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# Controllability of Impulsive Neutral Fractional Stochastic Systems

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**Abstract:** The study of dynamic systems appears in various aspects of dynamical structures such as decomposition, decoupling, observability, and controllability. In the present research, we study the controllability of fractional stochastic systems (FSF) and examine the Poisson jumps in finite dimensional space where the fractional impulsive neutral stochastic system is controllable. Sufficient conditions are demonstrated with the aid of fixed point theory. The Mittag-Leffler (ML) matrix function defines the controllability of the Grammian matrix (GM). The relation to symmetry is clear since the controllability Grammian is a hermitian matrix (since the integrand in its definition is hermitian) and this is the complex version of a symmetric matrix. In fact, such a Grammian becomes a symmetric matrix in the specific scenario where the controllability Grammian is a real matrix. Some examples are provided to demonstrate the feasibility of the present theory.

**Keywords:** fractional calculus; controllability; impulsive effect; stochastic calculus



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## 1. Introduction

The dual qualities of observability and controllability in linear systems are widely known. If a linear system is observable, the dual system is controllable, and vice versa. Additionally, filtering and control issues in linear systems are also dual. There are a few matrices that describe these issues in connection to one another, and these relationships have practical applications. Additionally, some dualities in nonlinear systems are well recognized. Controllability is significant in systems defined by partial/ordinary differential equations in finite-dimensional (FD) spaces as well as infinite-dimensional (InFD) spaces. In a dynamic system, controllability is one of the most fundamental characteristics. On the other hand, controllability is one of the key fundamental theories in mathematical control theory. It plays a significant role in deterministic as well as stochastic control systems. Generally, controllability means that it is probable to steer a dynamical control system from an initial state to a final state by means of the set of admissible controls. Controllability theory has been considered widely in the fields of FD linear/nonlinear systems as well as InFD systems (see [1–3] and references therein). In 1985, the controllability results were investigated using fixed point theorems in FD spaces. Further research has been carried out in detail on infinite dimensional spaces. Local null controllability of nonlinear functional differential systems has been investigated in [4], and it is also investigated that the systems of nonlinear integrodifferential equations in Banach space are controllable (see [2]). Saying that controllability is the key idea of the body of findings that make up the topic of systems and control is perhaps not overstating the case. Introduced by Kalman [5], it was rapidly recognized as being of vital importance, and the entire subject's structure was built largely around this idea. Controllability became the distinguishing characteristic of control theory as it was used to understand and solve critical problems such as stabilizability, trajectory tracking, disturbance rejection, etc. In recent days, it has become necessary to extend the theory for FD settings due to the appealing features of fractional calculus, stochastic theory, and impulsive condition.

Stochastic fractional ODEs and PDEs have received a lot of attention recently [6–8]. Differential equations with non-integer order derivatives have memory properties known as non-local qualities [9–12]. Because of the non-locality of the Caputo fractional (CF) derivatives in time, CF differential equations are essential for representing and characterizing the memory phenomenon. Controllability problems for stochastic differential equations have grown in popularity in recent years (see [13–16] and references therein). Many applications of stochastic differential equations can be observed in ecology, finance, and economics. The discussion on the stochastic systems on which deterministic controllability ideas are applicable is limited in the literature. In the recent study of the controllability of dynamical systems (DS), the nonlinear FSF with delays is part of the discussion. Neutral differential equations appear in many fields of practical mathematics, and as a result, they have gotten a lot of attention in recent decades. Many physical systems can store information on the state component's derivative, and these systems are referred to as neutral systems. The literature related to neutral FDE is very limited and we refer the reader to [13]. Many real-world systems and biological procedures exhibit some form of dynamic actions, with continuous and discrete properties.

Various evolutionary processes, with biological systems like biological neural models and in pathology, bursting rhythm, and further optimal control models in finance, frequency modulation in signal processing systems, and flying body motions are considered by sudden changes in states at definite times (see [17]). Impulsive systems are a distinct class of the DS that are hybrid, which syndicates continuous-time dynamics with instant state jumps. Impulsive systems have involved growing research consideration because of their wide applications in practical systems from the diversity of scenarios such as complex networks, sampled-data control systems, networked control systems, multi-agent systems [18], and neural network systems. Recently, the impulsive fractional integrodifferential systems with the nonlocal condition in Hilbert space (HS) have been proved controllable in [19]. Ulam-Hyers stability [20], partial asymptotic stability [21], and other related studies on stochastic equations can be seen in [22].

Rather than having continuous movement, a jump process features discontinuous movements, which are called jumps, with random arrival times. It is typically described as a simple or complex Poisson process. Poisson jumps have grown in popularity, and they are now used to describe a wide variety of phenomena. Many real-world systems (such as market crashes, earthquakes, and epidemics) can sometimes experience some jump-type stochastic perturbations. Such systems have the nonexistence of continuous sample paths. Consequently, stochastic processes with jumps are well-matched to modeling such models. Generally, these jump models are produced from Poisson random actions. The sample trails in these systems have left limits and are right-continuous. The analysis of stochastic differential equations with jumps has just achieved popularity [23,24]. Thus, the study of fractional neutral stochastic impulsive DS with Poisson jumps is achieving popularity, mostly in terms of controllability. It can be noticed that most researchers have focused on results on the controllability of stochastic equations with no jumps. To the best of our knowledge, there is no previous work on the controllability of fractional dynamical systems with jumps. In contrast, controllability problems for the proposed issues in this manuscript have not been tackled in the existing literature. This study explores the controllability of DS with Poisson jumps in FD space.

Using the Banach fixed point (BFP) theorem as well as the Schauder fixed point (SFP) theorem with a GM defined by the ML matrix function, sufficient conditions for controllability results are obtained. The relation to symmetry is clear since the controllability

Gramian is a Hermitian matrix (since the integrand in its definition is Hermitian) and this is the complex version of a symmetric matrix. In fact, in the particular case when the controllability Gramian is a real matrix (implying that conjugate transpose matrices are transpose matrices), such a Gramian becomes a symmetric matrix. In the Riemann-Liouville (RL) sense, FDEs require unusual initial conditions with no physical interpretation, and the RL fractional derivatives have a singularity at zero. To avoid this problem, Caputo (1967) proposed another definition, but neither the RL fractional operator nor the CF operator has the semi-group or commutative properties that are fundamental to the derivative on integer order. To address this issue, the concepts of sequential FDEs are addressed [25].

This work is structured in the following manner: Section 2 introduces certain well-known fractional operators and special functions, and basic definitions to be used. There is also a discussion on the solution interpretation of linear fractional stochastic impulsive DS. In Section 3, we have used controllability GM, and the controllability results for linear and nonlinear fractional neutral stochastic impulsive DS with Poisson jumps are constructed. In Section 4, an example is given to show the effectiveness of the theory used. Finally, in Section 5, we conclude our results.

## 2. Preliminaries

We suppose $a,\ b > 0$, such that $a, b \in (k-1, k)$, $k \in \mathbb{N}$, and $D$ be the traditional differential operator. We further suppose $\mathbb{R}^k$ be the Euclidean space of dimension $k$, $\mathbb{R}_+ = [0, \infty)$, and let $\mathbf{f} \in L^1(\mathbb{R}_+)$. The properties and definitions listed below are notable from fractional calculus for $a,\ b > 0$ and appropriate function $\mathbf{f}$ (see [25]). For convenience, we use the notions given below:

Let $L_2(\Theta, \mathcal{G}_t, \mathbb{R}^k)$ indicates the HS of all measurable square integrable random variables $\mathcal{G}_t$ with values in $\mathbb{R}^k$, where $(\Theta, \mathcal{G}, P)$ denotes probability space and $\mathcal{S} := [0, T]$ for some $T > 0$. Let $L_2^{\mathcal{G}}(\mathcal{S}, \mathbb{R}^k)$ be the HS of all square-integrable and $\mathcal{G}_t$-measurable processes with values in $\mathbb{R}^k$. Let $P\zeta(\mathcal{S}, \mathbb{R}^k) = \{p : p$ is a function from $\mathcal{T}$ into $\mathbb{R}^k$ such that $p(t)$ is continuous at $t \neq t_n$ and left continuous at $t = t_n$ and the right hand limit $p(t_n^+)$ exists for $n = 1, \cdots, \rho\}$.

Let $\zeta$ denote the Banach space $P\zeta^b_{\mathcal{G}_t}(\mathcal{S}, L_2(\Theta, \mathcal{G}_t, \mathbb{R}^n))$ the family of all bounded $\mathcal{G}_t$-measurable, $P\zeta(\mathcal{S}, \mathbb{R}^n)$-valued random variables $x$, satisfy the condition $\|x\|^2 = \sup\{E\|x(t)\|^2 : t \in \mathcal{S}\}$, where the mathematical expectation operator of stochastic process regarding the given probability measure $P$ is $E(\cdot)$. Let from $\mathbb{R}^k$ to $\mathbb{R}^i$, the space of all linear transformations is $L(\mathbb{R}^k, \mathbb{R}^i)$. Additionally, we suppose that the set of admissible controls $\mathcal{U}_{ad} := L_2^{\mathcal{G}}(\mathcal{S}, \mathbb{R}^i)$.

(a) Riemann-Liouville fractional operators:

$$\begin{aligned} (I^b_{0+}g)(\mu) &= \frac{1}{\Gamma(b)} \int_0^\mu (\mu - t)^{b-1} g(t)dt, \\ (D^b_{0+}g)(\mu) &= D^k(I^{k-b}_{0+}g)(\mu). \end{aligned}$$

(b) CF derivative:

$$({}^C D^b_{0+}g)(\mu) = (I^{k-b}_{0+}D^k g)(\mu).$$

In particular, $I^b_{0+}({}^C D^q_{0+})g(t) = g(t) - g(0)$, where $\quad 0 < b < 1$.

(c) For finite interval $[\alpha, \beta] \in \mathbb{R}_+$ :

$$(D^b_{\alpha+}g)(\mu) = ({}^C D^b_{\alpha+}g)(\mu) + \sum_{n=0}^{k-1} \frac{g^{(n)}(\alpha)}{\Gamma(1+n-b)} (\mu - \alpha)^{n-b},\ k = R(q) + 1.$$

Its Laplace transformation is

$$\mathcal{L}\{{}^{C}D_{0+}^{b}g(t)\}(s) = s^{q}G(s) - \sum_{n=0}^{k-1} g^{n}(0^{+})s^{b-1-n}.$$

(d) ML Function:

$$\mathfrak{M}_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(nb+a)} \quad \text{for } a, b > 0,$$

with

$$\int_{0}^{\infty} e^{-t}t^{a-1}\mathfrak{M}_{b,a}(t^{b}z)dt = \frac{1}{(1-z)}$$

exists for absolute values of z that are less than 1. The Laplace transformation of $\mathfrak{M}_{b,a}(z)$ follows from the integral

$$\int_{0}^{\infty} e^{-st}t^{a-1}\mathfrak{M}_{b,a}(\pm\alpha t^{b})dt = \frac{s^{b-a}}{(s^{b}\mp\alpha)}.$$

$$\text{That is,} \qquad \mathcal{L}\{t^{a-1}\mathfrak{M}_{b,a}(\pm\alpha t^{b})\}(s) = \frac{s^{b-a}}{(s^{b}\mp\alpha)},$$

for $|\alpha|^{\frac{1}{b}} < R(s)$ and $0 < R(a)$. Specifically, for $a = 1$,

$$\mathfrak{M}_{b,1}(\lambda z^{b}) = \mathfrak{M}_{b}(\lambda z^{b}) = \sum_{n=0}^{\infty} \frac{\lambda^{n}z^{nb}}{\Gamma(bn+1)}, \quad \lambda, z \in \mathbb{C}$$

have the compulsive property ${}^{C}D^{b}\mathfrak{M}_{b}(\lambda t^{b}) = \lambda\mathfrak{M}_{b}(\lambda t^{b})$ and $\mathcal{L}\{\mathfrak{M}_{b}(\pm\alpha t^{b})\}(s) = \frac{s^{b-1}}{(s^{b}\mp\alpha)}$, for $b = 1$.

(e) Solution's Presentation:

We consider following problem for $n = 1$ *to* $\rho$,

$$\begin{aligned} {}^{C}D^{b}(\mu(t)) &= A\mu(t) + Bu(t) + \int_{0}^{t}\omega(s)dw(s) + \int_{-\infty}^{\infty}\chi(s,z)\lambda(ds,dz),\ s,t \in \mathcal{S},\ \ t \neq t_{n}, \\ \Delta\mu(t_{n}) &= I_{n}(\mu(t_{n}^{-})), \quad t_{n} = t, \\ \mu(0) &= \mu_{0}, \end{aligned} \tag{1}$$

where $0 < b < 1$, $\mu \in \mathbb{R}^{k}$, $u(t) \in \mathbb{R}^{i}$, $A$ is an $k \times k$ matrix, $B$ is an $k \times i$ matrix and $\omega : \mathcal{S} \longrightarrow \mathbb{R}^{k\times i}$, $h : \mathcal{S} \times \mathbb{R} \longrightarrow \mathbb{R}^{k}$, $I_{n} : \mathcal{S} \longrightarrow \mathbb{R}^{k}$ are appropriate functions. Further $\Delta\mu(t) = \mu(t^{+}) - \mu(t^{-})$, where

$$\lim_{h\longrightarrow 0^{+}} \mu(t+h) = \mu(t^{+}); \qquad \lim_{h\longrightarrow 0^{+}} \mu(t-h) = \mu(t^{-})$$

and $0 = t_{0} < t_{1} < t_{2} <, \cdots, < t_{\rho} < t_{\rho+1} = T$, $I_{n}(\mu(t_{n}^{-})) = (I_{1n}(\mu(t_{n}^{-})), \cdots, I_{kn}(\mu(t_{n}^{-})))^{T}$ shows the impulsive perturbation of $\mu$ at time $t_{n}$ and $\mu(t_{n}) = \mu(t_{n}^{-})$, $n = 1$ *to* $\rho$ . Hence the solution of (1) is left continuous at $t_{n}$.

Let $\{\overline{\lambda}(ds,dz),\ s \in \mathcal{S},\ z \in \mathbb{R}\}$ be a centered Poisson random measure along the parameter $\pi(dz)ds$. Let $\int_{-\infty}^{\infty}\pi(dz) < \infty$ and $\lambda(ds,dz) = \overline{\lambda}(ds,dz) - \pi(dz)ds$ be the compensated Poisson random measure. By applying the Laplace transform, we get a solution for (1)

$$s^{b}\chi(s) - s^{b-1}\mu(0) \;=\; A\chi(s) + \mathcal{L}\{Bu(s)\} + \mathcal{L}\bigg\{\int_{0}^{t}\omega(s)dw(s)\bigg\} + \mathcal{L}\bigg\{\int_{-\infty}^{\infty}\chi(s,z)\lambda(ds,dz)\bigg\}.$$

Consequently, we can write (see [26])

$$\mathcal{L}^{-1}\{\chi(s)\} =$$
$$\mathcal{L}^{-1}\left\{s^{b-1}(s^bI-A)^{-1}\right\}\mu_0 + \mathcal{L}^{-1}\{Bu(s)\} * \mathcal{L}^{-1}\left\{(s^bI-A)^{-1}\right\}$$
$$+\left\{\int_0^t \omega(s)dw(s)\right\} * \mathcal{L}^{-1}\left\{(s^bI-A)^{-1}\right\} + \int_{-\infty}^{\infty} \chi(s,z)\lambda(ds,dz) * \mathcal{L}^{-1}\left\{(s^bI-A)^{-1}\right\}.$$

The stochastic impulsive system for ODE has been studied in Lemma 2.1 (see [27]). Substituting Laplace transform of ML function, we can write as follow

$$\begin{aligned} \mu(t) &= \mathfrak{M}_b(At^b)\mu_0 + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[Bu(s) + \int_0^s \omega(\kappa)dw(\kappa) \\ &\quad + \int_{-\infty}^{\infty} \chi(s,z)\lambda(ds,dz)\Big]ds + \sum_{0<t_n<t} \mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(\mu(t_n^-)), \end{aligned} \tag{2}$$

where $\mathfrak{M}_b(At^b)$ can be represented as:

$$\mathfrak{M}_b(At^b) = \sum_{n=0}^{\infty} \frac{A^n t^{nb}}{\Gamma(nb+1)},$$

with the property that ${}^CD^q\mathfrak{M}_q(At^q) = A\mathfrak{M}_q(At^q)$. The following definition will be helpful to prove our main results.

**Definition 1** (Controllability). *A system (1) is controllable if* $\forall\ \mu_0,\ \mu_1 \in \mathbb{R}^k\ \exists$ *a stochastic control* $u(t) \ni$ *the* $\mathcal{G}_t-$ *adapted processes* $\mu(t)$ *of system (1) that satisfies* $\mu(0) = \mu_0$ *and* $\mu(\vartheta) = \mu_1$.

## 3. Controllability Results

**Theorem 1.** *The system (1) is controllable on* $\mathcal{T}$ *iff the controllability GM*

$$W = \int_0^l (l-s)^{b-1}[\mathfrak{M}_{b,b}(A(l-s)^b)B][\mathfrak{M}_{b,b}(A(l-s)^b)B]^* ds > 0 \tag{3}$$

*for* $l > 0$.

**Proof.** Since $W > 0$, therefore its inverse is well-defined. Define the following $u(\cdot) \in \mathcal{U}_{ad}$

$$\begin{aligned} u(t) = B^*\mathfrak{M}_{b,b}(A^*(l-t)^b)\mathbb{E}\Big\{W^{-1}\Big(\mu_1 - \mathfrak{M}_b(AT^b)\mu_0 - \int_0^l (l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b) \\ \times\left[\int_0^s \omega(\kappa)dw(\kappa) + \int_{-\infty}^{\infty}\chi(s,z)\lambda(ds,dz)\right]ds\Big) - \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(\mu-t_n)^b)I_n(\mu(t_n^-))\Big|\mathcal{G}_t\Big\}. \end{aligned} \tag{4}$$

By substituting $t = l$ in (2) and using (3) one can get the following

$$
\begin{aligned}
&\mu(l)\\
&= \mathfrak{M}_b(Al^b)\mu_0 + \int^T l_0(l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b)BB^*\mathfrak{M}_{b,b}(A^*(l-s)^b)W^{-1}\Big(\mu_1 - \mathfrak{M}_b(Al^b)\mu_0\\
&\quad - \int_0^l (l-T)^{b-1}\mathfrak{M}_{b,b}(A(l-T)^b)\Big[\int_0^T \omega(\kappa)dw(\kappa) + \int_{-\infty}^{\infty}\chi(\xi,z)\lambda(d\xi,dz)\Big]dl\Big)ds\\
&\quad - \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(l-t_n)^b)I_k(\mu(t_n^-)) + \int_0^l (l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b)\Big[\int_0^s \omega(\kappa)dw(\kappa)\\
&\quad + \int_{-\infty}^{\infty}\chi(\xi,z)\lambda(d\xi,dz)\Big]ds + \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(l-t_n)^b)I_n(\mu(t_n^-))\\
&= WW^{-1}\mu_1 = \mu_1.
\end{aligned}
$$

Thus the dynamical system (1) is controllable.
If it is non positive definite, then $\exists$ a non zero $\phi \ni \phi^* W\phi = 0$, i.e.,

$$
\begin{aligned}
\phi^*\int_0^l (-s)^{b-1}\Big[\mathfrak{M}_{b,b}(A(l-s)^b)B\Big]\Big[\mathfrak{M}_{b,b}(A(l-s)^b)B\Big]^*\phi ds = 0,\\
\phi^*\Big[(l-s)^{b-1}\Big(\mathfrak{M}_{b,b}(A(l-s)^b)B\Big)\Big] = 0.
\end{aligned}
$$

Let $\mu_0 = [\mathfrak{M}_b(Al^b)]^{-1}\phi$. By assumption, $\exists$ a control $u \ni$ it steers $\mu_0$ to 0.

$$
\begin{aligned}
\mu(l) = 0 &= \mathfrak{M}_b(Al^b)\mu_0 + \int_0^l (l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b)Bu(s)ds.\\
0 &= \phi + \int_0^l (l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b)Bu(s)ds.
\end{aligned}
$$

Then $0 = \phi^*\phi + \phi^*\int_0^l (l-s)^{b-1}\mathfrak{M}_{b,b}(A(l-s)^b)Bu(s)ds$,
which implies that $0 = \phi^*\phi + 0$. That is, $\phi = 0$, contradicting the supposition that $\phi \neq 0$.
Therefore, $W$ is positive definite and it is proved. □

Consider the following proposed systems,

$$
\begin{aligned}
{}^C D^b[\mu(t) - g(t,\mu(t))] &= A\mu(t) + B\mu(t) + \psi(t,\mu(t)) + \int_0^t \omega(s,\mu(s))dw(s)\\
&\quad + \int_{-\infty}^{\infty}\chi(t,\mu(t),z)\lambda(dt,dz), \quad t \neq t_n, \quad t \in \mathcal{T},\\
\Delta\mu(t_n) &= I_n(t_n, n(t_n^-)), \quad t = t_n, \quad n = 1,2,\cdots,\rho, \\
\mu(0) &= \mu_0,
\end{aligned} \tag{5}
$$

where $b : \mathcal{T} \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is appropriate continuous function.

The solution of (5) can be written as,

$$
\begin{aligned}
\mu(t) &= \mathfrak{M}_b(At^b)(\mu_0 - g(0,\mu_0)) + g(t,\mu(t)) + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[Bu(s)\\
&\quad + Ag(s,\mu(s)) + \psi(s,\mu(s)) + \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa) + \int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds\\
&\quad + \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(t_n,\mu(t_n^-)).
\end{aligned} \tag{6}
$$

Introducing the following notation

$$\begin{aligned}\gamma(\mu_0,\mu_1:\mu) &= \mu_1 - \mathfrak{M}_b(A\vartheta^b)(\mu_0 - g(0,\mu_0)) - g(\vartheta,\mu(\vartheta)) - \int_0^{\vartheta}(\vartheta-s)^{b-1}\mathfrak{M}_{b,b}(A(\vartheta-s)^b)\\ &\times\Big[Ag(s,\mu(s)) + \psi(s,\mu(s)) + \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa)\\ &+\int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds - \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(\vartheta-t_n)^b)I_n(t_n,\mu(t_n^-)).\end{aligned} \tag{7}$$

Using (7) the controllability GM (3) and the control function (4), we have

$$u(t) = B^*\mathfrak{M}_{b,b}(A^*(\vartheta-t)^b)\mathbb{E}\Big\{W^{-1}\gamma(\mu_0,\mu_1:\mu)\Big|\mathcal{G}_t\Big\}, \tag{8}$$

where $\mu_0,\ \mu_1 \in \mathbb{R}^n$ are chosen arbitrarily and $*$ denotes the matrix transpose. Assume the following hypothesis hold:

**Hypothesis 1 (H1).** *The linear operators generated by $A$ are compact such that* $\max_t\Big\{\big\|\mathfrak{M}_q(At^q)\big\|^2\Big\} \leq S_1,\ \max_t\Big\{\Big\|\mathfrak{M}_{b,b}(A(t-s)^b)\Big\|^2\Big\} \leq S_2,\ \max_t\Big\{\Big\|\mathfrak{M}_{b,b}(A(t-t_n)^b)\Big\|^2\Big\} \leq S_3,$ *where $t\in\mathcal{T}$ and $S_1,\ S_2,\ S_3$ are constants.*

**Hypothesis 2 (H2).** *The Grammian matrices $W > 0$ and thus $W^{-1}$ is bounded. i.e., $\exists$ a $S_4 > \ni \|W^{-1}\| \leq S_4$ and $\|A\| \leq l_1$.*

**Hypothesis 3 (H3).** *The $\psi, g, \omega, I_n$ and $h$ satisfy the following:*

*(a)* $\|g(t,\mu)\|^2 \leq \overline{U}(1+\|\mu\|^2)$
*(b)* $\|\psi(t,\mu)\|^2 \leq \overline{V}(1+\|\mu\|^2)$
*(c)* $\|\omega(t,\mu)\|^2 \leq \overline{W}(1+\|\mu\|^2)$
*(d)* $\int_{-\infty}^{\infty}\|\chi(t,\mu,z)\|^2\lambda(dz) \leq \overline{Z}(1+\|\mu\|^2)$
*(e)* $\|I_n(t,\mu)\|^2 \leq \overline{\alpha}_n(1+\|\mu\|^2)$

*where $\overline{U},\ \overline{V},\ \overline{W},\ \overline{Z} > 0$ and $\overline{\alpha}_n > 0$.*

**Hypothesis 4 (H4).** *[(**H4**)] $\exists\ U,\ V,\ W,\ Z > 0$ and $\alpha_N > \ni$*

*(a)* $\|g(t,\mu) - g(t,\eta)\|^2 \leq U\|\mu-\eta\|^2$
*(b)* $\|\psi(t,\mu) - \psi(t,\eta)\|^2 \leq V\|\mu-\eta\|^2$
*(c)* $\|\omega(t,\mu) - \omega(t,\eta)\|^2 \leq W\|\mu-\eta\|^2$
*(d)* $\int_{-\infty}^{\infty}\|\chi(t,\mu(t),z) - \chi(t,\eta(t),z)\|^2\lambda(dz) \leq Z\|\mu-\eta\|^2$
*(e)* $\|I_n(t,\mu) - I_\mu(t,\eta)\|^2 \leq \alpha_n\|\mu-\eta\|^2$

**Theorem 2.** *Under the conditions (**H1**)–(**H4**), the controllability of the system (5) on $\mathcal{T}$ is provided that the following Equation (9) holds.*

$$\begin{aligned}&7\Big(U(1+6S_2^2S_4) + \frac{\vartheta^{2b}}{b^2}S_2\Big(1+6S_2^2S_4\Big)(l_1U+V+W+Z)\\ &+(1+6S_2^2S_4)S_3\sum_{0<t_n<t}\alpha_n\Big) \times \sup_{t\in\mathcal{T}}\mathbb{E}\|\mu(s)-\eta(s)\|^2 < 1.\end{aligned} \tag{9}$$

**Proof.** Under the stochastic control function $u(t)$ defined as (8), we can prove the existence of controllability results.
Define the operator $\Phi : \mathcal{C} \longrightarrow \mathcal{C}$ by $(\Phi\mu)(t)$

$$= \mathfrak{M}_b(At^b)(\mu_0 - g(0,\mu_0)) + g(t,\mu(t)) + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)BB^*\mathfrak{M}_{b,b}(A^*(\vartheta-s)^b)$$
$$\times \mathbf{W}^{-1}[\gamma(\mu_0,\mu_1:\mu)]ds + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[Ag(s,\mu(s)) + \psi(s,\mu(s))$$
$$+ \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa) + \int_{-\infty}^{\infty} \chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds + \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(t_n,\mu(t_n^-)).$$

To prove the system (5) is controllable on $\mathcal{T}$, it is sufficient to prove that $\Phi$ has a fixed point in $\mathcal{C}$.

**Step 1.** $\Phi$ maps into $\mathcal{C}$ into itself

$$\begin{aligned}
&\mathbb{E}\|(\Phi\mu)(t)\|^2 \\
&\leq 8\Big(2S_1\Big[\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big] + \mathbb{E}\|g(t,\mu(t))\|^2 + S_2^2S_4\mathbb{E}\Big\|\int_0^t \gamma(\mu_0,\mu_1:\mu)\Big\|^2 ds \\
&\quad + \frac{\vartheta^{2b}}{b^2}S_2\int_0^t \Big[l_1\mathbb{E}\|g(s,\mu(s))\|^2 + \mathbb{E}\|\psi(s,\mu(s))\|^2 + \mathbb{E}\Big\|\int_0^s \omega(\kappa,\mu(\kappa)dw(\kappa))\Big\|^2 \\
&\quad + \mathbb{E}\Big\|\int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big\|^2\Big]ds + S_3\mathbb{E}\Big\|\sum_{0<t_n<t} I_n(t,\mu)\Big\|^2\Big).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\Big\|\int_0^t \gamma(\mu_0,\mu_1:\mu)\Big\|^2 ds \\
&\leq 8\Big(\|\mu_1\|^2 + 2S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big) + \mathbb{E}\|g(\vartheta,\mu(\vartheta))\|^2 + \frac{\vartheta^{2b}}{b^2}S_2\int_0^\vartheta \Big[l_1\mathbb{E}\|g(s,\mu(s))\|^2 \\
&\quad + \mathbb{E}\|\psi(s,\mu(s))\|^2 + \mathbb{E}\Big\|\int_0^s \omega(\kappa,\mu(\kappa)dw(\kappa))\Big\|^2 + \mathbb{E}\Big\|\int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big\|^2\Big]ds \\
&\quad + S_3\mathbb{E}\Big\|\sum_{0<t_n<t} I_n(t,\mu)\Big\|^2\Big) \\
&\leq 8\Big(\|\mu_1\|^2 + 2S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big) + \overline{U}\Big(1+\mathbb{E}\|\mu\|^2\Big) + \frac{\vartheta^{2b}}{b^2}S_2\Big[l_1\overline{U} + \overline{V} + \overline{W} + \overline{Z} \\
&\quad + S_3\sum_{0<t_n<t}\overline{\alpha}_n\Big]\int_0^\vartheta\Big(1+\mathbb{E}\|\mu(s)\|^2\Big)ds\Big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}\|(\Phi\mu)(t)\|^2 \\
&\leq 16S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big) + 8\overline{U}\Big(1+\mathbb{E}\|\mu\|^2\Big) + 64S_2^2S_4\Big(\|\mu_1\|^2 + 2S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big) \\
&\quad + \overline{U}\Big(1+\mathbb{E}\|\mu\|^2\Big) + \Big[\frac{\vartheta^{2b}}{b^2}S_2(l_1\overline{U} + \overline{V} + \overline{W} + \overline{Z}) + S_3\sum_{0<t_n<t}\overline{\alpha}_n\Big]\int_0^\vartheta\Big(1+\mathbb{E}\|\mu(s)\|^2\Big)ds\Big) \\
&\quad + 8\Big[\frac{\mu^{2b}}{b^2}S_2(l_1\overline{U} + \overline{V} + \overline{W} + \overline{Z}) + S_3\sum_{0<t_n<t}\overline{\alpha}_n\Big]\int_0^\vartheta\Big(1+\mathbb{E}\|\mu(s)\|^2\Big)ds \\
&\leq 16S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big) + 64S_2^2S_4\Big[\|\mu_1\|^2 + 2S_1\Big(\|\mu_0\|^2 + \|g(0,\mu_0)\|^2\Big)\Big] \\
&\quad + 8\Big[\overline{U}(1+8S_2^2S_4) + \frac{\vartheta^{2b}}{b^2}S_2\Big(1+8S_2^2S_4\Big)(l_1\overline{U} + \overline{V} + \overline{W} + \overline{Z}) + (1+8S_2^2S_4)S_3\sum_{0<t_n<t}\overline{\alpha}_n\Big] \\
&\quad \times \int_0^\vartheta\Big(1+\mathbb{E}\|\mu(s)\|^2\Big)ds.
\end{aligned}$$

From $(H3)$ and the obtained inequality, there exist $c > 0$ such that $\mathbb{E}\|(\Phi\mu)(t)\|^2 \leq c\Big(1 + \int_0^\vartheta (1 + \mathbb{E}\|\mu(s)\|^2)ds\Big)$. So, we have $\mathbb{E}\|(\Phi\mu)(t)\|^2 \leq \bar{c}(1 + \mathbb{E}\|\mu\|^2) = r$ (say) for some $\bar{c} > 0$. Therefore $\Phi$ maps $\mathcal{C}$ into itself.

**Step 2.** To prove $\Phi$ is a contraction mapping on $\mathcal{C}$, for $x, y$ belongs to $\mathcal{C}$

$$\begin{aligned}
&\mathbb{E}\|(\Phi\mu)(t) - (\Phi\eta)(t)\|^2 \\
&= \mathbb{E}\Big\| g(t,\mu(t)) - g(t,\eta(t)) + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[BB^*\mathfrak{M}_{b,b}(A^*(\mu-s)^b)\mathbf{W}^{-1} \\
&\quad \times [\gamma(\mu_0,\mu_1:\mu) - \gamma(\mu_0,\mu_1:\eta)]ds + A[g(s,\mu(s)) - g(s,\eta(s))] + [\psi(s,\mu(s)) - \psi(s,\eta(s))] \\
&\quad + \int_0^s [\omega(\kappa,\mu(\kappa)) - \omega(\kappa,\eta(\kappa))]dw(\kappa) + \int_{-\infty}^{\infty} (\chi(\xi,\mu(\xi),z) - \chi(\xi,\eta(\xi),z))\lambda(d\xi,dz)\Big]ds \\
&\quad + \sum_{0<t_n<t} \mathfrak{M}_{b,b}(A(t-t_n)^q)[I_n(t_n,\mu(t_n^-)) - I_n(t_n,y(t_n^-))]\Big\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 7\bigg(U + \frac{\vartheta^{2b}}{b^2}S_2(l_1U + V + W + Z) + S_3\sum_{0<t_n<t}\alpha_n\bigg)\int_0^\vartheta \Big(\mathbb{E}\|\mu(s) - \eta(s)\|^2\Big)ds \\
&\quad + 42S_2^2S_4\bigg(U + \frac{\vartheta^{2b}}{b^2}S_2(l_1U + V + W + Z) + S_3\sum_{0<t_n<t}\alpha_n\bigg)\int_0^\vartheta \Big(\mathbb{E}\|\mu(s) - \eta(s)\|^2\Big)ds \\
&\leq 7\bigg(U(1 + 6S_2^2S_4) + \frac{\vartheta^{2b}}{b^2}S_2\Big(1 + 6S_2^2S_4\Big)(l_1U + V + W + Z) + (1 + 6S_2^2S_4)S_3\sum_{0<t_n<t}\alpha_n\bigg) \\
&\quad \times \int_0^\vartheta \Big(\mathbb{E}\|\mu(s) - \eta(s)\|^2\Big)ds \\
&\sup_{t\in\mathcal{T}} \mathbb{E}\|(\Phi\mu)(t) - (\Phi\eta)(t)\|^2 \\
&\leq 7\bigg(U(1 + 6S_2^2S_4) + \frac{\vartheta^{2b}}{b^2}S_2\Big(1 + 6S_2^2S_4\Big)(l_1U + V + W + Z) + (1 + 6S_2^2S_4)S_3\sum_{0<t_n<t}\alpha_n\bigg) \\
&\quad \times \sup_{t\in\mathcal{T}} \mathbb{E}\|\mu(s) - \eta(s)\|^2.
\end{aligned}$$

Hence $\Phi$ has a unique fixed point $\mu(\cdot) \in \mathcal{C}$ and contraction on $\mathcal{C}$. It is not difficult to check $\mu(\vartheta) = \mu_1$, i.e., the stochastic control function $u(t)$ steers the system (5) from $\mu_0$ to $\mu_1$ on $\mathcal{T}$. Consequently, the system (5) is completely controllable on $\mathcal{T}$.
□

**Note**: According to the above theorem, system (5) is controllable on $\mathcal{T}$ uniquely by using the BFP theorem. By using the SFP theorem, we obtain sufficient conditions for (5) and show that the system is controllable on $\mathcal{T}$.

**Theorem 3.** *If the conditions* **(H1)**–**(H3)** *are fulfilled, then the system (5) is controllable on* $\mathcal{T}$.

**Proof.** Using the hypotheses **(H2)**, we define the operator $(\Psi\mu)(t)$ by

$$\begin{aligned}
(\Psi\mu)(t) &= \mathfrak{M}_b(At^b)(\mu_0 - g(0,x_0)) + g(t,\mu(t)) + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b) \\
&\quad \times \Big[Bu(s) + Ag(s,\mu(s)) + \psi(s,\mu(s)) + \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa) \\
&\quad + \int_{-\infty}^{\infty} \chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds + \sum_{0<t_n<t} \mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(t_n,\mu(t_n^-)),
\end{aligned}$$

and prove it where $u(t)$ is the stochastic control function defined by (8) has a fixed point, $\mathcal{C}_r = \{\mu : \ \mu \in \mathcal{C}, \ \mu(0) = \mu_0, \ \mathbb{E}\|\mu(t)\|^2 \le r \text{ for } t \in \mathcal{T}\}$ where $r$ is the positive constant. From step 1 of Theorem 3, it is evident that $\mathcal{C}_r$ is totally bounded. We define the operator $\Psi : \mathcal{C} \longrightarrow \mathcal{C}_r$ by

$$\begin{aligned}
&(\Psi\mu)(t)\\
&= \mathfrak{M}_b(At^b)(\mu_0 - g(0,\mu_0)) + g(t,\mu(t)) + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)BB^*\mathfrak{M}_{b,b}(A^*(\vartheta-s)^q)\\
&\quad \times \mathbf{W}^{-1}[\gamma(\mu_0,\mu_1:\mu)]ds + \int_0^t (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[Ag(s,\mu(s)) + \psi(s,\mu(s))\\
&\quad + \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa)) + \int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds + \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(t_n,\mu(t_n^-)).
\end{aligned}$$

Since $f, \ g, \ \omega, \ I_k$ and $h$ are continuous functions and $\mathbb{E}\|(\Psi\mu)(t)\|^2 \le r$ it follows that $\Psi$ maps from $\mathcal{C}_r$ to itself is also continuous. Furthermore, $\Psi$ maps into a precompact subset of $\mathcal{C}_r$. To show this, $\forall \ t \in \mathcal{S}$, the set $\mathcal{C}_r(t) = \{(\Psi\mu)(t) : \mu \in \mathcal{C}_r\}$ is a precompact in $\mathbb{R}^k$. It is clear for $t = 0$, because $\mathcal{C}_r(0) = \{\mu_0\}$. Let $t > 0$ be fixed and for $\epsilon \in (0,t)$ define

$$\begin{aligned}
&(\Psi_\epsilon\mu)(t)\\
&= \mathfrak{M}_b(At^b)(\mu_0 - g(0,\mu_0)) + g(t,\mu(t)) + \int_0^{t-\epsilon} (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)BB^*\mathfrak{M}_{b,b}(A^*(\vartheta-s)^b)\\
&\quad \times \mathbf{W}^{-1}[\gamma(\mu_0,\mu_1:\mu)]ds + \int_0^{t-\epsilon} (t-s)^{b-1}\mathfrak{M}_{b,b}(A(t-s)^b)\Big[Ag(s,\mu(s)) + \psi(s,\mu(s))\\
&\quad + \int_0^s \omega(\kappa,\mu(\kappa))dw(\kappa)) + \int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\Big]ds + \sum_{0<t_n<t}\mathfrak{M}_{b,b}(A(t-t_n)^b)I_n(t_n,\mu(t_n^-)).
\end{aligned}$$

Since $\mathfrak{M}_b(At^b)$, $\mathfrak{M}_{b,b}(A(t-s)^b)$, $\mathfrak{M}_{b,b}(A(t-t_n)^b)$, for $t \in \mathcal{T}$ are compact $\forall \ t > 0$, the set $\mathcal{C}_\epsilon(t) = \{(\Psi_\epsilon\mu)(t) : \mu \in \mathcal{C}_r\}$ is precompact in $\mathbb{R}^k$ $\forall \ \epsilon, \ \epsilon \in (0,t)$. Moreover, for $\mu \in \mathcal{C}_r$, we get

$$\begin{aligned}
&\mathbb{E}\|(\Psi\mu)(t) - (\Psi_\epsilon\mu)(t)\|^2\\
&\le 5\int_{t-\epsilon}^t (t-s)^{b-1}\|\mathfrak{M}_{b,b}(A(t-s)^b)\|^2\|B\|^2\|B^*\|^2\|\mathfrak{M}_{b,b}(A^*(\vartheta-s)^q)\|^2\|\mathbf{W}^{-1}\|^2\mathbb{E}\|\gamma(\mu_0,\mu_1:\mu)\|^2 ds\\
&\quad + 5\int_{t-\epsilon}^t (t-s)^{b-1}\|\mathfrak{M}_{b,b}(A(t-s)^b)\|^2\Big[\|A\|\mathbb{E}\|g(s,\mu(s))\|^2 + \mathbb{E}\|\psi(s,\mu(s))\|^2\\
&\quad + \int_0^s \mathbb{E}\|\omega(\kappa,\mu(\kappa))\|^2 dw(\kappa)) + \int_{-\infty}^{\infty}\mathbb{E}\|\chi(\xi,\mu(\xi),z)\|^2\lambda(d\xi,dz)\Big]ds\\
&\le \epsilon r := r,
\end{aligned}$$

$\Longrightarrow$ $\mathcal{C}_r(t)$ is bounded i.e., pre-compact in $\mathbb{R}^k$. Now we show that $\Psi(\mathcal{C}_r) = Y = \{\Psi\mu : \mu \in \mathcal{C}_r\}$ is an equicontinuous family of function.

For that let us take $p_1, p_2 \in \mathcal{T}$ with $p_2 > p_1$, and for all $\mu \in \mathcal{C}_r$, then we have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi\mu)(p_2)-(\Psi\mu)(p_1)\|^2\\
&=\|\mathfrak{M}_b(At_2^b)(\mu_0-g(0,\mu_0))-\mathfrak{M}_b(At_1^b)(\mu_0-g(0,\mu_0))+g(p_2,\mu(p_2))-g(p_1,\mu(p_1))\\
&\quad+\int_0^{p_2}(p_2-s)^{b-1}\mathfrak{M}_{b,b}(A(p_2-s)^b)Bu(s)ds-\int_0^{p_1}(p_1-s)^{b-1}\mathfrak{M}_{b,b}(A(p_1-s)^b)Bu(s)ds\\
&\quad+\int_0^{p_2}(p_2-s)^{b-1}\mathfrak{M}_{b,b}(A(p_2-s)^b)Ag(s,\mu(s))ds-\int_0^{p_1}(p_1-s)^{b-1}\mathfrak{M}_{b,b}(A(p_1-s)^b)\\
&\quad\times Ag(s,\mu(s))ds+\int_0^{p_2}(p_2-s)^{b-1}\mathfrak{M}_{b,b}(A(p_2-s)^b)\psi(s,\mu(s))ds-\int_0^{p_1}(p_1-s)^{b-1}\mathfrak{M}_{b,b}(A(p_1-s)^b)\\
&\quad\times\psi(s,\mu(s))ds+\int_0^{p_2}(p_2-s)^{b-1}\mathfrak{M}_{b,b}(A(p_2-s)^b)\left(\int_0^s\omega(\kappa,b(\kappa))dw(\kappa)\right)ds\\
&\quad-\int_0^{p_1}(p_1-s)^{b-1}\mathfrak{M}_{b,b}(A(p_1-s)^b)\left(\int_0^s\omega(\kappa,\mu(\kappa))dw(\kappa)\right)ds+\int_0^{p_2}(p_2-s)^{b-1}\mathfrak{M}_{b,b}(A(p_2-s)^b)\\
&\quad\times\left(\int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\right)ds-\int_0^{p_1}(p_1-s)^{b-1}\mathfrak{M}_{b,b}(A(p_1-s)^b)\\
&\quad\times\left(\int_{-\infty}^{\infty}\chi(\xi,\mu(\xi),z)\lambda(d\xi,dz)\right)ds+\sum_{p_1\le t_n<p_2}\mathfrak{M}_{b,b}(A(p_2-t_n)^b)I_n(t_n,\mu(t_n^-))\\
&\qquad\qquad\qquad-\sum_{p_1\le t_n<p_2}\mathfrak{M}_{b,b}(A(p_1-t_n)^b)I_n(t_n,\mu(t_n^-))\|^2\\
&\qquad\qquad\qquad\le 28\Big[\mathfrak{M}_b(At_2^b)-\mathfrak{M}_b(At_1^b)\Big]\Big(\|\mu_0\|^2+\|g(0,\mu_0))\|^2\Big)+14\mathbb{E}\|g(p_2,\mu(p_2))-g(p_1,\mu(p_1))\|^2\\
&\qquad\qquad\qquad+14\int_{p_1}^{p_2}(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2\|B\|^2\mathbb{E}\|u(s)\|^2ds+14\int_0^{p_1}\Big[(p_2-s)^{2b-2}\\
&\qquad\qquad\qquad\times\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2-(p_1-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_1-s)^b)\|^2\Big]\|B\|^2\mathbb{E}\|u(s)\|^2ds\\
&\qquad\qquad\qquad+14\int_{p_1}^{p_2}(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2 l_1\overline{U}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds\\
&\qquad\qquad\qquad+14\int_0^{p_1}\Big[(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2-(p_1-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_1-s)^b)\|^2\Big]\\
&\qquad\qquad\qquad\times l_1\overline{U}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds+14\int_{p_1}^{p_2}(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2\overline{V}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds\\
&\qquad\qquad\qquad+14\int_0^{p_1}\Big[(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2-(p_1-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_1-s)^b)\|^2\Big]\\
&\qquad\qquad\qquad\times\overline{V}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds+14\int_{p_1}^{p_2}(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2\overline{W}\int_0^{\vartheta}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds\\
&\qquad\qquad\qquad+14\int_0^{p_1}\Big[(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2-(p_1-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_1-s)^b)\|^2\Big]\\
&\qquad\qquad\qquad\times\overline{W}\int_0^{\vartheta}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds+14\int_{p_1}^{p_2}(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2\overline{Z}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds\\
&\qquad\qquad\qquad+14\int_0^{p_1}\Big[(p_2-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_2-s)^b)\|^2-(p_1-s)^{2b-2}\|\mathfrak{M}_{b,b}(A(p_1-s)^b)\|^2\Big]\\
&\qquad\qquad\qquad\times\overline{Z}\Big(1+\mathbb{E}\|\mu\|^2\Big)ds+14\sum_{p_1\le t_n<p_2}\|\mathfrak{M}_{b,b}(A(p_2-t_n)^b)\|^2\overline{\alpha}_n\Big(1+\mathbb{E}\|\mu\|^2\Big)\\
&\qquad\qquad\qquad+14\sum_{0<t_n<p_1}\Big[\|\mathfrak{M}_{b,b}(A(p_2-t_n))^{\mu}\|^2-\|\mathfrak{M}_{b,b}(A(p_1-t_n))^b\|^2\Big]\overline{\alpha}_n\Big(1+\mathbb{E}\|\mu\|^2\Big).
\end{aligned}
$$

The right-hand side of the equation does not depend on $x \in \mathcal{C}_r$ and approaches zero as $(p_2-p_1) \longrightarrow 0$ as a result of the continuity of $\mathfrak{M}_b(At^b)$, $\mathfrak{M}_{b,b}(A(t-s)^b)$, $\mathfrak{M}_{b,b}(A(t-t_n)^b)$ for $t>0$ in the uniform operator topology which in turn follows from the compactness of $\mathfrak{M}_b(At^b)$, $\mathfrak{M}_{b,b}(A(t-s)^b)$ and $\mathfrak{M}_{b,b}(A(t-t_n)^b)$, $t>0$, so $\Psi(\mathcal{C}_r)$ is the family of continuous functions. $\Upsilon$ is bounded in $\mathcal{C}$, by the Arzela-Ascoli's theorem $\Upsilon = \Psi(\mathcal{C}_r)$ is precompact. Therefore, from the SFP theorem, $\Psi$ has a fixed point. Thus, $\mu(t)$ is solution of (5). It is simple to validate that $\mu(\vartheta) = \mu_1$. Hence, (5) is controllable on $\mathcal{T}$. □

## 4. Examples

Now, we apply the obtained results for the following stochastic fractional DS.

**Example 1.** *We take the non-linear system, denoted by the scalar FDE as follows*

$$\begin{aligned} {}^{C}D^{1/2}[\mu(t)-\sin\mu(t)] &= 3\mu(t)+4u(t)+t\cos\mu(t)+(1+5t^2)\mu(t)e^{-t}dw(t)+\frac{1}{4}\mu(t), \\ & \quad t\neq t_n, \quad t\in[0,\vartheta], \\ \Delta\mu(t_n) &= 0.5e^{-0.1n}\mu(t_n^-), \quad t=t_n, \quad n=1,2,\cdots,\rho, \\ \mu(0) &= \mu_0, \end{aligned} \tag{10}$$

*where* $t_n = t_{n-1} + 0.5n$ *for* $n = 1, 2, \cdots, \rho$. *Here we have* $A = 3$, $B = 4$, $q = \frac{1}{2}$, $\vartheta = 1$, $\psi(t,\mu(t)) = t\cos\mu(t)$, $g(t,\mu(t)) = \sin\mu(t), \omega(t,\mu(t)) = (1+5t^2)\mu(t)e^{-t}$, $\Delta\mu(t_n) = 0.5e^{-0.1n}\mu(t_n^-)$ *and* $\chi(t,\mu(t),z) = \frac{1}{4}\mu(t)$.

*Provided the ML matrix function as*

$$\mathfrak{M}_b(At^b) = \sum_{n=0}^{\infty}\frac{3^n t^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

*Further,*

$$\mathfrak{M}_{b,b}(A(\vartheta-s)^b) = \sum_{n=0}^{\infty}\frac{3^n(1-s)^{\frac{n}{2}}}{\Gamma(n+1)/2}.$$

*After calculation, the following controllability Gramian is obtained:*

$$\begin{aligned} \boldsymbol{W} &= \int_0^1 (1-s)^{1/2}\left(\sum_{n=0}^{\infty}\frac{3^n(1-s)^{n/2}}{\Gamma(n+1)/2}(4)\right)\left(\sum_{i=0}^{\infty}\frac{3^i(1-s)^{i/2}}{\Gamma(i+1)/2}(4)\right)ds \\ &= 32\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{3^{n+i}}{(n+i+1)\Gamma(n+1)\Gamma(i+1)} \\ &> 0, \end{aligned}$$

*which is positive definite.*

*Further* $\psi(t,\mu(t))$, $g(t,\mu(t))$, $\omega(t,\mu(t))$, $\Delta\mu(t_n)$ *and* $\chi(t,\mu(t),z)$ *satisfy the hypothesis (**H3**) and so the fractional systems (10) is controllable on* $[0,\vartheta]$. *This example shows that the nonlinear fractional stochastic dynamical system with Poisson jumps given by example 1 is controllable on* $[0,\vartheta]$ *provided the conditions are satisfied in (**H3**).*

**Example 2.** *We consider the following nonlinear fractional stochastic dynamical system with Poisson jumps*

$$\begin{aligned} {}^{C}D^{b}[\mu_1(t)-\cos\mu_1(t)] &= \mu_2(t)+\mu_1(t)+\frac{5\mu_1}{1+\mu_1^2(t)+\mu_2^2(t)}+(1+5t^2)\mu_1(t)e^{-t}dw(t) \\ & \quad +(2t^2+1)\mu_2(t)e^{-t} \\ {}^{C}D^{q}[\mu_2(t)-\frac{1}{2}\mu_2(t)] &= -\mu_1(t)+\mu_2(t)+\frac{\mu_2(t)}{1+\mu_2^2(t)}+\mu_2(t)dw(t)-\mu_1(t)e^{-t} \\ \begin{pmatrix}\Delta\mu_1(t_n)\\ \Delta\mu_2(t_n)\end{pmatrix} &= e^{-0.1n}\begin{pmatrix}0.6 & 0.4\\ -0.2 & 0.5\end{pmatrix}\begin{pmatrix}\mu_1(t_n)\\ \mu_2(t_n)\end{pmatrix}, t=t_n,\ n=1,2,\cdots,\rho \\ \mu(0) &= 0 \end{aligned}$$

*where $t \in \mathcal{T}$, $0 < b < 1$, $t_n = t_{n-1} + 0.2$ . We can write the above Equation (5) with $\mu(t) = (\mu_1(t), \mu_2(t)) \in \mathbb{R}^2$, $t_0 = 0$,*

$$g(t,\mu(t)) = \begin{pmatrix} \cos\mu_1(t) \\ \frac{1}{2}\mu_2(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \psi(t,\mu(t)) = \begin{pmatrix} \frac{5\mu_1}{1+\mu_1^2(t)+\mu_2^2(t)} \\ \frac{\mu_2}{1+\mu_2^2(t)} \end{pmatrix},$$

$$\omega(t,\mu(t)) = \begin{pmatrix} (1+5t^2)\mu_1(t)e^{-t} & 0 \\ 0 & \mu_2(t) \end{pmatrix}, \quad \Delta\mu(t_n) = e^{-0.1n}\begin{pmatrix} 0.6 & 0.4 \\ -0.2 & 0.5 \end{pmatrix}\mu(t_n^-)$$

$$\text{and} \quad \chi(t,\mu(t),z) = \begin{pmatrix} 0 & (2t^2+1)\mu_2(t)e^{-t} \\ -\mu_1(t)e^{-t} & 0 \end{pmatrix}.$$

*The following ML matrix function of the systems is given by (see [28])*

$$\mathfrak{M}_b(At^b) = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2nb}}{\Gamma[1+2nb]} & \sum_{n=0}^{\infty} \frac{(-1)^n t^{(2n+1)b}}{\Gamma[1+(2n+1)b]} \\ -\sum_{n=0}^{\infty} \frac{(-1)^n t^{(2k+1)b}}{\Gamma[1+(2n+1)b]} & \sum_{n=0}^{\infty} \frac{(-1)^n t^{2nb}}{\Gamma[1+(2nb)]} \end{pmatrix}.$$

*By the controllability matrix*

$$\boldsymbol{W} = \int_0^{\vartheta} (\vartheta - s)^{b-1} \begin{pmatrix} R_1^2 + R_2^2 & R_1R_3 + R_2R_4 \\ R_1R_3 + R_2R_4 & R_1^2 + R_2^2 \end{pmatrix},$$

*where*

$$R_1 = R_4 = \sum_{n=0}^{\infty} \frac{(-1)^n(\vartheta - s)^{2nb}}{\Gamma[(2n+1)q]}, \quad R_2 = -R_3 = \sum_{n=0}^{\infty} \frac{(-1)^n(\vartheta - s)^{(2n+1)b}}{\Gamma[(n+1)2b]},$$

*is positive definite.*

*Further $\psi(t,\mu(t))$, $g(t,\mu(t))$, $\omega(t,\mu(t))$, $\Delta\mu(t_n)$ and $\chi(t,\mu(t),z)$ satisfy the hypothesis (**H3**), so example 2 is controllable on $[0,\vartheta]$. This example shows that the nonlinear fractional stochastic dynamical system with Poisson jumps given by example 2 is controllable on $[0,\vartheta]$ provided the conditions are satisfied in (**H3**).*

## 5. Conclusions

Controllability is one of the most fundamental properties of a dynamical system. In systems characterized by partial/ordinary differential equations in FD spaces as well as InFD spaces, controllability is important. In this paper, the controllability of fractional neutral stochastic impulsive dynamical systems is investigated by considering Poisson jumps in finite-dimensional space. The BFP theorem and the SFP theorem have been used to find sufficient criteria for controllability results. ML matrix function defines the controllability Grammian matrix. To demonstrate the efficacy of the projected findings, an example has been provided.

## 6. Future Recommendation

In this research, we examine the fractional stochastic systems' controllability. We investigate whether the fractional impulsive neutral stochastic system is controlled using Poisson jumps in finite-dimensional space. Poisson jumps have grown in popularity, and they are now used to describe a wide variety of phenomena. Many real-world systems such as market crashes, earthquakes, and epidemics can sometimes experience some jump-type stochastic perturbations. Consequently, stochastic processes with jumps are well-matched to modeling such models. In the future, the same approach can be used for other kinds of noises or disturbances in dynamic systems. Moreover, stochastic equations can be considered as well.

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## Abbreviations

The following abbreviations are used in this manuscript:

| | |
|---|---|
| FSF | fractional stochastic systems |
| M-L | Mittag-Leffler |
| GM | Grammian matrix |
| FD | finite-dimensional |
| InFD | infinite-dimensional |
| CF | Caputo fractional |
| DS | Dynamical systems |
| SFP | Schauder fixed point |
| RL | Riemann-Liouville |
| BFP | Banach Fixed Point |

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