractional Calculus and Related Inequalities. *Symmetry* **2021**, *13*, 673. [CrossRef]
24. Khan, M.B.; Mohammed, P.O.; Noor, M.A.; Alsharif, A.M.; Noor, K.I. New fuzzy-interval inequalities in fuzzy-interval fractional calculus by means of fuzzy order relation. *AIMS Math.* **2021**, *6*, 10964–10988. [CrossRef]
25. Khan, M.B.; Noor, M.A.; Mohammed, P.O.; Guirao, J.L.; Noor, K.I. Some Integral Inequalities for Generalized Convex Fuzzy-Interval-Valued Functions via Fuzzy Riemann Integrals. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 158. [CrossRef]
26. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.M. New Hermite-Hadamard type inequalities for (h1, h2)-convex fuzzy-interval-valued functions. *Adv. Differ. Equations* **2021**, *2021*, 6–20. [CrossRef]
27. Noor, M.A. Fuzzy preinvex functions. *Fuzzy Sets Syst.* **1994**, *64*, 95–104. [CrossRef]
28. Khan, M.B.; Noor, M.A.; Abdullah, L.; Chu, Y.M. Some new classes of preinvex fuzzy-interval-valued functions and inequalities. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 1403–1418. [CrossRef]
29. Khan, M.B.; Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.G. Fuzzy Mixed Variational-like and Integral Inequalities for Strongly Preinvex Fuzzy Mappings. *Symmetry* **2021**, *13*, 1816. [CrossRef]
30. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.M. Higher-order strongly preinvex fuzzy mappings and fuzzy mixed variational-like inequalities. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 1856–1870. [CrossRef]
31. Khan, M.B.; Noor, M.A.; Noor, K.I.; Almusawa, H.; Nisar, K.S. Exponentially preinvex fuzzy mappings and fuzzy exponentially mixed variational-like inequalities. *Int. J. Anal. Appl.* **2021**, *19*, 518–541.
32. Liu, P.; Khan, M.B.; Noor, M.A.; Noor, K.I. On Strongly Generalized Preinvex Fuzzy Mappings. *J. Math.* **2021**, *2021*, 6657602. [CrossRef]
33. Khan, M.B.; Noor, M.A.; Noor, K.I.; Ab Ghani, A.T.; Abdullah, L. Extended Perturbed Mixed Variational-Like Inequalities for Fuzzy Mappings. *J. Math.* **2021**, *2021*, 6652930. [CrossRef]
34. Liu, P.; Khan, M.B.; Noor, M.A.; Noor, K.I. New Hermite–Hadamard and Jensen inequalities for log-convex fuzzy-interval-valued functions in the second sense. *Complex Intell. Syst.* **2021**, *2021*, 1–15. [CrossRef]
35. Khan, M.B.; Noor, M.A.; Abdullah, L.; Noor, M.A.; Noor, K.I. New Hermite–Hadamard and Jensen Inequalities for Log-h-Convex Fuzzy-Interval-Valued Functions. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 155. [CrossRef]
36. Bede, B.; Gal, S.G. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy dif-ferential equations. *Fuzzy Sets Syst.* **2005**, *151*, 581–599. [CrossRef]
37. Kaleva, O. Fuzzy differential equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
38. Kulish, U.; Miranker, W. *Computer Arithmetic in Theory and Practice*; Academic Press: Cambridge, MA, USA, 2014.
39. Awan, M.U.; Noor, M.; Noor, K.I.; Costache, S. Some Integral Inequalities for Harmonically logarithmic h-convex functions. *Sohag J. Math.* **2018**, *5*, 57–62. [CrossRef]
40. Allahviranloo, T.; Salahshour, S.; Abbasbandy, S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput.* **2011**, *16*, 297–302. [CrossRef]
41. Hadamard, J. Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *7*, 171–215.
42. Hermite, C. Sur deux limites d'une intégrale définie. *Mathesis* **1883**, *3*, 82–97.
43. Moore, R.E. *Interval Analysis*; Prentice Hall: Hoboken, NJ, USA, 1966.
44. Stefanini, L.; Bede, B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal.* **2010**, *71*, 1311–1328. [CrossRef]
45. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
46. Fejer, L. Uberdie Fourierreihen II. Math. Naturwise. Anz. *Ungar Akad Wiss.* **1906**, *24*, 369–390.
47. Costa, T.; Román-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
48. Khan, M.B.; Treanță, S.; Soliman, M.S.; Nonlaopon, K.; Zaini, H.G. Some Hadamard–Fejér Type Inequalities for LR-Convex Interval-Valued Functions. *Fractal Fract.* **2022**, *6*, 6. [CrossRef]
49. Macías-Díaz, J.E.; Khan, M.B.; Noor, M.A.; Mousa, A.A.A.; Alghamdi, S.M. Hermite-Hadamard inequalities for generalized convex functions in interval-valued calculus. *AIMS Math.* **2022**, *7*, 4266–4292. [CrossRef]
50. Khan, M.B.; Noor, M.A.; Al-Shomrani, M.M.; Abdullah, L. Some Novel Inequalities for LR-h-Convex Interval-Valued Func-tions by Means of Pseudo Order Relation. *Math. Meth. Appl. Sci.* **2021**, *45*, 1310–1340. [CrossRef]
51. Khan, M.B.; Noor, M.A.; Abdeljawad, T.; Abdalla, B.; Althobaiti, A. Some fuzzy-interval integral inequalities for harmonically convex fuzzy-interval-valued functions. *AIMS Math.* **2022**, *7*, 349–370. [CrossRef]
52. Fernandez, A.; Mohammed, P. Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. Math. *Meth. Appl. Sci.* **2020**, *2020*, 8414–8431.
53. Mohammed, P.O.; Abdeljawad, T. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. *Adv. Differ. Equ.* **2020**, *2020*, 363. [CrossRef]
54. Khan, M.B.; Zaini, H.G.; Treanță, S.; Soliman, M.S.; Nonlaopon, K. Riemann–Liouville Fractional Integral Inequalities for Generalized Pre-Invex Functions of Interval-Valued Settings Based upon Pseudo Order Relation. *Mathematics* **2022**, *10*, 204. [CrossRef]

55. Todorcevic, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer Nature: Basingstoke, UK, 2019.
56. Debnath, P.; Konwar, N.; Radenovic, S. *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Science*; Springer: Berlin/Heidelberg, Germany, 2021.