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Duality on q-Starlike Functions Associated with Fractional q-Integral Operators and Applications

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Abstract: In this paper, we make use of the Riemann–Liouville fractional q-integral operator to discuss the class $S^*_{q,\delta}(\alpha)$ of univalent functions for $\delta > 0, \alpha \in \mathbb{C} - \{0\}$, and $0 < |q| < 1$. Then, we develop convolution results for the given class of univalent functions by utilizing a concept of the fractional q-difference operator. Moreover, we derive the normalized classes $\mathcal{P}^{\zeta}_{\delta,q}(\beta,\gamma)$ and $\mathcal{P}_{\delta,q}(\beta)$ $(0 < |q| < 1, \delta \geq 0, 0 \leq \beta \leq 1, \zeta > 0)$ of analytic functions on a unit disc and provide conditions for the parameters q, δ, ζ, β, and γ so that $\mathcal{P}^{\zeta}_{\delta,q}(\beta,\gamma) \subset S^*_{q,\delta}(\alpha)$ and $\mathcal{P}_{\delta,q}(\beta) \subset S^*_{q,\delta}(\alpha)$ for $\alpha \in \mathbb{C} - \{0\}$. Finally, we also propose an application to symmetric q-analogues and Ruscheweh's duality theory.

Keywords: Riemann–Liouville; q-analogue; difference operator; q-starlike functions; duality principle; dual set; q-hypergeometric function

MSC: 05A15; 11B68; 26B10; 33E20

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1. Introduction

In recent decades, the theory of q-calculus has been applied to various areas of science and computational mathematics. The concept of q-calculus was used in quantum groups, q-deformed super algebras, q-transform analysis, q-integral calculus, optimal control, and many other fields, to mention but a few [1–4]. Soon after the concept of q-calculus was furnished, many basic q-hypergeometric functions, q-hypergeometric symmetric functions, and q-hypergeometric and hypergeometric symmetric function polynomials were discussed in geometric function theory [5]. Jackson [6] was the first to introduce and analyze the q-derivative and the q-integral operator. Later, various researchers applied the concept of the q-derivative to various sub-collections of univalent functions. Srivastava [7] used the q-derivative operator to describe some properties of a subclass of univalent functions. Agrawal et al. [8] extended a class of q-starlike functions to certain subclasses of q-starlike functions. Kanas et al. [9] used convolutions to define a q-analogue of the Ruscheweyh operator and studied some useful applications of their operator. Srivastava et al. [10] defined the q-Noor integral operator by following the concept of convolution. Purohit [11] introduced a subclass of univalent functions by using a certain operator of a fractional q-derivative. Aouf et al. [12] employed subordination results to discuss analytic functions associated with a new fractional q-analogue of certain operators. However, many extensions of different operators can be found in [13–29] and the references cited therein.

Here, we will make use of definitions and notations used in the literature [30,31]. For $a, q \in \mathbb{C}$, the q-analogue of the Pochhammer symbol is defined by

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-aq^j), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ \prod_{j=0}^{\infty}(1-aq^j), & \text{if } k \to \infty, \end{cases}$$

and, hence, it is very natural to write $(a;q)_k = \frac{(a;q)_\infty}{(aq^k,q)_\infty}$, $(k \in \mathbb{N} \cup \{\infty\})$. The extension of the Pochhammer symbol to a real number δ is given as

$$(a;q)_\delta = \frac{(a;q)_\infty}{(aq^\delta;q)_\infty}, \quad (\delta \in \mathbb{R}).$$

Therefore, for any real number $\delta > 0$, the q-analogue of the gamma function is defined by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\delta;q)\infty}(1-q)^{1-\delta}.$$

The q-analogue of the natural number n and the multiple q-shifted factorial for complex numbers $a_1, \cdots, a_k$ are, respectively, defined by

$$[n]_q = \frac{1-q^n}{1-q}, \quad 0 < |q| < 1, \text{ and } (a_1, \cdots, a_k; q)_n = \prod_{j=1}^{k}(a_j;q)_n.$$

Let $a_1, ..., a_r, b_1, ...b_s$ be complex numbers; then, the q-hypergeometric series ${}_r\phi_s$ is denoted as

$${}_r\phi_s\left(\begin{matrix} a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix} ;q,z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r;q)n}{(b_1, \cdots, b_s;q)n} z^n \left(-q^{\frac{n-1}{2}}\right)^{n(s-r+1)}. \tag{1}$$

It is clear that the series representation of the function ${}_r\phi_s$ converges absolutely for all $z \in \mathbb{C}$ if $r \le s$ and converges only for $|z| < 1$ if $r = s+1$. Now, let $\mathcal{A}$ be the collection of all analytic functions in the open unit disc $\mathcal{U} = \{z \in \mathbb{C};\ |z| < 1\}$ expressed in the normalized form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

and let $\mathcal{A}_0$ be a collection comprising all functions g such that $zg \in \mathcal{A}$ and $g(0) = 1, z \in \mathbb{C}$. Then, the sub-collection of $\mathcal{A}$ of functions that are univalent in $\mathcal{U}$ is denoted by S. However, in geometric function theory, a variety of sub-collections of univalent functions have been discussed. See the monographs published by [32,33] for details.

Let us consider the Riemann–Liouville fractional q-integral operator of a non-integer of order δ defined by [34]

$$I_q^\delta f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (x - [qt])^{\delta-1} f(t) d_q t. \tag{3}$$

Then, $I_q^\delta f \longrightarrow I_q$ when $\delta \longrightarrow 1$, where $I_q f$ is the q-Jackson integral defined by [6]

$$I_q f(z) = \int_0^z f(t) d_q(t) \ \ z \in \mathcal{U}, z \neq 0,\ |q| < 1.$$

With the concept of the Riemann–Liouville fractional q-integral of the non-integer order δ, we recall some rules associated with I_q^δ by (3):

(i) $I_q^\delta(cf) = cI_q^\delta f, \quad c \in \mathbb{C} - \{0\}, f \in \mathcal{A},$

(ii) $I_q^{\delta}(f+g) = I_q^{\delta}f + I_q^{\delta}g, \qquad f,g \in \mathcal{A},$
(iii) $I_q^{\delta}|f| \leq |I_q^{\delta}f|.$

Agarwal [34] defined the q-analogue difference operator of a non-integer order δ as follows:

$$D_q^{\delta}f(z) = \frac{1}{(1-q)^{\delta}z^{\delta}} \sum_{n=0}^{\infty} \frac{(q^{-\delta};q)_n}{(q;q)_n} q^n f(q^n z). \tag{4}$$

Note that $D_q^{\delta}f \longrightarrow D_q f$ when $\delta \longrightarrow 1$. $D_q f$ is the q-derivative of the function f introduced in [6] in the subsequent form:

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathcal{U}, z \neq 0, \ |q| < 1. \tag{5}$$

Thus, for $n \in \mathbb{N}$, through simple computations, we obtain

$$D_q^{\delta}z^n = \frac{z^{n-\delta}}{(1-q)^{\delta}} \frac{(q^{1+n-\delta};q)_{\infty}}{(q^{1+n};q)_{\infty}} \quad \text{and} \quad I_q^{\delta}z^n = \frac{(q^{n+1+\delta};q)_{\infty}}{(q^{n+1};q)_{\infty}} z^{n+\delta}.$$

Let $0 < |q| < 1$, $\delta \geq 0$ $\zeta > 0$, $0 \leq \beta \leq 1$, and $0 < \gamma \leq 1$. By the definition of the q-analogue difference operator with the non-integer order δ, the following rules of D_q^{δ} hold:

(i) $D_q^{\delta}(cf) = cD_q^{\delta}f, \quad c \in \mathbb{C} - \{0\}, f \in \mathcal{A},$
(ii) $D_q^{\delta}(f+g) = D_q^{\delta}f + D_q^{\delta}g, \qquad f,g \in \mathcal{A}.$

We define $\mathcal{P}_{\delta,q}^{\zeta}(\beta,\gamma)$ as the class of all functions $f \in \mathcal{A}$ satisfying the following condition:

$$Re\left\{ \frac{(1-q)^{\delta}\left(D_q^{\delta+1}I_q^{\delta}f(z) + \frac{1-\gamma}{\zeta\gamma}zD_q^{\delta+2}I_q^{\delta}f(z)\right) - \beta}{1-\beta} \right\} > 0, \qquad |z| < 1.$$

For $0 < |q| < 1$, $\delta \geq 0$, and $0 \leq \beta \leq 1$, the class $\mathcal{P}_{\delta,q}(\beta)$ consists of functions satisfying the following condition:

$$Re\left\{ \frac{(1-q)^{\delta}\left(D_q^{\delta+1}I_q^{\delta}f(z) + qzD_q^{\delta+2}I_q^{\delta}f(z)\right) - \beta}{1-\beta} \right\} > 0, \qquad |z| < 1.$$

Now, for two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

we recall the convolution (or the Hadamard product) of f and g, denoted by $f * g$, which is given by

$$(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \ \ z \in \mathcal{U}.$$

For a set $\mathcal{V} \subseteq \mathcal{A}_0$, the dual set $\mathcal{V}^*$ is defined by

$$\mathcal{V}^* = \{g \in \mathcal{A}_0 : (f*g)(z) \neq 0, \ \forall f \in \mathcal{V}, \ z \in \mathcal{U}\}.$$

However, the second dual of $\mathcal{V}$ is defined as $\mathcal{V}^{**} = (\mathcal{V}^*)^*$. However, $\mathcal{V} \subseteq \mathcal{V}^{**}$. For basic reference to this theory, we may refer to the book by Ruscheweyh [35] (see also [36–38]).

In this paper, we define the class $S^*_{q,\delta}$ for $\delta > 0$, $0 < |q| < 1$, and establish the convolution condition of this class. Furthermore, we find conditions for q, δ, ζ, β, and γ so that $\mathcal{P}^{\zeta}_{\delta,q}(\beta,\gamma) \subset S^*_{q,\delta}(\alpha)$ and $\mathcal{P}_{\delta,q}(\beta) \subset S^*_{q,\delta}(\alpha)$.

2. Preliminary Lemmas

The following lemmas are very useful in our investigation.

Lemma 1 (Duality principle; see [35]). *Let $\mathcal{V} \subseteq \mathcal{A}_0$ be compact; it has the following property:*

$$f \in \mathcal{V} \Longrightarrow \forall |x| \leq 1 : f_x \in \mathcal{V}, \tag{6}$$

where $f_x(z) = f(xz)$. Then,

$$\varphi(\mathcal{V}) = \varphi(\mathcal{V}^{**}),$$

for all continuous linear functionals φ on $\mathcal{A}$, and

$$\bar{co}(\mathcal{V}) \subseteq \bar{co}(\mathcal{V}^{**}),$$

where $\bar{co}$ stands for the closed convex hull of a set.

Lemma 2 ([35]). *Let $0 \leq \gamma < 1$ and $\beta \in \mathbb{R}$, $\beta \neq 1$. If*

$$V_{\beta,\gamma} = \left\{\gamma(1-\beta)\frac{1+xz}{1-xz} + (1-\gamma)(1-\beta)\frac{1+yz}{1-yz} + \beta,\ \ |x| = |y| = 1,\ \ z \in \mathcal{U}\right\}, \tag{7}$$

then

$$V^*_{\beta,\gamma} = \left\{f \in \mathcal{A}_0 : \exists\zeta \in \mathbb{R}, Re\left\{g(z) - \frac{1-2\beta}{2(1-\beta)}\right\} > 0,\ g(z) = f_x(z),\ |x| \leq 1\right\},$$

and

$$V^{**}_{\beta,\gamma} = \left\{f \in \mathcal{A}_0;\ Re\left\{\frac{g(z)-\beta}{1-\beta}\right\} > 0,\ g(z) = f_x(z),\ |x| \leq 1\right\}.$$

We see that the set $V_{\beta,\gamma}$ in (7) does not satisfy the property (6), i.e., if $f \in V_{\beta,\gamma}$, then $f(xz) \in V_{\beta,\gamma}$ for all $|x| \leq 1$, as is required in the Duality Principle. However, the Duality Principle can be stated with a slightly weaker but more complicated condition that $V_{\beta,\gamma}$ can be seen to satisfy (see [35] for more details).

3. Main Results

Definition 1. *Let $f \in \mathcal{A}$, $\delta > 0$, and $\alpha \in \mathbb{C} - \{0\}$. Then, a function f is said to be in the class $S^*_{q,\delta}(\alpha)$ if it satisfies the following inequality:*

$$Re\left\{1 + \frac{1}{\alpha}\left(\frac{zD_q^{\delta+1}I_q^{\delta}f(z)}{f(z)} - \frac{1}{(1-q)^{\delta}}\right)\right\} > 0,$$

where the operators D_q^{δ} and I_q^{δ} are given by (4) and (3), respectively.

Putting $\delta = 0$ into Definition 1 leads to the following definition.

Definition 2. *The function $f \in \mathcal{A}$ is said to be in the class of q-starlike functions of order α, $S^*_q(\alpha)$, if it satisfies the following inequality:*

$$Re\left\{1 + \frac{1}{\alpha}\left(\frac{zD_qf(z)}{f(z)} - 1\right)\right\} > 0, \quad \alpha \in \mathbb{C} - \{0\},$$

where $D_q f(z)$ is given by (5).

Theorem 1. *Let $f \in \mathcal{A}$, $\delta > 0$, $\alpha \in \mathbb{C} - \{0\}$, and $|z| < R < 1$. Then, $f \in S^*_{q,\delta}(\alpha)$ if and only if*

$$\frac{f(z)}{z} * \frac{1 + qz\left(\frac{x+1}{2\alpha(1-q)^\delta} - 1\right)}{(1-z)(1-qz)} \neq 0,$$

where $|x| = 1$ and $x \neq -1$.

Proof. Since $\frac{zD_q^{\delta+1} I_q^\delta f(z)}{f(z)} - \frac{1}{(1-q)^\delta} = 0$ at $z = 0$, we have

$$1 + \frac{1}{\alpha}\left(\frac{zD_q^{\delta+1} I_q^\delta f(z)}{f(z)} - \frac{1}{(1-q)^\delta}\right) \neq \frac{x-1}{x+1}, \ \ |x| = 1, \ \ x \neq -1.$$

By following simple computations, we can rewrite this as

$$(x+1)(1-q)^\delta zD_q^{\delta+1} I_q^\delta f(z) - \left(2\alpha(1-q)^\delta - x - 1\right) f(z) \neq 0. \tag{8}$$

Since the function f satisfies (2), we obtain

$$zD_q^{\delta+1} I_q^\delta f(z) = \frac{1}{(1-q)^\delta}\left(z + \sum_{n=2}^{\infty} [n]_q a_n z^n\right) = \frac{1}{(1-q)^\delta}\left(f(z) * \frac{z}{(1-z)(1-qz)}\right).$$

Now, as Equation (8) is equivalent to

$$\left(f(z) * \frac{(x+1)z}{(1-z)(1-qz)}\right) + \left(f(z) * \frac{z(2\alpha(1-q)^\delta - x - 1)}{1-z}\right) \neq 0,$$

it simplifies to

$$f(z) * \frac{(x+1)z + z(1-qz)(2\alpha(1-q)^\delta - x - 1)}{(1-z)(1-qz)} \neq 0.$$

Hence, the required result has been proven. □

Putting $\delta = 0$ into Theorem 1, we get the following corollary.

Corollary 1. *Let $\alpha \in \mathbb{C} - \{0\}$, $|x| = 1$, and $x \neq -1$. Then, the function f is a q-starlike function of order α if and only if*

$$\frac{f(z)}{z} * \frac{1 + qz\left(\frac{x+1}{2\alpha} - 1\right)}{(1-z)(1-qz)} \neq 0, |z| < R \leq 1. \tag{9}$$

Theorem 2. *Let $\delta > 0$, $0 < q < 1$, $\alpha \in \mathbb{C} - \{0\}$, $\zeta > 0$, $0 \leq \beta < 1$, $0 < \gamma < 1$, and $|x| = 1$ with $x \neq -1$. Then, $\mathcal{P}^\zeta_{q,\delta}(\beta, \gamma) \subseteq S^*_{q,\delta}(\alpha)$ if and only if*

$$Re\left\{F(x,z)\right\} > -\frac{(1-q)^\delta}{\zeta\gamma(1-\beta)}, \tag{10}$$

where

$$F(x,z) = \frac{1}{\alpha}\sum_{n=1}^{\infty} \frac{(x+1)[n+1]_q + 2\alpha(1-q)^\delta - (x+1)}{[n+1]_q(\zeta\gamma + (1-\gamma)[n]_q)} z^n, |z| < R \leq 1, |z| < R \leq 1. \tag{11}$$

Proof. Let the function f be in the class $\mathcal{P}^{\zeta}_{q,\delta}(\beta,\gamma), |z| < R \leq 1$. If we denote

$$g(z) = (1-q)^{\delta}\left(D_q^{\delta+1}I_q^{\delta}f(z) + \frac{1-\gamma}{\zeta\gamma}zD_q^{\delta+2}I_q^{\delta}f(z)\right),$$

then we have $g \in V^{**}_{\beta,\gamma}$. If f satisfies (2), then we obtain

$$\begin{aligned} g(z) &= 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1} + \sum_{n=2}^{\infty}\frac{1-\gamma}{\zeta\gamma}[n]_q[n-1]_q a_n z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty}[n]_q a_n\left(1 + \frac{1-\gamma}{\zeta\gamma}[n-1]_q\right)z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty}[n]_q a_n\left(\frac{\zeta\gamma + (1-\gamma)[n-1]_q}{\zeta\gamma}\right)z^{n-1}. \end{aligned}$$

Therefore,

$$\frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty}a_n z^{n-1} = g(z) * \left(1 + \sum_{n=2}^{\infty}\frac{\zeta\gamma}{[n]_q(\zeta\gamma + (1-\gamma)[n-1]_q)}z^{n-1}\right).$$

We now obtain a one-to-one correspondence between $\mathcal{P}^{\zeta}_{q,\delta}(\beta,\gamma)$ and $V^{**}_{\beta,\gamma}$. Thus, by Theorem 1, $\mathcal{P}^{\zeta}_{q,\delta}(\beta,\gamma) \subseteq S^{*}_{q,\delta}(\alpha)$ if and only if

$$g(z) * \left(1 + \sum_{n=2}^{\infty}\frac{\zeta\gamma}{[n]_q(\zeta\gamma + (1-\gamma)[n-1]_q)}z^{n-1}\right) * \frac{1 + qz\left(\frac{x+1}{2\alpha(1-q)^{\delta}} - 1\right)}{(1-z)(1-qz)} \neq 0. \tag{12}$$

For $z \in \mathcal{U}$, consider the continuous linear functional $\lambda_z : \mathcal{A}_0 \longrightarrow \mathbb{C}$ such that

$$\lambda_z(h) = h(z) * \left(1 + \sum_{n=2}^{\infty}\frac{\zeta\gamma}{[n]_q(\zeta\gamma + (1-\gamma)[n-1]_q)}z^{n-1}\right) * \frac{1 + qz\left(\frac{x+1}{2\alpha(1-q)^{\delta}} - 1\right)}{(1-z)(1-qz)} \neq 0.$$

By the Duality Principle, we have $\lambda_z(V) = \lambda_z(V^{**}_{\beta,\gamma})$. Therefore, (12) holds if and only if

$$\begin{aligned} \left(1 + 2(1-\beta)\sum_{k=1}^{\infty}z^k\right) &* \left(1 + \sum_{n=1}^{\infty}\frac{\zeta\gamma}{[n+1]_q(\zeta\gamma + (1-\gamma)[n]_q)}z^n\right) \\ &* \left(1 + \sum_{n=1}^{\infty}\left([n+1]_q + \left(\frac{(x+1)q}{2\alpha(1-q)^{\delta}} - q\right)[n]_q\right)z^n\right) \neq 0. \end{aligned}$$

Using the properties of convolution, we obtain

$$1 + \frac{2(1-\beta)\zeta\gamma}{2\alpha(1-q)^{\delta}}\sum_{n=1}^{\infty}\frac{2\alpha(1-q)^{\delta}[n+1]_q + \left(q(x+1) - 2\alpha q(1-q)^{\delta}\right)[n]_q}{[n+1]_q(\zeta\gamma + (1-\gamma)[n]_q)}z^n \neq 0.$$

Since $[n+1]_q = 1 + q[n]_q$, we get

$$1 + \frac{2(1-\beta)\zeta\gamma}{2\alpha(1-q)^{\delta}}\sum_{n=1}^{\infty}\frac{(x+1)[n+1]_q + 2\alpha(1-q)^{\delta} - (x+1)}{[n+1]_q(\zeta\gamma + (1-\gamma)[n]_q)}z^n \neq 0.$$

Hence, we have

$$\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(x+1)[n+1]_q + 2\alpha(1-q)^{\delta} - (x+1)}{[n+1]_q(\zeta\gamma + (1-\gamma)[n]_q)}z^n \neq -\frac{(1-q)^{\delta}}{\zeta\gamma(1-\beta)}, \tag{13}$$

where $z \in \mathcal{U}$, $|x| = 1$. The equality on the right side of Equation (13) takes its value on the line $Rew \neq -\frac{(1-q)^{\delta}}{\zeta\gamma(1-\beta)}$, and so (13) is equivalent to (10). $\square$

Remark 1. *Under the hypothesis of Theorem 2, the inequality (10) can be written in the form*

$$\begin{aligned}
&\frac{(1-q)^{\delta}}{\zeta\gamma(1-\beta)} + Re\left\{\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{\zeta\gamma+(1-\gamma)[n]_q}\right\} \\
&+(1-q)^{\delta}Re\left\{\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right\} - Re\left\{\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right\} \\
&\geq Re\left\{\frac{x}{\alpha}\left(\sum_{n=1}^{\infty}\frac{z^n}{\zeta\gamma+(1-\gamma)[n]_q} - \sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right)\right\}.
\end{aligned}$$

Therefore, for more clarification, we can see that this satisfies the inequality when

$$\begin{aligned}
&\frac{(1-q)^{\delta}}{\zeta\gamma(1-\beta)} + Re\left\{\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{\zeta\gamma+(1-\gamma)[n]_q}\right\} \\
&+(1-q)^{\delta}Re\left\{\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right\} - Re\left\{\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right\} \\
&\geq \left|\frac{1}{\alpha}\left(\sum_{n=1}^{\infty}\frac{z^n}{\zeta\gamma+(1-\gamma)[n]_q} - \sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}\right)\right|. \qquad (14)
\end{aligned}$$

Assume that the function ψ is given by

$$\psi(z) = \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}.$$

Then, inequality (14) can be written in the form

$$\frac{(1-q)^{\delta}}{\zeta\gamma(1-\beta)} + Re\{D_q z\psi(z) + (2\alpha(1-q)^{\delta}-1)\psi(z)\} \geq |D_q z\psi(z) + \psi(z)|. \qquad (15)$$

Hence, $\psi(z) \in S^*_{q,\delta}(\alpha)$ if and only if (15) is satisfied.

Putting $\delta = 0$ into Theorem 2 leads to the following corollary.

Corollary 2. *Let $0 < q < 1$, $\alpha \in \mathbb{C} - \{0\}$, $\zeta > 0$, $0 \leq \beta < 1$, $0 < \gamma < 1$, and $|x| = 1$ with $x \neq -1$. Then, $\mathcal{P}^{\zeta}_{q,0}(\beta,\gamma) \subseteq S^*_q(\alpha)$ if and only if*

$$Re\Big\{F_1(x,z)\Big\} > \frac{1}{\zeta\gamma(1-\beta)},$$

where

$$F_1(x,z) = \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(x+1)[n+1]_q + 2\alpha - (x+1)}{[n+1]_q(\zeta\gamma+(1-\gamma)[n]_q)}z^n,\ |z| < R \leq 1.$$

Similarly, from Theorem 2, we get the following theorem.

Theorem 3. *Let $\delta > 0$, $0 < q < 1$, $\alpha \in \mathbb{C} - \{0\}$, $0 \leq \beta < 1$, and $|x| = 1$ with $x \neq -1$. Then, $\mathcal{P}_{q,\delta}(\beta) \subseteq S^*_{q,\delta}(\alpha)$ if and only if*

$$Re\Big\{F_1(x,z)\Big\} > -\frac{(1-q)^{\delta}}{(1-\beta)}, \qquad (16)$$

where

$$F_1(x,z) = \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(x+1)[n+1]_q + 2\alpha(1-q)^{\delta} - (x+1)}{[n+1]_q^2}z^n, |z| < R \le 1. \quad (17)$$

Putting $\delta = 0$ into Theorem 3 leads to the following corollary.

Corollary 3. *Let* $0 < q < 1$, $\alpha \in \mathbb{C} - \{0\}$, $0 \le \beta < 1$, *and* $|x| = 1$ *with* $x \neq -1$. *Then,* $\mathcal{P}_{q,0}(\beta,) \subseteq S_q^*(\alpha)$ *if and only if*

$$Re\Big\{F_2(x,z)\Big\} > -\frac{1}{(1-\beta)},$$

where

$$F_2(x,z) = \frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(x+1)[n+1]_q + 2\alpha - (x+1)}{[n+1]_q^2}z^n, |z| < R \le 1.$$

Remark 2. *The function* $F_1(x,z)$ *can be represented in terms of a* q*-hypergeometric function as follows:*

$$F_1(x,z) = \frac{x+1}{\alpha}\,{}_2\phi_1\left(\begin{matrix} q & q \\ q^2 & \end{matrix};q,z\right) + \frac{2\alpha(1-q)^{\delta} - (x+1)}{\alpha}\,{}_2\phi_1\left(\begin{matrix} q & q & q \\ q^2 & q^2 & \end{matrix};q,z\right).$$

Proof. From the definition of $F_1(x,z)$ introduced in (11), we infer that

$$\begin{aligned} F_1(x,z) &= \frac{x+1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q} + \frac{2\alpha(1-q)^{\delta} - (x+1)}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{[n+1]_q^2} \\ &= \frac{x+1}{\alpha}\sum_{n=0}^{\infty}\frac{z^n}{[n+1]_q} + \frac{2\alpha(1-q)^{\delta} - (x+1)}{\alpha}\sum_{n=0}^{\infty}\frac{z^n}{[n+1]_q^2} - 2(1-q). \quad (18) \end{aligned}$$

Since $[n+1]_q = \frac{(q^2;q)_n}{(q;q)_n}$, we have

$$F_1(x,z) = \frac{x+1}{\alpha}\sum_{n=0}^{\infty}\frac{(q;q)_n}{(q^2;q)_n}z^n + \frac{2\alpha(1-q)^{\delta} - (x+1)}{\alpha}\sum_{n=0}^{\infty}\frac{(q;q)_n(q;q)_n}{(q^2;q)_n(q^2;q)_n}z^n - 2(1-q).$$

Hence, by using the definition of ${}_r\phi_s$ from (1), the proof of the corollary is complete. □

Putting $\delta = 0$ into Remark 2 leads to the following corollary.

Corollary 4. *The function* $F_2(x,z)$ *can be expressed in terms of the* q*-hypergeometric function as follows:*

$$F_2(x,z) = \frac{x+1}{\alpha}\,{}_2\phi_1\left(\begin{matrix} q & q \\ q^2 & \end{matrix};q,z\right) + \frac{2\alpha - (x+1)}{\alpha}\,{}_2\phi_1\left(\begin{matrix} q & q & q \\ q^2 & q^2 & \end{matrix};q,z\right).$$

We now consider the Riemann–Liouville fractional q-integral and obtain the following corollary.

Remark 3. *The function* $F_1(x,z)$ *can be expressed in terms of the Riemann–Liouville fractional* q*-integral as follows:*

$$F_1(x,z) = \frac{x+1}{\alpha}\int_0^1 \frac{1}{1-tz}d_qt + \frac{2\alpha(1-q)^{\delta} - (x+1)}{\alpha}\int_0^1\int_0^1 \frac{1}{1-vtz}d_qvd_qt - 2(1-q)$$

Proof. Since Equation (18) is satisfied, we have

$$\begin{aligned} F_1(x,z) &= \frac{x+1}{\alpha}\sum_{n=0}^{\infty}\int_0^1 t^n d_q t z^n + \frac{2\alpha(1-q)^{\delta}-(x+1)}{\alpha}\sum_{n=0}^{\infty}\int_0^1 v^n d_q v\int_0^1 t^n d_t z^n \\ &= \frac{x+1}{\alpha}\int_0^1\left(\sum_{n=0}^{\infty} t^n z^n\right) d_q t + \frac{2\alpha(1-q)^{\delta}-(x+1)}{\alpha}\int_0^1\int_0^1\left(\sum_{n=0}^{\infty} v^n t^n z^n\right) d_q v d_q t. \end{aligned}$$

This completes the proof of the corollary. □

Putting $\delta = 0$ into Remark 3 leads to the following corollary.

Corollary 5. *The function $F_2(x,z)$ can be expressed in terms of the Riemann–Liouville fractional q-integral in the following form:*

$$F_2(x,z) = \frac{x+1}{\alpha}\int_0^1 \frac{1}{1-tz} d_q t + \frac{2\alpha-(x+1)}{\alpha}\int_0^1\int_0^1 \frac{1}{1-vtz} d_q v d_q t - 2(1-q).$$

Theorem 4. *Let $0 < |q| < 1, \delta > 0, 0 \le \beta \le 1$, and $f \in \mathcal{P}_{q,\delta}(\beta)$. We define*

$$K_q = \int_0^1 \frac{d_q t}{1-t}. \tag{19}$$

If

$$\beta \ge \frac{1-2K_q}{2(1-K_q)},$$

then $f \in \mathcal{P}_{q,\delta}(0)$, and, hence, it is univalent.

Proof. Let $\zeta > 0$ and $\gamma > 0$; we define

$$\phi_q(z) = 1 + \sum_{n=1}^{\infty} [n+1]_q z^n,$$

and

$$\begin{aligned} \psi_q(z) &= 1 + \sum_{n=1}^{\infty} \frac{1}{[n+1]_q} z^n = 1 + \sum_{n=1}^{\infty}\int_0^1 t^n d_q t z^n \\ &= \int_0^1\left(1 + \sum_{n=1}^{\infty} t^n z^n\right) d_q t = \int_0^1 \frac{1}{1-tz} d_q t. \end{aligned}$$

In view of these representations, we can write

$$D_q^{\delta+1} I_q^{\delta} f(z) + q D_q^{\delta+2} I_q^{\delta} f(z) = D_q^{\delta+1} I_q^{\delta} f(z) * \phi_q(z)$$

and

$$\left(D_q^{\delta+1} I_q^{\delta} f(z) + q D_q^{\delta+2} I_q^{\delta} f(z)\right) * \psi_q(z) = D_q^{\delta+1} I_q^{\delta} f(z).$$

Let $f \in \mathcal{P}_{q,\delta}(\beta)$. Then, by using Lemma 2, we may restrict our attention to the function $f \in P_{\zeta}(\beta,\gamma)$ for which

$$(1-q)^{\delta}\left(D_q^{\delta+1} I_q^{\delta} f(z) + q D_q^{\delta+2} I_q^{\delta} f(z)\right) = \gamma(1-\beta)\frac{1+xz}{1-xz} + (1-\beta)(1-\gamma)\frac{1+yz}{1-yz} + \beta.$$

Thus, we obtain

$$(1-q)^{\delta} D_q^{\delta+1} I_q^{\delta} f(z) = \left(\gamma(1-\beta)\frac{1+xz}{1-xz} + (1-\beta)(1-\gamma)\frac{1+yz}{1-yz} + \beta\right) * \psi_q(z). \tag{20}$$

Hence, Equation (20) is equivalent to

$$\begin{aligned}(1-q)^{\delta} D_q^{\delta+1} I_q^{\delta} f(z) &= \left(\gamma\frac{1+xz}{1-xz} + (1-\gamma)\frac{1+yz}{1-yz}\right) * ((1-\beta)\psi_q(z) + \beta). \\ &= \left(\gamma\frac{1+xz}{1-xz} + (1-\gamma)\frac{1+yz}{1-yz}\right) * \left(\int_0^1 \left((1-\beta)\frac{1}{1-tz} + \beta\right) d_q t\right) \\ &= \left(\gamma\frac{1+xz}{1-xz} + (1-\gamma)\frac{1+yz}{1-yz}\right) * G_q(z), \end{aligned} \tag{21}$$

where

$$G_q(z) = \int_0^1 \left((1-\beta)\frac{1}{1-tz} + \beta\right) d_q t.$$

Therefore,

$$Rel\big(G_q(z)\big) = \int_0^1 \left((1-\beta)\frac{1}{1-t} + \beta\right) d_q t = (1-\beta)k_q + \beta,$$

where K_q is defined by (19). Note that if $\beta \geq (1-2K_q)/2(1-K_q)$, then $ReG(z) \geq 1/2$. Functions with real parts greater than $1/2$ are known to preserve the closed convex hull under convolution [10], p. 23. Therefore, from (21), we have

$$\begin{aligned}(1-q)^{\delta} D_q^{\delta+1} I_q^{\delta} f(z) &= \gamma\left(\frac{2}{1-xz} - 1\right) * G_q(z) + (1-\gamma)\left(\frac{2}{1-yz} - 1\right) * G_q(z) \\ &= 2\gamma G_q(xz) - \gamma + 2(1-\gamma)G_q(yz) - (1-\gamma) \\ &= 2\gamma G_q(xz) + 2(1-\gamma)G_q(yz) - 1.\end{aligned}$$

In addition, since $Re\{D_q^{\delta+1} I_q^{\delta} f(z)\} > 0$, we have $f \in \mathcal{P}_{q,\delta}(0)$. This completes the proof of the theorem. □

4. Conclusions

In this article, a new class of univalent functions was introduced by using Riemann–Liouville fractional q-integrals and q-difference operators of non-integer orders. Then, some convolution results for such a class of univalent functions were obtained. In addition, two classes of normalized analytic functions in the unit disc were derived, and some conditions on q, δ, ζ, β, and γ were given so that the new classes satisfied $\mathcal{P}^{\zeta}_{\delta,q}(\beta,\gamma) \subset S^*_{q,\delta}(\alpha)$ and $\mathcal{P}_{\delta,q}(\beta) \subset S^*_{q,\delta}(\alpha)$.

The result obtained during this research can be further used for writing fractional differential and integral operators in order to extend the results of analytic functions.

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References

1. Kac, V.; Cheung, P. Quantum Calculus (Universitext). In *Softcover Reprint of the Original*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2001.
2. Gauchman, H. Integral inequalities in q-calculus. *Comput. Math. Appl.* **2004**, *47*, 281–300. [CrossRef]
3. Hristov, J. Approximate solutions to fractional subdiffusion equations. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 229–243. [CrossRef]
4. Tang, Y.; Zhang, T. A remark on the q-fractional order differential equations. *Appl. Math. Comput.* **2019**, *350*, 198–208. [CrossRef]
5. Srivastava, H.M.; Owa, S. *Univalent Functions, Fractional Calculus, and Their Applications*; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1989.
6. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
7. Al-Omari, S. Estimates and properties of certain q-Mellin transform on generalized q-calculus theory. *Adv. Differ. Equ.* **2021**, *242*, 1–11. [CrossRef]
8. Agrawal, S.; Sahoo, S.K. A generalization of starlike functions of order alpha. *Hokkaido Math. J.* **2017**, *46*, 15–27. [CrossRef]
9. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
10. Srivastava, H.M.; Khan, S.; Ahmad, Q.Z.; Khan, N.; Hussain, S. The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator. *Stud. Univ. Babes-Bolyai Mat.* **2018**, *63*, 419–436. [CrossRef]
11. Purohit, S.D. A new class of multivalently analytic functions associated with fractional q-calculus operators. *Fract. Differ. Calc.* **2012**, *2*, 129–138. [CrossRef]
12. Aouf, M.K.; Mostafa, A.O. Subordination results for analytic functions associated with fractional q-calculus operators with complex order. *Afr. Mat.* **2020**, *31*, 1387–1396. [CrossRef]
13. Al-Omari, S. On a q-Laplace-type integral operator and certain class of series expansion. *Math. Methods Appl. Sci.* **2021**, *240*, 8322–8332. [CrossRef]
14. Amini, E.; Al-Omari, S.; Nonlaopon, K.; Baleanu, D. Estimates for Coefficients of Bi-Univalent Functions Associated with a Fractional q-Difference Operator. *Symmetry* **2022**, *14*, 879. [CrossRef]
15. Seoudy, T.; Aouf, M.K. Fekete-Szegö Problem for Certain Subclass of Analytic Functions with Complex Order Defined by q-Analogue of Ruscheweyh Operator. *Constr. Math. Anal.* **2020**, *3*, 36–44. [CrossRef]
16. Seoudy, T.M.; Aouf, M.K. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order. *J. Math. Inequalities* **2016**, *1*, 135–145. [CrossRef]
17. Özkan Uçar, H.E. Coefficient inequality for q-starlike functions. *Appl. Math. Comput.* **2016**, *276*, 122–126. [CrossRef]
18. Al-Omari, S.; Baleanu, D.; Purohit, D. Some results for Laplace-type integral operator in quantum calculus. *Adv. Differ. Equ.* **2018**, *124*, 1–10. [CrossRef]
19. Al-Omari, S. On q-analogues of Mangontarum transform of some polynomials and certain class of H-functions. *Nonlinear Stud.* **2016**, *23*, 51–61.
20. Cheng, W.T.; Nasiruzzaman, M.; Mohiuddine, S.A. Stancu-Type Generalized q-Bernstein–Kantorovich Operators Involving Bézier Bases. *Mathematics* **2022**, *10*, 2057. [CrossRef]
21. Al-Omari, S. On q-analogues of the Mangontarum transform for certain q-Bessel functions and Some application. *J. King Saud Univ. Sci.* **2016**, *28*, 375–379. [CrossRef]
22. Al-Omari, S. q-analogues and properties of the Laplace-type integral operator in the quantum calculus theory. *J. Inequalities Appl.* **2014**, *2020*, 203. [CrossRef]
23. Srivastava, H.; Karlsson, P. *Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester)*; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985.
24. Al-Omari, S. The q-Sumudu transform and its certain properties in a generalized q-calculus theory. *Adv. Differ. Equ.* **2021**, *2021*, 10. [CrossRef]
25. Alotaibi, A.; Yaying, T.; Mohiuddine, S.A. Sequence Spaces and Spectrum of q-Difference Operator of Second Order. *Symmetry* **2022**, *14*, 1155. [CrossRef]
26. Chandak, S.; Suthar, D.; Al-Omari, S.; Gulyaz-Ozyurt, S. Estimates of classes of generalized special functions and their application in the fractional (k,s)-calculus theory. *J. Funct. Spaces* **2022**, *2022*, 9582879. [CrossRef]
27. Amini, E.; Fardi, M.; Al-Omari, S.; Nonlaopon, K. Results on Univalent Functions Defined by q-Analogues of Salagean and Ruscheweh Operators. *Symmetry* **2022**, *14*, 1725. [CrossRef]
28. Srivastava, H. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
29. Al-Omari, S.; Suthar, D.; Araci, S. A Fractional q-integral Operator Associated with Certain Class of q-Bessel Functions and q-generating Series. *Advan. Contin. Discr. Models Theo. Appli.* **2021**, *41*, 1–13. [CrossRef]

30. Gasper, G. *Basic Hypergeometric Series (Encyclopedia of Mathematics and its Applications)*, 2nd ed.; Series Number 96; Cambridge University Press: Cambridge, UK, 2004.
31. Annaby, M.H.; Mansour, Z.S. *q-Fractional Calculus and Equations*, 1st ed.; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 2012.
32. Duren, P.L. *Univalent Functions Grundlehren der Mathematischen Wissenschaften 259*, 1983 ed.; Springer: Berlin/Heidelberg, Germany, 2001.
33. Graham, I. *Geometric Function Theory in One and Higher Dimensions Pure and Applied Mathematics*, 1st ed.; CRC Press: Boca Raton, FL, USA, 2019.
34. Agarwal, R.P. Certain fractional q-integrals and q-derivatives. *Math. Proc. Camb. Philos. Soc.* **1969**, *66*, 365–370. [CrossRef]
35. Ruscheweyh, S. *Convolutions in Geometric Function Theory*; Gaetan Morin Editeur Ltee: Quebec, Canada, 1982.
36. Ruscheweyh, S. Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc. *Trans. Am. Math. Soc.* **1975**, *210*, 63–74. [CrossRef]
37. Ruscheweyh, S. Some convexity and convolution theorems for analytic functions. *Math. Ann.* **1978**, *238*, 217–228. [CrossRef]
38. Ponnusamy, S.; Ronning, F. Duality for Hadamard products applied to certain integral transforms. *Complex Var. Theory Appl. Int. J.* **1997**, *32*, 263–287. [CrossRef]