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On the Solutions of Quaternion Difference Equations in Terms of Generalized Fibonacci-Type Numbers

Kübra Gül

Department of Computer Engineering, Faculty of Engineering and Architecture, Kafkas University, Kars 36100, Turkey; kubra.gul@kafkas.edu.tr

Abstract: The aim of this paper is to investigate the solution of the following difference equation $z_{n+1} = (p_n)^{-1}, n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $p_n = a + bz_n + cz_{n-1}z_n$ with the parameters a, b, c and the initial values z_{-1}, z_0 are nonzero quaternions such that their solutions are associated with generalized Fibonacci-type numbers. Furthermore, we deal with the solutions to the following symmetric system of difference equations given by $z_{n+1} = (q_n)^{-1}, w_{n+1} = (r_n)^{-1}, n \in \mathbb{N}_0$ where $q_n = a + bw_n + cz_{n-1}w_n$ and $r_n = a + bz_n + cw_{n-1}z_n$. We provide the solution to the third-order quaternion linear difference equation in terms of the zeros of the characteristic polynomial associated with the linear difference equation.

Keywords: quaternion difference equation; solution of difference equation; recurrence sequence

MSC: 39A05; 11R52; 40A05

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1. Introduction

Difference equations and systems of difference equations have been presented in the mathematical modeling of mathematics, as well as in the fields of science and engineering (see [1,2]). The theory of differential and difference equations was comprehensively studied in [3–5]. In recent years, as seen in [6,7], a wide body of literature on the study of solving difference equations has developed. Moreover, the systems of difference equations, especially those that are symmetric and close to symmetric, have been studied by several authors; see, for example, [8–10]. There are several methods for finding general solutions to difference equations or some systems of difference equations in the literature. Many authors have recently dealt with finding closed-form solutions for difference equations and systems of difference equations (see, e.g., [7,11]). Some of the solutions of these equations are representable in terms of well-known integer sequences such as Fibonacci numbers, Lucas numbers, Padovan numbers and Tribonacci numbers (see, e.g., [12–17]).

In [15], Tollu et al. examined the dynamics of the solutions of the two different cases of Riccati difference equations, finding that their solutions are associated with Fibonacci numbers

$$x_{n+1} = \frac{1}{1+x_n}, y_{n+1} = \frac{1}{-1+y_n}, \ n \in \mathbb{N}_0$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions x_0 and y_0 are nonzero real numbers. Stevic, in [14], gave an extension of formulas in [15] for the solutions of difference equations

$$z_{n+1} = \frac{\alpha z_n + \beta}{\gamma z_n + \delta}, n \in \mathbb{N}_0$$

where their solutions were associated with the generalized Fibonacci numbers. The formulas were explained in [15] using some mathematical techniques. Moreover, the authors obtained results for a two-dimensional system of bilinear difference equations. In many papers on difference equations in the literature, such as [13,14,16,18], some nonlinear difference equations and the systems of difference equations have been solved by transforming them using some suitable changes of variables to linear difference equations with constant coefficients.

For example, in [18], the authors presented a representation of the solutions for the following nonlinear second-order difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, n \in \mathbb{N}_0$$

where parameters a, b, and c and the initial values x_{-1} and x_0 are complex numbers such that $c \neq 0$. The difference equation was solved by the change in variables

$$x_n = \frac{y_{n+1}}{y_n},$$

and the third-order linear difference equation was obtained with constant coefficients

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2},$$

for $n \in \mathbb{N}_0$. Furthermore, a special solution was given for the third-order homogeneous linear difference equation by the characteristic polynomial associated with the linear difference equation, and then the solutions to the nonlinear difference equation were examined.

In this study, motivated by the above mentioned papers, we investigate the solutions of the following difference equation

$$z_{n+1} = (p_n)^{-1}, n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad (1)$$

where $p_n = a + bz_n + cz_{n-1}z_n$ and the initial values z_{-1}, z_0 are nonzero quaternions such that their solutions are associated with generalized Fibonacci-type numbers. We also examine the solution form of the following symmetric system of difference equations

$$z_{n+1} = (q_n)^{-1}, w_{n+1} = (r_n)^{-1}, n \in \mathbb{N}_0 \quad (2)$$

for $q_n = a + bw_n + cz_{n-1}w_n$ and $r_n = a + bz_n + cw_{n-1}z_n$. Furthermore, we present the solution to the third-order quaternion linear difference equation in terms of the zeros of the characteristic polynomial associated with the linear difference equation.

Quaternions bear many applications in several areas of science [19–22]. In [21], quaternions were investigated by Hamilton as an extension to the complex numbers. Since quaternions and their particular structure are a form of noncommutative algebra, the quaternion difference equations are quite different from that presented by the classical theory of difference equations. Recently, quaternion differential and difference equations have been used to cover a wide area of interest in modern mathematics and have become a important research topic owing to their comprehensive applications for natural phenomena, for example, see [23,24]. In [25,26], Wang et al. dealt with the general solutions for a class of quaternion matrix equations. In [27], the authors investigated the general solutions of the higher-order linear quaternion difference equations with both variable and constant coefficients. Several methods for finding the general solutions for the higher-order linear quaternion difference equations were given and some examples were presented to illustrate the feasibility of the obtained results. Moreover, by using the quaternion characteristic polynomial, they examined the particular solutions of given quaternion difference equations. The quaternion difference equations have been used for describing discrete dynamic behavior in quaternion space. For more information, see, for example, [28–30]. It is known that there exist many inconsistencies between the elements in the quaternion space and in the real space such as the non-commutativity of the quaternion multiplication. The algebraic structure of the quaternion space brings many difficulties to the study of quaternion difference equations. However, they play an important role in analysing the discrete quaternion dynamical behaviour in quantum mechanics, fluid mechanics, geometry and kinematic modelling, (see [31–33]).

2. Preliminaries

Quaternions form a four-dimensional non-commutative associative algebra. A quaternion is defined by

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where q_0, q_1, q_2, and q_3 are real numbers and i, j, and k are quaternionic units which satisfy the following rules:

$$i^2 = j^2 = k^2 = ijk = -1 \text{ and } ij = k = -ji, jk{=}i = -kj, ki{=}j{=}-ik.$$

Furthermore, the quaternion q can be written as follows:

$$q = q_0 + u = q_0 + q_1 i + q_2 j + q_3 k$$

where $u = q_1 i + q_2 j + q_3 k$. q_0 denotes the scalar part of the quaternion q and u represent the vector part of the quaternion q. The conjugate of the quaternion q is denoted by $\overline{q}$ and $\overline{q} = q_0 - u$. For the quaternions $q, p \in \mathbb{H}$, from [22], the following properties hold:

$$\overline{q+p} = \overline{q} + \overline{p}, \overline{qp} = \overline{p}\overline{q}.$$

The norm of the quaternion q is defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

For the quaternions $q, p \in \mathbb{H}$, we can write

$$|pq| = |p||q|.$$

From [22], if two quaternions $q, q' \in \mathbb{H}$ are congruent, then for some quaternion $w \neq 0$, we have $q' = wqw^{-1}$, written $q' \sim q$. For $q \in \mathbb{H}$, denote the set $[q] = \{q' \in \mathbb{H} : q' \sim q\}$. The product of two quaternions q, p is given by the product rules of the quaternionic units and the distributive law as follows:

$$\begin{aligned} qp \;=\; & q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)i \\ & +(q_0p_2 + q_2p_0 + q_3p_1 - q_1p_3)j + (q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1)k. \end{aligned}$$

The multiplicative inverse of q is given by

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

The inverse operation satisfies the properties $qq^{-1} = q^{-1}q = 1$, $(q^{-1})^{-1} = q$ and $(pq)^{-1} = q^{-1}p^{-1}$.

The quaternions q and p can be divided into two different cases (when p is nonzero). The quotient $\frac{q}{p}$ of these quaternions can be either $p^{-1}q$ or qp^{-1}. For more details on quaternions, we refer the reader to [21].

By $\mathbb{H}[r]$, we denote the polynomial ring with respect to $\mathbb{H}$, i.e., the set of all polynomials

$$P(r) = a_m r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0,\ a_m \neq 0, m \geq 0,$$

where $a_l \in \mathbb{H}, l = 1, 2, \ldots, m$. Then, the quaternion conjugate of $P(r)$ is denoted by

$$\overline{P}(r) = \overline{a}_m r^m + \overline{a}_{m-1} r^{m-1} + \cdots + \overline{a}_1 r + \overline{a}_0,\ \overline{a}_m \neq 0, m \geq 0,$$

where $\bar{a}_l \in \mathbb{H}, l = 1, 2, \ldots, m$.

Lemma 1 ([34])**.** *Consider the quaternion polynomial* $P(r) = \sum_{u=0}^{m} a_u r^u$ *and its conjugate* $\overline{P}(r) = \sum_{l=0}^{m} \bar{a}_l r^l$. *Define*

$$F(r) = P(r)\overline{P}(r) = \sum_{w=0}^{2m} \sum_{u+l=w} a_u \bar{a}_l r^w.$$

Then, $F(r)$ *has real coefficients and the coefficient of* r^w *satisfies*

$$\sum_{u+l=w} a_u \bar{a}_l = \sum_{u+l=w} a_l \bar{a}_u = \sum_{u+l=w} \overline{a_u \bar{a}_l}.$$

Let $v \in \mathbb{C}$ *be a root of* $F(r)$*. Then,* $P(r)$ *has a root in* $[v]$*. Furthermore, if* $q \in \mathbb{H}$ *is a root of* $P(r)$*, then there exists a complex number* $v \in [q]$*, which is a root of* $F(r)$*. In particular, we have the following cases:*

(i) Suppose $v \in \mathbb{C}$ *is a root of* $F(r)$*. If* v *is not a root of* $\overline{P}(r)$*, then* $\overline{P}(v)v\overline{P}(v)^{-1}$ *is a root of* $P(r)$*.*
(ii) Suppose $v \in \mathbb{C}$ *is a root of* $F(r)$*. If* v *is a root of* $\overline{P}(r)$ *and* $\bar{v}$ *is not a root of* $\overline{P}(r)$*, then* $\overline{P}(\bar{v})\bar{v}\overline{P}(\bar{v})^{-1}$ *is a root of* $P(r)$*.*
(iii) Suppose $v \in \mathbb{C}$ *is a root of* $F(r)$*. If* v *and* $\bar{v}$ *are roots of* $\overline{P}(r)$*, then* v *and* $\bar{v}$ *are also roots of* $P(r)$*. In particular, if* $v \in \mathbb{R}$*, then* $P(v) = 0$*.*
(iv) Suppose $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$ *is a root of* $P(r)$*. Then,* $v = q_0 + i\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ *is a root of* $F(r)$*.*

Consider m-order quaternion linear difference equations, given as follows:

$$x(n+m) + a_{m-1}x(n+m-1) + \cdots + a_1 x(n+1) + a_0 x(n) = f(n), n \in \mathbb{Z}, m \in \mathbb{N}, \quad (3)$$

where $a_{m-1}, \ldots, a_1, a_0 \in \mathbb{H}$, $f : \mathbb{Z} \to \mathbb{H}$ is a quaternion-valued function. If $f(n) = 0$ for all $n \in \mathbb{Z}$, the m-order quaternion homogeneous linear difference equation is

$$x(n+m) + a_{m-1}x(n+m-1) + \cdots + a_1 x(n+1) + a_0 x(n) = 0, n \in \mathbb{Z}, m \in \mathbb{N}. \quad (4)$$

The form of the general solution of (4) is presented by the following definition and theorem.

Theorem 1 (Superposition principle [27])**.** *Let* $x_1(n), x_2(n), \ldots, x_m(n)$ *be right linearly independent solutions of (4), then for any* $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_m \in \mathbb{H}$*,*

$$x(n) = x_1(n)\tilde{c}_1 + x_2(n)\tilde{c}_2 + \cdots + x_m(n)\tilde{c}_m$$

is also the solution of (4).

Definition 1 ([27])**.** *Let* $\{x_1(n), x_2(n), \ldots, x_m(n)\}$ *be a fundamental set of solutions of (4). Then,* $x_t(n) = \sum_{i=1}^{m} x_i(n)\tilde{c}_i$ *is called the general solution of (4), where* $\tilde{c}_i \in \mathbb{H}$ *for* $l = 1, 2, \ldots, m$*.*

In [27], the method was established to solve the higher-order quaternion linear difference equations with both constant and variable coefficients through adopting the quaternion characteristic polynomial. For this method, the results of [34] were employed.

Lemma 2 ([19])**.** *Let* $P(r) = a_m r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0$ *be a quaternion polynomial of degree k, where the coefficients* $a_m, a_{m-1}, \ldots, a_0 \in \mathbb{H}$*. Then, there exist integers* $K \geq 0$ *and* $L \geq 0$*,*

$\lambda_1, \lambda_2, \ldots, \lambda_K \in \mathbb{H}, \lambda_l \neq \lambda_u, l \neq u$, and $l, u = 1, 2, \ldots, K$, non-negative integers $m_1, m_2, \ldots, m_K$. $\tau_1, \tau_2, \ldots, \tau_L \in \mathbb{H}, \tau_l \neq \tau_u, l \neq u$ and $l, u = 1, 2, \ldots, L$, and non-negative integers $l_1, l_2, \ldots, l_L$, and there exists a quaternion polynomial $P_0(r)$ of degree n_0 such that

$$\begin{aligned} P(r) &= a_m r^m + a_{m-1} r^{m-1} + \ldots + a_1 r + a_0 \\ &= P_0(r)(r-\lambda_1)^{m_1} \ldots (r-\lambda_K)^{m_K} (r-\tau_1)^{l_1} (r-\overline{\tau}_1)^{l_1} \ldots (r-\tau_L)^{l_L} (r-\overline{\tau}_L)^{l_L}. \end{aligned} \quad (5)$$

This indicates that the λ_K, τ_L, and $\overline{\tau}_L$ are the distinct roots of (5), in which case $P_0(r)$ has no root in common with $P(r)$. For $l = 1, 2, \ldots, K, u = 1, 2, \ldots, L$, the integers m_l and $2l_u$ are called the multiplicities of the zeroes. We have

$$n_0 + \sum_{l=1}^{K} m_l + 2 \sum_{u=1}^{L} l_u = m.$$

By the following theorem, it is seen that the solutions of m-order quaternion linear homogeneous difference equations can be obtained by the roots of its corresponding quaternion characteristic polynomials.

Theorem 2 ([27]). *Let $P(r) = r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0$. Then, the solutions of (4) can be given by $x(n) = r^n$, where $r \in \mathbb{H}$ is the root of $P(r)$.*

For more detailed information on the solutions of the quaternion difference equations, see [27].

3. Main Results

In this section, we introduce a new type of sequences, called the generalized Fibonacci-type numbers and present some results.

Definition 2. *The generalized Fibonacci-type numbers T_n are defined by the recurrent relation, for $n \geq -1$,*

$$T_{n+3} = aT_{n+2} + bT_{n+1} + cT_n \quad (6)$$

where a, b, and c are nonzero quaternions with initial conditions

$$T_{-1} = c^{-1}, T_0 = 0, T_1 = 0.$$

It can be clearly obtained that the characteristic equation of (6) has the form

$$r^3 - ar^2 - br - c = 0.$$

Now we derive the solution of the difference equation in (1) through an analytical approach. Inspired by the above mentioned papers, we give the following theorem which investigates the solution of (1) by transforming the difference equation to linear quaternion difference equations by employing some suitable changes of variables.

Theorem 3. *Let $\{z_n\}_{n=-1}^{\infty}$ be a solution of the Equation (1). Then, we have*

$$z_n = Q_n (Q_{n+1})^{-1} \quad (7)$$

where the initial conditions $z_{-1}, z_0 \in \mathbb{H} - F$, with F are the forbidden set of the Equation (1), given by

$$F = \overset{\infty}{\cup} \{(z_{-1}, z_0) : Q_{n+1} = T_{n+2} + (T_{n+3} - T_{n+2} a) z_0 + (T_{n+1} c) z_{-1} z_0 = 0\}$$

and $Q_n = T_{n+1} + (T_{n+2} - T_{n+1} a) z_0 + (T_n c) z_{-1} z_0$ such that T_n is the nth generalized Fibonacci-type number.

Proof. First, we derive the solution form of the Equation (1) by the change in variables

$$z_n = \frac{y_{n-1}}{y_n}. \tag{8}$$

Suppose that this quotient is $y_{n-1}y_n^{-1}$. This means that this quotient is taken as $y_{n-1}y_n^{-1}$by the change in variables in (8) and throughout this study. Then, Equation (1) is reduced to a linear third-order difference equation

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2}. \tag{9}$$

By the iterative method in [18], we define the initial values of three sequences, which will be recursively defined and used in the rest of the proof. Let

$$a_1 := a, b_1 := b, c_1 := c.$$

We use a recurrent (iterative) method such that $2 \le m \le n-1$, $a_m := a_{m-1}a + b_{m-1}$, $b_m := a_{m-1}b + c_{m-1}$, $c_m := a_{m-1}c$. Thus, we obtain

$$\begin{aligned} y_n &= a_1y_{n-1} + b_1y_{n-2} + c_1y_{n-3} \\ &= a_1(ay_{n-2} + by_{n-3} + cy_{n-4}) + b_1y_{n-2} + c_1y_{n-3} \\ &= (a_1a + b_1)y_{n-2} + (a_1b + c_1)y_{n-3} + a_1cy_{n-4} \\ &= a_2y_{n-2} + b_2y_{n-3} + c_2y_{n-4} \end{aligned} \tag{10}$$

where

$$a_2 := a_1a + b_1, b_2 := a_1b + c_1, c_2 := a_1c. \tag{11}$$

By continuing the iteration, it follows that

$$\begin{aligned} y_n &= a_2(ay_{n-3} + by_{n-4} + cy_{n-5}) + b_2y_{n-3} + c_2y_{n-4} \\ &= (a_2a + b_2)y_{n-3} + (a_2b + c_2)y_{n-4} + a_2cy_{n-5} \\ &= a_3y_{n-3} + b_3y_{n-4} + c_3y_{n-5} \end{aligned} \tag{12}$$

where

$$a_3 := a_2a + b_2, b_3 := a_2b + c_2, c_3 := a_2c. \tag{13}$$

Based on the relations (10) and (12) for $m \in \mathbb{N}$ such that $2 \le m \le n-1$, we have

$$y_n = a_my_{n-m} + b_my_{n-m-1} + c_my_{n-m-2} \tag{14}$$

and

$$a_m := a_{m-1}a + b_{m-1}, b_m := a_{m-1}b + c_{m-1}, c_m := a_{m-1}c. \tag{15}$$

By continuing the iteration, it follows that

$$\begin{aligned} y_n &= a_m(ay_{n-m-1} + by_{n-m-2} + cy_{n-m-3}) + b_my_{n-m-1} + c_my_{n-m-2} \\ &= (a_ma + b_m)y_{n-m-1} + (a_mb + c_m)y_{n-m-2} + a_mcy_{n-m-3} \\ &= a_{m+1}y_{n-m-1} + b_{m+1}y_{n-m-2} + c_{m+1}y_{n-m-3} \end{aligned}$$

where

$$a_{m+1} := a_ma + b_m, b_{m+1} := a_mb + c_m, c_{m+1} := a_mc.$$

Consider the sequences a_k, b_k and c_k for some nonpositive values of k. The recurrent relations in (15) can be used for computing the values of sequences a_k, b_k and c_k for all

$k \leq 0$. Using the recurrent relations with the indices $k = 1, k = 0$ and $k = -1$, respectively, we obtain the following values of these sequences:

$$\begin{array}{lll} a_0 = c_1 c^{-1} = 1, & a_{-1} = c_0 c^{-1} = 0, & a_{-2} = c_{-1} c^{-1} = 0, \\ b_0 = a_1 - a_0 a = 0, & b_{-1} = a_0 - a_{-1} a = 1, & b_{-2} = a_{-1} - a_{-2} a = 0, \\ c_0 = b_1 - a_0 b = 0, & c_{-1} = b_0 - a_{-1} b = 0, & c_{-2} = b_{-1} - a_{-2} b = 1. \end{array} \tag{16}$$

From (15), for $n \in \mathbb{N}$,

$$a_n = a_{n-1} a + a_{n-2} b + a_{n-3} c, \tag{17}$$

$$b_n = a_{n-1} b + a_{n-2} c = a_{n+1} - a_n a \tag{18}$$

$$c_n = a_{n-1} c. \tag{19}$$

When we take as $m = n$ in (14), for $n > 0$, we have

$$y_n = a_n y_0 + b_n y_{-1} + c_n y_{-2}. \tag{20}$$

From (17)–(19), we can write

$$y_n = a_n y_0 + (a_{n+1} - a_n a) y_{-1} + (a_{n-1} c) y_{-2}. \tag{21}$$

By using (21) in (8), we obtain

$$\begin{aligned} z_n &= (a_{n-1} y_0 + (a_n - a_{n-1} a) y_{-1} + (a_{n-2} c) y_{-2})(a_n y_0 + (a_{n+1} - a_n a) y_{-1} + (a_{n-1} c) y_{-2})^{-1} \\ &= (a_{n-1} y_0 y_0^{-1} + (a_n - a_{n-1} a) y_{-1} y_0^{-1} + (a_{n-2} c) y_{-2} y_0^{-1})(a_n y_0 y_0^{-1} + \\ &\quad (a_{n+1} - a_n a) y_{-1} y_0^{-1} + (a_{n-1} c) y_{-2} y_0^{-1})^{-1} \\ &= (a_{n-1} + (a_n - a_{n-1} a) z_0 + (a_{n-2} c) z_{-1} z_0)(a_n + (a_{n+1} - a_n a) z_0 + (a_{n-1} c) z_{-1} z_0)^{-1}. \end{aligned}$$

From initial values (16) and the definitions of sequences a_n and T_n, for $n \geq -2$, we have

$$a_n = T_{n+2}.$$

Hence, we obtain

$$z_n = (T_{n+1} + (T_{n+2} - T_{n+1} a) z_0 + (T_n c) z_{-1} z_0)(T_{n+2} + (T_{n+3} - T_{n+2} a) z_0 + (T_{n+1} c) z_{-1} z_0)^{-1}.$$

It is seen that

$$z_n = Q_n (Q_{n+1})^{-1}.$$

Thus, the proof is completed. □

Now, we will analyze a special case of the above theorem for $a = k, b = j, c = i$. In this case, the (a_n) sequence has the following recurrence relation

$$a_n = a_{n-1} k + a_{n-2} j + a_{n-3} i \tag{22}$$

such that a few terms of this sequence are

$$a_{-3} = -i, a_{-2} = 0, a_{-1} = 0, a_0 = 1, a_1 = k, a_2 = j - 1, a_3 = -k + i. \tag{23}$$

It is seen that

$$a_n = T_{n+2}.$$

Then, we have

$$z_n = (T_{n+1} + (T_{n+2} - T_{n+1} k) z_0 + (T_n i) z_{-1} z_0)(T_{n+2} + (T_{n+3} - T_{n+2} k) z_0 + (T_{n+1} i) z_{-1} z_0)^{-1}.$$

Now, we investigate the solution form of the system in (2). The following theorem introduces the general solution in explicit form of system (2).

Theorem 4. *Let $\{z_n, w_n\}_{n=-1}^{\infty}$ be a solution of (2). Then, for $n = 1, 2, \dots$, we have*

$$\begin{aligned} z_{2n+1} &= A_n(B_n)^{-1}, \\ z_{2n} &= C_n(D_n)^{-1} \end{aligned} \tag{24}$$

and

$$\begin{aligned} w_{2n+1} &= A_n^*(B_n^*)^{-1}, \\ w_{2n} &= C_n^*(D_n^*)^{-1} \end{aligned} \tag{25}$$

where the initial conditions are z_{-1}, z_0, $w_{-1}, w_0 \notin \mathbb{H} \setminus (F_1 \cup F_2)$ with F_1 and F_2 are the forbidden sets of system (2) given by $F_1 = \bigcup_{n=-1}^{\infty} \{(z_{-1}, z_0, w_{-1}, w_0) : B_n = 0, D_n = 0\}$ and $F_2 = \bigcup_{n=-1}^{\infty} \{(z_{-1}, z_0, w_{-1}, w_0) : B_n^* = 0, D_n^* = 0\}$
where

$$\begin{aligned} A_n &= T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + (T_{2n+1}c)z_{-1}w_0, A_n^* = T_{2n+2} + (T_{2n+3} - T_{2n+2}a)z_0 + (T_{2n+1}c)w_{-1}z_0, \\ B_n &= T_{2n+3} + (T_{2n+4} - T_{2n+3}a)w_0 + (T_{2n+2}c)z_{-1}w_0, B_n^* = T_{2n+3} + (T_{2n+4} - T_{2n+3}a)z_0 + (T_{2n+2}c)w_{-1}z_0, \\ C_n &= T_{2n+1} + (T_{2n+2} - T_{2n+1}a)z_0 + (T_{2n}c)w_{-1}z_0, C_n^* = T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + (T_{2n}c)z_{-1}w_0, \\ D_n &= T_{2n+2} + (T_{2n+3} - T_{2n+2}a)z_0 + (T_{2n+1}c)w_{-1}z_0, D_n^* = T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + (T_{2n+1}c)z_{-1}w_0. \end{aligned}$$

Proof. The closed-form solution of (2) can be given similarly to the approach used in the proof of Theorem 1. Both for convenience and to use a different method from the iterative approach, we prove the theorem by the principle of mathematical induction on n. For $n = 0$, it is clear that the result is true since we have

$$z_1 = (a + bw_0 + cz_{-1}w_0)^{-1} \text{ and } w_1 = (a + bz_0 + cw_{-1}z_0)^{-1}.$$

Now, we assume that $n > 0$ and that the Equations (24) and (25) are true for $n - 1$. That is,

$$\begin{aligned} z_{2n-2} &= (T_{2n-1} + (T_{2n} - T_{2n-1}a)z_0 + (T_{2n-2}c)w_{-1}z_0)(T_{2n} + (T_{2n+1} - T_{2n}a)z_0 + (T_{2n-1}c)w_{-1}z_0)^{-1}, \\ z_{2n-1} &= (T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + (T_{2n-1}c)z_{-1}w_0)(T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + (T_{2n}c)z_{-1}w_0)^{-1}, \\ w_{2n-2} &= (T_{2n-1} + (T_{2n} - T_{2n-1}a)w_0 + (T_{2n-2}c)z_{-1}w_0)(T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + (T_{2n-1}c)z_{-1}w_0)^{-1}, \\ w_{2n-1} &= (T_{2n} + (T_{2n+1} - T_{2n}a)z_0 + (T_{2n-1}c)w_{-1}z_0)(T_{2n+1} + (T_{2n+2} - T_{2n+1}a)z_0 + (T_{2n}c)w_{-1}z_0)^{-1}. \end{aligned}$$

Since the following equations exist

$$\begin{aligned} A_{n-1} &= T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + (T_{2n-1}c)z_{-1}w_0, \\ B_{n-1} &= T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + (T_{2n}c)z_{-1}w_0, \\ C_{n-1} &= T_{2n-1} + (T_{2n} - T_{2n-1}a)z_0 + (T_{2n-2}c)w_{-1}z_0, \\ D_{n-1} &= T_{2n} + (T_{2n+1} - T_{2n}a)z_0 + (T_{2n-1}c)w_{-1}z_0, \\ A_{n-1}^* &= T_{2n} + (T_{2n+1} - T_{2n}a)z_0 + (T_{2n-1}c)w_{-1}z_0, \\ B_{n-1}^* &= T_{2n+1} + (T_{2n+2} - T_{2n+1}a)z_0 + (T_{2n}c)w_{-1}z_0, \\ C_{n-1}^* &= T_{2n-1} + (T_{2n} - T_{2n-1}a)w_0 + (T_{2n-2}c)z_{-1}w_0, \\ D_{n-1}^* &= T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + (T_{2n-1}c)z_{-1}w_0, \end{aligned}$$

we can write the following equations:

$$\begin{aligned} z_{2n-2} &= C_{n-1}D_{n-1}^{-1}, \\ z_{2n-1} &= A_{n-1}B_{n-1}^{-1}, \\ w_{2n-2} &= C_{n-1}^*(D_{n-1}^*)^{-1}, \\ w_{2n-1} &= A_{n-1}^*(B_{n-1}^*)^{-1}. \end{aligned}$$

It is clearly seen that

$$\begin{array}{ll} A_{n-1} = D^*_{n-1}, & A^*_{n-1} = D_{n-1}, \\ B_{n-1} = C^*_{n-1}, & B^*_{n-1} = C_{n-1}. \end{array}$$

From system (2), by some mathematical computations, it is seen that

$$\begin{aligned} z_{2n} &= (a + bw_{2n-1} + cz_{2n-2}w_{2n-1})^{-1} \\ &= (a + bA^*_{n-1}(B^*_{n-1})^{-1} + cC_{n-1}D^{-1}_{n-1}A^*_{n-1}(B^*_{n-1})^{-1})^{-1} \\ &= B^*_{n-1}(aB^*_{n-1} + bA^*_{n-1}(B^*_{n-1})^{-1}B^*_{n-1} + cC_{n-1}D^{-1}_{n-1}A^*_{n-1}(B^*_{n-1})^{-1}B^*_{n-1})^{-1} \\ &= B^*_{n-1}(aB^*_{n-1} + bA^*_{n-1} + cC_{n-1})^{-1} \end{aligned}$$

Thus, we have

$$z_{2n} = (T_{2n+1} + (T_{2n+2} - T_{2n+1}a)z_0 + T_{2n}cw_{-1}z_0)(T_{2n+2} + (T_{2n+3} - T_{2n+2}a)z_0 + T_{2n+1}cw_{-1}z_0)^{-1}.$$

It implies that

$$z_{2n} = C_n(D_n)^{-1}$$

where $C_n = T_{2n+1} + (T_{2n+2} - T_{2n+1}a)z_0 + (T_{2n}c)w_{-1}z_0$ and $D_n = T_{2n+2} + (T_{2n+3} - T_{2n+2}a)z_0 + (T_{2n+1}c)w_{-1}z_0$. Furthermore, it follows from (2) that

$$\begin{aligned} w_{2n} &= (a + bz_{2n-1} + cw_{2n-2}z_{2n-1})^{-1} \\ &= (a + bA_{n-1}B^{-1}_{n-1} + cC^*_{n-1}(D^*_{n-1})^{-1}A_{n-1}B^{-1}_{n-1}) \\ &= B_{n-1}(aB_{n-1} + bA_{n-1} + cC^*_{n-1})^{-1}. \end{aligned}$$

So, we get

$$\begin{aligned} w_{2n} &= (T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + T_{2n}cz_{-1}w_0)(a(T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + T_{2n}cz_{-1}w_0) \\ &\quad + b(T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + T_{2n-1}cz_{-1}w_0) + c(T_{2n-1} + (T_{2n} - T_{2n-1}a)w_0 + T_{2n-2}cz_{-1}w_0))^{-1} \\ &= (T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + T_{2n}cz_{-1}w_0)(T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + T_{2n+1}cz_{-1}w_0)^{-1}. \end{aligned}$$

It is seen that

$$w_{2n} = C^*_n(D^*_n)^{-1}$$

where $C^*_n = T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + (T_{2n}c)z_{-1}w_0$ and $D^*_n = T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + (T_{2n+1}c)z_{-1}w_0$. Using the above arguments, we similarly obtain the following equations:

$$\begin{aligned} z_{2n+1} &= (a + bw_{2n} + cz_{2n-1}w_{2n})^{-1} \\ &= (a + bC^*_n(D^*_n)^{-1} + cA_{n-1}B^{-1}_{n-1}C^*_n(D^*_n)^{-1})^{-1} \\ &= D^*_n(aD^*_n + bC^*_n + cA_{n-1})^{-1}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} z_{2n+1} &= (T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + T_{2n+1}cz_{-1}w_0)(a(T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + T_{2n+1}cz_{-1}w_0) \\ &\quad + b(T_{2n+1} + (T_{2n+2} - T_{2n+1}a)w_0 + T_{2n}cz_{-1}w_0) + c(T_{2n} + (T_{2n+1} - T_{2n}a)w_0 + T_{2n-1}cz_{-1}w_0))^{-1} \\ &= (T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + (T_{2n+1}c)z_{-1}w_0)(T_{2n+3} + (T_{2n+4} - T_{2n+3}a)w_0 + (T_{2n+2}c)z_{-1}w_0)^{-1}. \end{aligned}$$

Then we have

$$z_{2n+1} = A_n(B_n)^{-1}$$

where $A_n = T_{2n+2} + (T_{2n+3} - T_{2n+2}a)w_0 + (T_{2n+1}c)z_{-1}w_0$ and $B_n = T_{2n+3} + (T_{2n+4} - T_{2n+3}a)w_0 + (T_{2n+2}c)z_{-1}w_0$. Similarly, we obtain

$$w_{2n+1} = A^*_n(B^*_n)^{-1}.$$

This completes the proof of the theorem. □

It is known that the difference Equation (1) is solved by the change of variables (8) which transforms it to the following third-order linear quaternion difference equation with quaternion coefficients (9)

$$y_{n+1} = ay_n + by_{n-1} + cy_{n-2}.$$

Its characteristic polynomial is

$$p(r) = r^3 - ar^2 - br - c,$$

and its conjugate is

$$\overline{p}(r) = r^3 + ar^2 + br + c.$$

In [27], it is seen that

$$F(r) = p(r)\overline{p}(r).$$

The roots of $p(r)$ are obtained by using Lemma 1, Lemma 2 and Theorem 2 and then the solution of (9) is found by these roots.

Consider the following third-order quaternion linear difference equation in the case $a = k, b = j, c = i$ of Equation (9)

$$y_{n+1} = ky_n + jy_{n-1} + iy_{n-2} \tag{26}$$

and its characteristic polynomial is

$$p(r) = r^3 - kr^2 - jr - i = (r-k)(r - \frac{1}{\sqrt{2}}(1+j))(r + \frac{1}{\sqrt{2}}(1+j)). \tag{27}$$

Its conjugate is given by

$$\overline{p}(r) = r^3 + kr^2 + jr + i = (r-\overline{k})(r - \frac{1}{\sqrt{2}}(\overline{1+j}))(r + \frac{1}{\sqrt{2}}(\overline{1+j})). \tag{28}$$

Then, we have

$$\begin{aligned} F(r) &= p(r)\overline{p}(r) \\ &= (r-k)(r-\overline{k})(r - \frac{1}{\sqrt{2}}(1+j))(r - \frac{1}{\sqrt{2}}(\overline{1+j}))(r + \frac{1}{\sqrt{2}}(1+j))(r + \frac{1}{\sqrt{2}}(\overline{1+j})) \\ &= \left(r^2+1\right)(r^2 - \sqrt{2}r + 1)(r^2 + \sqrt{2}r + 1). \end{aligned} \tag{29}$$

Thus, the roots of (29) are

$$i, -i, j, -j, k, -k, \frac{1}{\sqrt{2}}(-1-i), \frac{1}{\sqrt{2}}(-1+i), \frac{1}{\sqrt{2}}(-1-j), \frac{1}{\sqrt{2}}(-1+j), \frac{1}{\sqrt{2}}(-1-k),$$
$$\frac{1}{\sqrt{2}}(-1+k), \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(1-i), \frac{1}{\sqrt{2}}(1+j), \frac{1}{\sqrt{2}}(1-j), \frac{1}{\sqrt{2}}(1+k), \frac{1}{\sqrt{2}}(1-k).$$

Then, we obtain

$$\begin{array}{ll}
\overline{p}(i) = i^3 + ki^2 + ji + i = 0, & \overline{p}(-i) = -i^3 + ki^2 - ji + i = 2i, \\
\overline{p}(j) = j^3 + kj^2 + j^2 + i = -j - k + i - 1, & \overline{p}(-j) = -j^3 + kj^2 - jj + i = j - k + i + 1 \\
\overline{p}(k) = k^3 + k^3 + jk + i = -2k + 2i, & \overline{p}(-k) = -k^3 + k^3 - jk + i = 0, \\
\overline{p}(\frac{1}{\sqrt{2}}(1+i)) = \frac{(\sqrt{2}+1)}{\sqrt{2}}(i+j) - \frac{1}{\sqrt{2}}(k+1), & \overline{p}(\frac{1}{\sqrt{2}}(-1-i)) = \frac{1}{\sqrt{2}}((\sqrt{2}-1)(i+j)+k+1), \\
\overline{p}(\frac{1}{\sqrt{2}}(1+j)) = \frac{1}{\sqrt{2}}(2j-2), & \overline{p}(\frac{1}{\sqrt{2}}(-1-j)) = \frac{1}{\sqrt{2}}(2-2j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1+k)) = \frac{(\sqrt{2}+1)}{\sqrt{2}}(i-1) + \frac{1}{\sqrt{2}}(k+j), & \overline{p}(\frac{1}{\sqrt{2}}(-1-k)) = \frac{1}{\sqrt{2}}((\sqrt{2}-1)(1+i)-(k+j)), \\
\overline{p}(\frac{1}{\sqrt{2}}(1-i)) = \frac{(\sqrt{2}-1)}{\sqrt{2}}(i-j) + \frac{1}{\sqrt{2}}(k-1), & \overline{p}(\frac{1}{\sqrt{2}}(-1+i)) = \frac{(\sqrt{2}+1)}{\sqrt{2}}(i-j) + \frac{1}{\sqrt{2}}(-k+1), \\
\overline{p}(\frac{1}{\sqrt{2}}(1-j)) = 2i, & \overline{p}(\frac{1}{\sqrt{2}}(-1+j)) = 2i, \\
\overline{p}(\frac{1}{\sqrt{2}}(1-k)) = \frac{(\sqrt{2}-1)}{\sqrt{2}}(1+i) + \frac{1}{\sqrt{2}}(j-k), & \overline{p}(\frac{1}{\sqrt{2}}(-1+k)) = \frac{(\sqrt{2}+1)}{\sqrt{2}}(1+i) + \frac{1}{\sqrt{2}}(k-j).
\end{array}$$

It is clearly seen that $-i$, j, $-j$, and k are not the roots of $\overline{p}(r)$, and $i, -k$ are the roots of $\overline{p}(r)$. Using (i) of Lemma 1, we can obtain the roots of $p(r)$ as follows:

$$\begin{aligned}
\overline{p}(-i)(-i)\overline{p}(-i)^{-1} &= -i \\
\overline{p}(j)(j)\overline{p}(j)^{-1} &= -i \\
\overline{p}(-j)(-j)\overline{p}(-j)^{-1} &= -i \\
\overline{p}(k)(k)\overline{p}(k)^{-1} &= -i
\end{aligned}$$

$-i$ is not the root of $\overline{p}(r)$, while i is the root of $\overline{p}(r)$. Thus, by using Lemma 1(ii), we can write

$$\overline{p}(\bar{i})(\bar{i})\overline{p}(\bar{i})^{-1} = \overline{p}(-i)(-i)\overline{p}(-i)^{-1} = -i$$

where it is the root of $p(r)$. Similarly, k is not the root of $\overline{p}(r)$ while $-k$ is the root of $\overline{p}(r)$. Thus, by using Lemma 1(ii), we can write

$$\overline{p}(\overline{-k})(\overline{-k})\overline{p}(\overline{-k})^{-1} = \overline{p}(k)(k)\overline{p}(k)^{-1} = -i$$

where it is the root of $p(r)$. So, $p(r)$ has $r_1 = -i$ as a root. Similarly, $-1-i, -1+i, -1-j, -1+j, -1-k, -1+k, \frac{1}{2}(1+i), \frac{1}{2}(1-i), \frac{1}{2}(1+j), \frac{1}{2}(1-j), \frac{1}{2}(1+k)$, and $\frac{1}{2}(1-k)$ are not the roots of $\overline{p}(r)$. Then, by similar computations, we can write the following equations:

$$\begin{array}{l}
\overline{p}(\frac{1}{\sqrt{2}}(1+i))(\frac{1}{\sqrt{2}}(1+i))\overline{p}(\frac{1}{\sqrt{2}}(1+i))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1+j))(\frac{1}{\sqrt{2}}(1+j))\overline{p}(\frac{1}{\sqrt{2}}(1+j))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1+k))(\frac{1}{\sqrt{2}}(1+k))\overline{p}(\frac{1}{\sqrt{2}}(1+k))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1-i))(\frac{1}{\sqrt{2}}(1-i))\overline{p}(\frac{1}{\sqrt{2}}(1-i))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1-j))(\frac{1}{\sqrt{2}}(1-j))\overline{p}(\frac{1}{\sqrt{2}}(1-j))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(1-k))(\frac{1}{\sqrt{2}}(1-k))\overline{p}(\frac{1}{\sqrt{2}}(1-k))^{-1} = \frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1-i))(\frac{1}{\sqrt{2}}(-1-i))\overline{p}(\frac{1}{\sqrt{2}}(-1-i))^{-1} = -\frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1-j))(\frac{1}{\sqrt{2}}(-1-j))\overline{p}(\frac{1}{\sqrt{2}}(-1-j))^{-1} = -\frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1-k))(\frac{1}{\sqrt{2}}(-1-k))\overline{p}(\frac{1}{\sqrt{2}}(-1-k))^{-1} = -\frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1+i))(\frac{1}{\sqrt{2}}(-1+i))\overline{p}(\frac{1}{\sqrt{2}}(-1+i))^{-1} = -\frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1+j))(\frac{1}{\sqrt{2}}(-1+j))\overline{p}(\frac{1}{\sqrt{2}}(-1+j))^{-1} = -\frac{1}{\sqrt{2}}(1+j), \\
\overline{p}(\frac{1}{\sqrt{2}}(-1+k))(\frac{1}{\sqrt{2}}(-1+k))\overline{p}(\frac{1}{\sqrt{2}}(-1+k))^{-1} = -\frac{1}{\sqrt{2}}(1+j).
\end{array}$$

Hence, the roots r_1, r_2, r_3 of $p(r)$ are $r_1 = -i$, $r_2 = \frac{1}{\sqrt{2}}(1+j)$, and $r_3 = -\frac{1}{\sqrt{2}}(1+j)$. Then, the solutions of (26) can be represented by

$$y_1(n) = (-i)^n, y_2(n) = (\frac{1}{\sqrt{2}}(1+j))^n \text{ and } y_3(n) = (-\frac{1}{\sqrt{2}}(1+j))^n.$$

Hence, from Theorem 1 and Definition 1, we obtain the general solution of (26) as follows:

$$\begin{aligned} y(n) &= y_1(n)\widetilde{c}_1 + y_2(n)\widetilde{c}_2 + y_3(n)\widetilde{c}_3 \\ &= (-i)^n\widetilde{c}_1 + (\frac{1}{\sqrt{2}}(1+j))^n\widetilde{c}_2 + (-\frac{1}{\sqrt{2}}(1+j))^n\widetilde{c}_3 \end{aligned}$$

where $\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3 \in \mathbb{H}$.

4. Conclusions

In recent years, quaternion differential and difference equations have been used to cover a wide area of interest in modern mathematics (see [27–30]). In [27], the authors investigated the general solutions of the higher-order linear quaternion difference equations with variable and constant coefficients. In this paper, we deal with the solutions of the following difference equation

$$z_{n+1} = (p_n)^{-1}, n \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

where $p_n = a + bz_n + cz_{n-1}z_n$ and the parameters a, b, c and the initial values z_{-1}, z_0 are nonzero quaternions such that their solutions are associated with generalized Fibonacci-type numbers. Furthermore, we derive the solution form of the following symmetric system of difference equations

$$z_{n+1} = (q_n)^{-1}, w_{n+1} = (r_n)^{-1}, n \in \mathbb{N}_0$$

where $q_n = a + bw_n + cz_{n-1}w_n$ and $r_n = a + bz_n + cw_{n-1}z_n$. By using the zeros of the quaternion characteristic polynomial associated with the linear difference equation, we also give the solution to the third-order linear quaternion difference equation, in which the difference equation (1) is transformed by a change of variables.

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