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Multivalent Prestarlike Functions with Respect to Symmetric Points

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Abstract: A class of p-valent functions of complex order is defined with the primary motive of unifying the concept of prestarlike functions with various other classes of multivalent functions. Interesting properties such as inclusion relations, integral representation, coefficient estimates and the solution to the Fekete–Szegő problem are obtained for the defined function class. Further, we extended the results using quantum calculus. Several consequences of our main results are pointed out.

Keywords: multivalent functions; Jackson's q-derivative operator; prestarlike; starlike and convex functions; subordination; Fekete–Szegő problem; coefficient inequalities; q-calculus

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1. Introduction

Let Π_p denote the class of all analytic functions of the form

$$\chi(\zeta) = \zeta^p + \sum_{k=1}^{\infty} a_{p+k}\zeta^{p+k} \quad (p = 1, 2, 3, \dots) \tag{1}$$

and let $\Pi = \Pi_1$. Further, let $\mathcal{R}$ denote the class of functions $k(\zeta)$ analytic in $\Omega = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and satisfy $\Re\{(k(\zeta)\} > 0$ for all ζ in Ω. Aouf ([1], Equation (1.4)) defined the class $h(\zeta) \in \mathcal{R}(X, Y, p, \alpha)$ if and only if

$$h(\zeta) = \frac{p + [pY + (X - Y)(p - \alpha)]w(\zeta)}{[1 + Yw(\zeta)]}, \quad (-1 \le Y < X \le 1, 0 \le \alpha < 1) \tag{2}$$

where $w(\zeta)$ is the Schwartz function. The class $\mathcal{R}(X, Y, p, \alpha)$ is an extension of the famous Janowski class of functions [2]. Further, it was proved, in [1] (Theorem 5), that, if $h(\zeta) = p + \sum_{k=1}^{\infty} b_k\zeta^k$ is in $\mathcal{R}(X, Y, p, \alpha)$, then

$$|b_k| \le (X - Y)(p - \alpha), \quad k = 1, 2, \dots.$$

Throughout this paper, we let $\Psi \in \mathcal{R}$ and Ψ, which has a power series expansion of the form

$$\Psi(\zeta) = 1 + \ell_1\zeta + \ell_2\zeta^2 + \ell_3\zeta^3 + \cdots, \ \zeta \in \Omega, \ \ell_1 \neq 0. \tag{3}$$

Letting $w(\zeta) = \frac{\Psi(\zeta)-1}{\Psi(\zeta)+1}$ for some $\Psi(\zeta) = 1 + \ell_1\zeta + \ell_2\zeta^2 + \cdots \in \mathcal{R}$ in (2), we have the following relation (see [1], Equation (1.6)):

$$
\begin{aligned}
&\aleph(\zeta) \in \mathcal{R}(X, Y, p, \alpha) \\
\Leftrightarrow \quad &\aleph(\zeta) = \frac{[(1+X)p + \alpha(Y-X)]\Psi(\zeta) + [(1-X)p - \alpha(Y-X)]}{[(Y+1)\Psi(\zeta) + (1-Y)]}.
\end{aligned} \tag{4}
$$

From (4), we see that

$$
\begin{aligned}
&\frac{[\alpha(Y-X)-(1-X)p]}{(Y-1)}\left[1-\frac{[(1+X)p+\alpha(Y-X)]\Psi(\zeta)}{[\alpha(Y-X)-(1-X)p]}\right]\left[1-\frac{(Y+1)\Psi(\zeta)}{Y-1}\right]^{-1} \\
&\quad = \frac{[\alpha(Y-X)-(1-X)p]}{(Y-1)} + \left\{\frac{[\alpha(Y-X)-(1-X)p](Y+1)^2}{(Y-1)} - \right. \\
&\qquad \left. \frac{[(1+X)p+\alpha(Y-X)]}{(Y-1)}\right\}\Psi(\zeta) \\
&+\left\{\frac{[\alpha(Y-X)-(1-X)p](Y+1)^2}{(Y-1)^3} - \frac{[(1+X)p+\alpha(Y-X)](Y+1)}{(Y-1)^2}\right\}[\Psi(\zeta)]^2 + \cdots \\
&\quad = \sum_{n=1}^{\infty}\frac{-2p(Y+1)^{n-1}}{(Y-1)^n} + \sum_{n=1}^{\infty}\frac{2n(X-Y)(p-\alpha)(Y+1)^{n-1}}{(Y-1)^{n+1}}\ell_1\zeta + \cdots.
\end{aligned} \tag{5}
$$

The first and second terms of infinite series (1) are convergent to p and $\frac{(X-Y)(p-\alpha)\ell_1}{2}$, provided that $|(Y+1)/(Y-1)| < 1$. Hence, (4) can be rewritten as

$$
\aleph(\zeta) = p + \frac{(X-Y)(p-\alpha)\ell_1}{2}\zeta + \cdots \qquad \text{if } \left|\frac{(Y+1)}{Y-1}\right| < 1. \tag{6}
$$

For the functions χ and φ that are analytic in Ω, we say that the function χ is *subordinate* to φ if there exits a function w, analytic in Ω with $w(0) = 0$ and $|w(\zeta)| < 1, \zeta \in \Omega$, such that $\chi = \varphi \circ w$. We denote this subordination by $\chi \prec \varphi$ or $\chi(\zeta) \prec \varphi(\zeta)$. In particular, if the function φ is univalent in Ω, the above subordination is equivalent to (see [3,4]) $\chi(0) = \varphi(0)$ and $\chi(\Omega) \subset \varphi(\Omega)$. For the functions $\chi(\zeta)$ of the form (1) and $\varphi(\zeta) = \zeta^p + \sum_{k=1}^{\infty} b_{p+k}\zeta^{p+k}$, the Hadamard product (or convolution) of χ and φ is defined by $(\chi * \varphi)(\zeta) = \zeta^p + \sum_{k=1}^{\infty} a_{p+k}\, b_{p+k}\, \zeta^{p+k}$.

We let $\mathcal{S}_p^*(\delta)$ and $\mathcal{C}_p(\delta)$ denote the familiar subclasses of $\mathcal{A}_p$ consisting of functions which are respectively p-valent starlike of order δ and p-valent convex of order δ in Ω. In addition, we let $\mathcal{S}_p(\delta)$ denote the class of p-valent starlike functions of order δ satisfying the condition

$$
\Re\left(\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right) > \delta, \quad (0 \le \delta < p).
$$

The extremal function for the class $\mathcal{K}_p(\delta)$ is given by

$$
\mathcal{K}_\delta(\zeta) = \zeta^p(1-\zeta)^{-2(1-\delta)} = \zeta^p + \sum_{k=1}^{\infty}\Gamma_\delta^k\zeta^{p+k}. \tag{7}
$$

with $\Gamma_\delta^k = \frac{\prod_{i=2}^{k+1}(i-2\delta)}{k!}$, $(0 \le \delta < 1,\ p = 1, 2, 3, \ldots)$. A function $\chi(\zeta) \in \Pi_p$ is said to be p-valent prestarlike of order δ if

$$
\chi(\zeta) * \mathcal{K}_p(\delta) \in \mathcal{S}_p(\delta), \quad (0 \le \delta < p).
$$

We denote by $\mathcal{PS}_p(\delta)$ the class of all p-valent prestarlike functions of order δ. The class of univalent *prestarlike functions* was introduced by Ruscheweyh ([5], Section 2). The so-called class of prestarlike functions was further extended and studied by various authors; refer to [6–8].

In the present section, we define a new differential operator motivated by the concept of convex combination of analytic functions and we use the operator to define presumably a new class of multivalent functions of complex order with respect to symmetric points.

We focus on the coefficient estimates, inclusion results and solution to the Fekete–Szegő problem of the defined function class. In the subsequent section, we have extended the study using quantum calculus.

For $\chi \in \Pi_p$, we now define following operator $\mathcal{T}^m_{\lambda,\delta}\chi(\zeta) : \Omega \longrightarrow \Omega$ by

$$\mathcal{T}^0_{\lambda,\delta}\chi(\zeta) = \chi(\zeta) * \mathcal{K}_\delta(\zeta)$$

$$\mathcal{T}^1_{\lambda,\delta}\chi(\zeta) = (1-\lambda)(\chi(\zeta) * \mathcal{K}_\delta(\zeta)) + \frac{\lambda}{p}\zeta(\chi(\zeta) * \mathcal{K}_\delta(\zeta))' \tag{8}$$

$$\mathcal{T}^m_{\lambda,\delta}\chi(\zeta) = \mathcal{T}^1_{\lambda,\delta}(\mathcal{T}^{m-1}_{\lambda,\delta}\chi(\zeta)). \tag{9}$$

If $\chi \in \Pi_p$, then, from (8) and (9), we may easily deduce that

$$\mathcal{T}^m_{\lambda,\delta}\chi(\zeta) = \zeta^p + \sum_{k=1}^{\infty}\left[\frac{p+\lambda k}{p}\right]^m \Gamma^k_\delta\, a_{p+k}\,\zeta^{p+k}, \tag{10}$$

where $m \in N_0 = N \cup \{0\}$ and $0 \le \lambda \le 1$. For $p = 1$, $\mathcal{T}^m_{\lambda,\delta}\chi$ is a special case of the operator $D^m_\lambda(\alpha_1, \beta_1)\chi$ (see [9], Equation (5)). If we let $\delta = 1/2$ and $p = \lambda = 1$ in (10), then $\mathcal{T}^m_{\lambda,\delta}\chi$ reduces to $D^m\chi$, the well-known Sălăgean differential operator [10].

Unless otherwise mentioned,

$$-1 \le Y < X \le 1,\ 0 \le \alpha < 1,\ 0 \le \lambda \le 1,\ 0 \le \delta < p.$$

Definition 1. *For $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that the function χ belongs to the class $\Lambda^m_p(b; \Psi; \alpha; \lambda, \delta; X, Y)$ if it satisfies the subordination condition*

$$p + \frac{1}{b}\left\{\frac{[1-(-1)^p]\zeta\left[\mathcal{T}^m_{\lambda,\delta}\chi(\zeta)\right]'}{\left[\mathcal{T}^m_{\lambda,\delta}\chi(\zeta) - \mathcal{T}^m_{\lambda,\delta}\chi(-\zeta)\right]} - p\right\}$$
$$\prec \frac{[(1+X)p + \alpha(Y-X)]\Psi(\zeta) + [(1-X)p - \alpha(Y-X)]}{[(Y+1)\Psi(\zeta) + (1-Y)]}, \tag{11}$$

where "$\prec$" denotes subordination and $\Psi(\zeta)$ is defined as in (3).

Remark 1. *In the literature, for p = 1, numerous study of Janowski starlike and convex functions of complex order with respect to symmetric points can be found. Here, we give some recent studies as special cases of $\Lambda^m_p(b; \Psi; \alpha; \lambda, \delta; X, Y)$.*

1. *If we let $p = \lambda = 1$, $\delta = 1/2$, $\alpha = 0$ and $\Psi(\zeta) = (1+\zeta)/(1-\zeta)$, the class $\Lambda^m_p(b; \Psi; \alpha; \lambda, \delta; X, Y)$ reduces to*

$$\mathcal{Q}^b_s(X, Y, m) = \left\{\chi \in \Pi : 1 + \frac{1}{b}\left(\frac{2D^{m+1}\chi(\zeta)}{D^m\chi(\zeta) - D^m\chi(-\zeta)} - 1\right) \prec \frac{1+X\zeta}{1+Y\zeta}\right\}.$$

   The class $\mathcal{Q}^b_s(X, Y, m)$ was recently introduced by Arif et al. in [11].

2. *If we let $m = 0$, $p = 1$, $X = 1$, $Y = -1$ and $\Psi(\zeta) = (1+\zeta)/(1-\zeta)$ in Definition 1, then $\Lambda^m_p(b; \Psi; \alpha; \lambda, \delta; X, Y)$ reduces to*

$$\mathcal{P}^b_s(\alpha, \delta) = \left\{\chi \in \Pi : \Re\left[1 + \frac{1}{b}\left(\frac{2\zeta[(\chi * \mathcal{K}_\delta)(\zeta)]}{(\chi * \mathcal{K}_\delta)(\zeta) - (\chi * \mathcal{K}_\delta)(-\zeta)} - 1\right)\right] > \alpha\right\}.$$

   Letting $b = 1$ in $\mathcal{P}^1_s(\alpha, \delta)$, we obtain the class of all prestarlike functions of order α *with respect to symmetric points.*

For studies pertaining to the classes of functions with respect to symmetric points, refer to [12] and references provided therein.

2. Inclusion Relationship and Initial Coefficient Estimates

Throughout this section, we let

$$\aleph(\zeta) = \frac{[(1+X)p + \alpha(Y-X)]\Psi(\zeta) + [(1-X)p - \alpha(Y-X)]}{[(Y+1)\Psi(\zeta) + (1-Y)]}. \tag{12}$$

We use the following results to obtain the solution of the Fekete–Szegő problem for the functions that belong to those classes we define in the first section.

Lemma 1 ([4], p. 41)**.** *If $p(\zeta) = 1 + \sum_{k=1}^{\infty} p_k \zeta^k \in \mathcal{R}$, then $|p_k| \leq 2$ for all $k \geq 1$ and the inequality is sharp for $p_\mu(\zeta) = \frac{1+\mu\zeta}{1-\mu\zeta}$, $|\mu| \leq 1$.*

Lemma 2 ([13])**.** *If $p(\zeta) = 1 + \sum_{k=1}^{\infty} p_k \zeta^k \in \mathcal{R}$ and v is complex number, then*

$$\left| p_2 - v p_1^2 \right| \leq 2\max\{1; |2v-1|\},$$

and the result is sharp for the functions

$$p_1(\zeta) = \frac{1+\zeta}{1-\zeta} \quad \textit{and} \quad p_2(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}.$$

If $\chi \in \Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$, then, by Definition 1, we have

$$\frac{[1-(-1)^p]\zeta\left[\mathcal{T}_{\lambda,\delta}^m \chi(\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^m \chi(\zeta) - \mathcal{T}_{\lambda,\delta}^m \chi(-\zeta)\right]} = p + b[\aleph(w(\zeta)) - p]. \tag{13}$$

Replacing ζ by $-\zeta$ in (13),

$$\frac{[1-(-1)^p]\zeta\left[\mathcal{T}_{\lambda,\delta}^m \chi(-\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^m \chi(\zeta) - \mathcal{T}_{\lambda,\delta}^m \chi(-\zeta)\right]} = p + b[\aleph(w(-\zeta)) - p]. \tag{14}$$

Subtracting (13) and (14), we have the following after-simplification:

$$\frac{\left[\mathcal{T}_{\lambda,\delta}^m L(\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^m L(\zeta)\right]} - \frac{1}{\zeta} = \frac{b[\aleph(w(\zeta)) - \aleph(w(-\zeta))]}{\zeta[1-(-1)^p]} - \frac{1}{\zeta}, \tag{15}$$

with $\mathcal{T}_{\lambda,\delta}^m L(\zeta) = \frac{\mathcal{T}_{\lambda,\delta}^m \chi(\zeta) - \mathcal{T}_{\lambda,\delta}^m \chi(-\zeta)}{2}$. Integrating the equality (15), we obtain

$$\log\left\{\frac{\mathcal{T}_{\lambda,\delta}^m L(\zeta)}{\zeta}\right\} = \int_0^\zeta \left(\frac{b[\aleph(w(t)) - \aleph(w(-t))]}{t[1-(-1)^p]} - \frac{1}{t}\right) dt,$$

or, equivalently,

$$\mathcal{T}_{\lambda,\delta}^m L(\zeta) = \zeta\, \exp\left\{\int_0^\zeta \left(\frac{b[\aleph(w(t)) - \aleph(w(-t))]}{t[1-(-1)^p]} - \frac{1}{t}\right) dt\right\}.$$

On summarizing the above discussion, we have the following.

Theorem 1. *Let $\chi \in \Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$ and $\aleph(\zeta)$ be defined as in (12), then we have*

$$\mathcal{T}_{\lambda,\delta}^{m} L(\zeta) = \zeta \, \exp\left\{ \int_0^{\zeta} \left(\frac{b[\aleph(w(t)) - \aleph(w(-t))]}{t[1-(-1)^p]} - \frac{1}{t} \right) dt \right\} \tag{16}$$

where the odd function $L(\zeta)$ is defined by the equality $L(\zeta) = \frac{1}{2}[\chi(\zeta) - \chi(-\zeta)]$, $w(\zeta)$ is analytic in Ω and $w(0) = 0$, $| \, w(\zeta) \, | < 1$.

Remark 2. *Letting $p = \lambda = 1$, $\delta = 1/2$, $\alpha = 0$ and $\Psi(\zeta) = (1+\zeta)/(1-\zeta)$ in Theorem 1, we can obtain the integral representation for the odd function $L(\zeta)$ in the class $\mathcal{Q}_s^b(X, Y, m)$.*

Theorem 2. *If $\chi(\zeta) = \zeta^p + a_{p+1}\zeta^{p+1} + a_{p+2}\zeta^{p+2} + \cdots \in \Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$, then, for odd values of p, we have*

$$|a_{p+1}| \le \frac{p|b|(X-Y)(p-\alpha)|\ell_1|}{4(p+\lambda)(1-\delta)}, \tag{17}$$

and

$$\begin{aligned} |a_{p+2}| \le & \frac{p|b|(X-Y)(p-\alpha)|\ell_1|}{2(p+2)(p+2\lambda)(1-\delta)(3-2\delta)} \\ & \max\left\{1;\, \left| \frac{(Y+1)\ell_1}{2} - \frac{\ell_2}{\ell_1} - \frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{4(p+\lambda)(1-\delta)} \right| \right\}. \end{aligned} \tag{18}$$

Additionally, for all $\mu \in \mathbb{C}$, we have

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| \le \frac{p(X-Y)(p-\alpha)|b\ell_1|}{2(p+2)(p+2\lambda)(1-\delta)(3-2\delta)} \max\{1, |2\mathcal{Q}_1 - 1|\},$$

where $\mathcal{Q}_1$ is given by

$$\begin{aligned} \mathcal{Q}_1 = \frac{1}{4}\bigg\{ & (Y+1)\ell_1 + 2\left(1 - \frac{\ell_2}{\ell_1}\right) - \frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{2(p+\lambda)(1-\delta)} \\ & + \frac{\mu p b \ell_1 (X-Y)(p-\alpha)(p+2)(p+2\lambda)(3-2\delta)}{4(p+\lambda)^2(1-\delta)} \bigg\}. \end{aligned}$$

The inequality is sharp for each $\mu \in \mathbb{C}$.

Proof. As $\chi \in \Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$, by (11), we have

$$p + \frac{1}{b}\left\{ \frac{[1-(-1)^p]\zeta\left[\mathcal{T}_{\lambda,\delta}^{m}\chi(\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^{m}\chi(\zeta) - \mathcal{T}_{\lambda,\delta}^{m}\chi(-\zeta)\right]} - p \right\} = \aleph[w(\zeta)]. \tag{19}$$

Thus, let $\vartheta \in \mathcal{R}$ be of the form $\vartheta(\zeta) = 1 + \sum_{k=1}^{\infty} \vartheta_n \zeta^n$ and defined by

$$\vartheta(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)}, \quad \zeta \in \Omega.$$

On computation, we have

$$w(\zeta) = \frac{1}{2}\vartheta_1\zeta + \frac{1}{2}\left(\vartheta_2 - \frac{1}{2}\vartheta_1^2\right)\zeta^2 + \frac{1}{2}\left(\vartheta_3 - \vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_1^3\right)\zeta^3 + \cdots, \; \zeta \in \Omega.$$

The right-hand side of (19):

$$p + b\{\aleph[w(\zeta)] - p\} = p + \frac{b\ell_1\vartheta_1(X-Y)(p-\alpha)}{4}\zeta + \frac{b(X-Y)(p-\alpha)\ell_1}{4}\left[\vartheta_2 - \vartheta_1^2\left(\frac{(Y+1)\ell_1 + 2\left(1-\frac{\ell_2}{\ell_1}\right)}{4}\right)\right]\zeta^2 + \cdots. \tag{20}$$

For odd values of p, the left hand side of (19) is given by

$$\frac{[1-(-1)^p]\zeta\left[\mathcal{T}^m_{\lambda,\delta}\chi(\zeta)\right]'}{\left[\mathcal{T}^m_{\lambda,\delta}\chi(\zeta) - \mathcal{T}^m_{\lambda,\delta}\chi(-\zeta)\right]} = p + 2\left(1+\frac{\lambda}{p}\right)(1-\delta)a_{p+1}\zeta + (p+2)(1-\delta)\left[(3-2\delta)\left(1+\frac{2\lambda}{p}\right)a_{p+2} - 2\left(1+\frac{\lambda}{p}\right)a_{p+1}^2\right]\zeta^2 + \cdots. \tag{21}$$

From (20) and (21), we obtain

$$a_{p+1} = \frac{pb\ell_1\vartheta_1(X-Y)(p-\alpha)}{8(p+\lambda)(1-\delta)} \tag{22}$$

and

$$a_{p+2} = \frac{b(X-Y)(p-\alpha)\ell_1 p}{4(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left[\vartheta_2 - \frac{1}{4}\left\{(Y+1)\ell_1 + 2\left(1-\frac{\ell_2}{\ell_1}\right) - \frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{2(p+\lambda)(1-\delta)}\right\}\vartheta_1^2\right]. \tag{23}$$

Applying Lemma 1 on (22), we can obtain (17). Using (23) together with Lemma 2, we have

$$\begin{aligned}|a_{p+2}| &= \frac{p|b|(X-Y)(p-\alpha)|\ell_1|}{4(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left|\vartheta_2 - \frac{1}{4}\left\{(Y+1)\ell_1 + 2\left(1-\frac{\ell_2}{\ell_1}\right) - \frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{2(1+\lambda)(1-\delta)}\right\}\vartheta_1^2\right| \\ &\leq \frac{p|b|(X-Y)(p-\alpha)|\ell_1|}{2(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\max\left\{1; \left|\frac{(Y+1)\ell_1}{2} - \frac{\ell_2}{\ell_1} - \frac{pb(X-Y)(p-\alpha)(p+2)\ell_1}{4(p+\lambda)(1-\delta)}\right|\right\}.\end{aligned}$$

Hence, the proof of (18).

Now, to prove the Fekete–Szegő inequality for the class $\Lambda_p^m(b;\Psi;\alpha;\lambda,\delta;X,Y)$, we consider

$$\begin{aligned}\left|a_{p+2} - \mu a_{p+1}^2\right| &= \left|\frac{b(X-Y)(p-\alpha)\ell_1 p}{4(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left[\vartheta_2 - \frac{1}{4}\left\{(Y+1)\ell_1 + 2\left(1-\frac{\ell_2}{\ell_1}\right) - \frac{pb(X-Y)(p-\alpha)(p+2)\ell_1}{2(p+\lambda)(1-\delta)}\right\}\vartheta_1^2\right] - \frac{\mu p^2 b^2 \ell_1^2 \vartheta_1^2 (X-Y)(p-\alpha)^2}{64(p+\lambda)^2(1-\delta)^2}\right| \\ &= \left|\frac{b(X-Y)(p-\alpha)\ell_1 p}{4(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left[\vartheta_2 - \frac{\vartheta_1^2}{2} + \frac{1}{4}\vartheta_1^2\left(\frac{2\ell_2}{\ell_1} - (Y+1)\ell_1 + \frac{pb(X-Y)(p-\alpha)(p+2)\ell_1}{2(p+\lambda)(1-\delta)} - \frac{\mu pb\ell_1(X-Y)(p-\alpha)(p+2)(p+2\lambda)(3-2\delta)}{4(p+\lambda)^2(1-\delta)}\right)\right]\right|\end{aligned}$$

$$\leq \frac{|b|(X-Y)(p-\alpha)|\ell_1|p}{4(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left[2+\frac{|\vartheta_1|^2}{4}\left(\left|\frac{2\ell_2}{\ell_1}-(Y+1)\ell_1\right.\right.\right.$$
$$+\frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{2(p+\lambda)(1-\delta)}$$
$$\left.\left.\left.-\frac{\mu pb\ell_1(X-Y)(p-\alpha)(p+2)(p+2\lambda)(3-2\delta)}{4(p+\lambda)^2(1-\delta)}\right|-2\right)\right]. \quad (24)$$

Denoting

$$H:=\left|\frac{2\ell_2}{\ell_1}-(Y+1)\ell_1+\frac{pb(X-Y)(p-\alpha)(p+2)\ell_1}{2(p+\lambda)(1-\delta)}\right.$$
$$\left.-\frac{\mu pb\ell_1(X-Y)(p-\alpha)(p+2)(p+2\lambda)(3-2\delta)}{4(p+\lambda)^2(1-\delta)}\right|,$$

if $H\leq 2$, from (24), we obtain

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leq \frac{|b|(X-Y)(p-\alpha)|\ell_1|p}{2(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}. \quad (25)$$

Further, if $H\geq 2$, from (24), we deduce

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leq \frac{|b|(X-Y)(p-\alpha)|\ell_1|p}{2(p+2)(p+2\lambda)(1-\delta)(3-2\delta)}\left(\left|\frac{2\ell_2}{\ell_1}-(Y+1)\ell_1+\right.\right.$$
$$\frac{pb(p+2)(X-Y)(p-\alpha)\ell_1}{2(p+\lambda)(1-\delta)}$$
$$\left.\left.-\frac{\mu pb\ell_1(X-Y)(p-\alpha)(p+2)(p+2\lambda)(3-2\delta)}{4(p+\lambda)^2(1-\delta)}\right|\right). \quad (26)$$

An examination of the proof shows that the equality for (25) holds if $p_1=0$, $p_2=2$. Equivalently, by Lemma 2, we have $p(\zeta^2)=p_2(\zeta)=\frac{1+\zeta^2}{1-\zeta^2}$. Therefore, the extremal function of the class $\Lambda_p^m(b;\,\Psi;\,\alpha;\,\lambda,\delta;\,X,Y)$ is given by

$$p+\frac{1}{b}\left\{\frac{[1-(-1)^p]\zeta\left[\mathcal{T}_{\lambda,\delta}^m\chi(\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^m\chi(\zeta)-\mathcal{T}_{\lambda,\delta}^m\chi(-\zeta)\right]}-p\right\}$$
$$=\frac{[(1+X)p+\alpha(Y-X)]p(\zeta^2)+[(1-X)p-\alpha(Y-X)]}{[(Y+1)p(\zeta^2)+(1-Y)]}.$$

Similarly, the equality for (25) holds if $p_2=2$. Equivalently, by Lemma 2, we have $p(\zeta)=p_1(\zeta)=\frac{1+\zeta}{1-\zeta}$. Therefore, the extremal function in $\Lambda_p^m(b;\,\Psi;\,\alpha;\,\lambda,\delta;\,X,Y)$ is given by

$$p+\frac{1}{b}\left\{\frac{[1-(-1)^p]\zeta\left[\mathcal{T}_{\lambda,\delta}^m\chi(\zeta)\right]'}{\left[\mathcal{T}_{\lambda,\delta}^m\chi(\zeta)-\mathcal{T}_{\lambda,\delta}^m\chi(-\zeta)\right]}-p\right\}$$
$$=\frac{[(1+X)p+\alpha(Y-X)]p_1(\zeta)+[(1-X)p-\alpha(Y-X)]}{[(Y+1)p_1(\zeta)+(1-Y)]},$$

and the proof of the theorem is complete. □

3. Subclasses of Analytic Functions Using Quantum Derivative

In this section, we define a q-analogue of the operator $\mathcal{T}^m_{\lambda,\delta}\chi(\zeta)$ defined in Section 1. The study of *Geometric Function Theory* in dual with quantum calculus was initiated by Srivastava [14]. For recent developments and applications of quantum calculus in *Geometric Function Theory*, refer to the recent survey-cum-expository article by Srivastava [15] and references provided therein.

Now, we give a very brief introduction of the q-calculus. We let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C}).$$

For $\chi \in \Pi_p$, the *Jackson's q-derivative operator* or *q-difference operator* for a function $\chi \in \Pi_p$ is defined by

$$D_q\chi(\zeta) := \begin{cases} \chi'(0), & \text{if } \zeta = 0, \\ \dfrac{\chi(\zeta) - \chi(q\zeta)}{(1-q)\zeta}, & \text{if } \zeta \neq 0. \end{cases} \tag{27}$$

From (27), if $\chi \in \Pi_p$, we can easily see that $D_q\chi(\zeta) = [p]_q\zeta^{p-1} + \sum\limits_{k=p+1}^{\infty} [p+k]_q a_{p+k}\zeta^{p+k-1}$, for $\zeta \neq 0$ and note that $\lim\limits_{q\to 1-} D_q\chi(\zeta) = \chi'(\zeta)$. The *q-Jackson integral* is defined by (see [16])

$$I_q[\chi(\zeta)] := \int_0^{\zeta} \chi(t)d_qt = \zeta(1-q)\sum_{k=0}^{\infty} q^k\chi(\zeta q^k) \tag{28}$$

provided the q-series converges. Further, we observe that

$$D_qI_q\chi(\zeta) = \chi(\zeta) \qquad \text{and} \qquad I_qD_q\chi(\zeta) = \chi(\zeta) - \chi(0),$$

where the second equality holds if χ is continuous at $\zeta = 0$.

The class of q-starlike functions introduced by Ismail et al. in [17] is defined as the class of functions which satisfies the condition

$$\left| \frac{\zeta D_q\chi(\zeta)}{\chi(\zeta)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \qquad (\chi \in \mathcal{S}).$$

Here, we let $\mathcal{S}^*_q$ denote the class of q-starlike functions. Equivalently, a function $\chi \in \mathcal{S}^*_q$, if and only if the subordination condition (see ([18], Definition 7))

$$\frac{zD_q\chi(\zeta)}{\chi(\zeta)} \prec \frac{1+\zeta}{1-q\zeta},$$

holds.

The q-analogue of the function $\mathcal{K}_p(\delta)$ defined as in (7) is given by

$$q - \mathcal{K}_\delta(\zeta) = \frac{\zeta^p}{(1-q\zeta)^{1-\delta}(1-\zeta)^{1-\delta}} = \zeta^p + \sum_{k=1}^{\infty} \Theta^k_\delta \zeta^{p+k}, \tag{29}$$

with $\Theta^k_\delta = \frac{\prod_{i=2}^{k+1}([i]_q - [2]_q\delta)}{k!}$, $0 \leq \delta < 1$, $p = 1, 2, 3, \ldots$. Srivastava et al. [18,19] introduced function classes of q-starlike functions related with conic regions and also studied the impact of Janowski functions on those conic regions. Inspired by the aforementioned works on q-calculus, we now define the q-analogue of the operator $\mathcal{QT}^m_{\lambda,\delta}\chi(\zeta)$ as follows:

$$\mathcal{QT}^m_{\lambda,\delta}\chi(\zeta) = \zeta^p + \sum_{k=1}^{\infty}\left[(1-\lambda) + \frac{\lambda[p+k]_q}{[p]_q}\right]^m \Theta^k_\delta\, a_{p+k}\,\zeta^{p+k}. \tag{30}$$

The function $\hat{p}_{\nu,\sigma}(\zeta)$ plays the role of those extremal functions related to the conic domain and is given by

$$\hat{p}_{\nu,\sigma}(\zeta) = \begin{cases} \frac{1+(1-2\sigma)\zeta}{1-\zeta}, & \text{if } \nu = 0 \\ 1 + \frac{2(1-\sigma)}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2, & \text{if } \nu = 1 \\ 1 + \frac{2(1-\sigma)}{1-\nu^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\nu\right)\arctan h\sqrt{\zeta}\right], & \text{if } 0 < \nu < 1 \\ 1 + \frac{2(1-\sigma)}{1-\nu^2}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(\zeta)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{\nu^2-1}, & \text{if } \nu > 1 \end{cases} \tag{31}$$

where $u(\zeta) = \frac{\zeta-\sqrt{t}}{1-\sqrt{t\zeta}}$, $t \in (0,1)$ and t is chosen such that $\nu = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, where $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{\nu,\sigma}(\zeta)$ is in $\mathcal{R}$, with the expansion of the form

$$\hat{p}_{\nu,\sigma}(\zeta) = 1 + \tau_1\zeta + \tau_2\zeta^2 + \cdots, \qquad (\tau_j = p_j(\nu,\sigma),\ j = 1,2,3,\ldots), \tag{32}$$

we obtain

$$\tau_1 = \begin{cases} \frac{8(1-\sigma)(\arccos\nu)^2}{\pi^2(1-\nu^2)}, & \text{if } 0 \le \nu < 1, \\ \frac{8(1-\sigma)}{\pi^2}, & \text{if } \nu = 1 \\ \frac{\pi^2(1-\sigma)}{4\sqrt{t}(\nu^2-1)R^2(t)(1+t)}, & \text{if } \nu > 1. \end{cases} \tag{33}$$

Instead of defining the same class of functions defined in Definition 1 involving quantum derivative, we define a class (motivated by the study of [12] (Definition 1.2)) involving additional parameters.

Definition 2. *For $u, v \in \mathbb{C}$, with $u \neq v$, $|v| \le 1$, let the class $Q_p^m(u,v;\, b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$ consist of a function in Π_p satisfying the subordination condition*

$$[p]_q + \frac{1}{b}\left\{\frac{[u^p - v^p]\zeta D_q\left[\mathcal{QT}^m_{\lambda,\delta}\chi(\zeta)\right]}{\left[\mathcal{QT}^m_{\lambda,\delta}\chi(u\zeta) - \mathcal{QT}^m_{\lambda,\delta}\chi(v\zeta)\right]} - [p]_q\right\} \prec$$
$$\frac{\left[(1+X)[p]_q + \alpha(Y-X)\right]\Psi(\zeta) + \left[(1-X)[p]_q - \alpha(Y-X)\right]}{\left[(Y+1)\Psi(\zeta) + (1-Y)\right]} := \aleph_q(\zeta), \tag{34}$$

where $b \in \mathbb{C}\setminus\{0\}$ and $\aleph_q(\zeta) = [p]_q + \sum_{k=1}^{\infty} b_k\zeta^k \in \mathcal{R}$.

Remark 3. *Unlike the function class $\Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$, the presence of u and v in the function class $Q_p^m(u,v;\, b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$ unifies the various subclasses of analytic functions and classes of functions with respect to symmetric points. Now, we list some special cases:*

1. *For a choice of the parameters $u = 1, v = -1$ and $q \to 1-$ in (34), the class $Q_p^m(u,v;\, b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$ reduces to the class $\Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$ defined in Definition 1.*
2. *If we let $q \to 1-$, $p = u = b = 1$, $\delta = 1/2$, $m = \alpha = 0$ and $\Psi(\zeta) = p_{k,\sigma}(\zeta)$ (see (31)) in Definition 2, the class $Q_p^m(u,v;\, b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$ reduces to class $k-\mathcal{US}(X,Y,\sigma,t)$ introduced by Arif et al. ([12], Definition 1.3).*

For other special cases of our classes, see [12] (p. 264).

Coefficient Estimates of $Q_p^m(u,v;\, b;\, \Psi;\, \alpha;\, \lambda,\delta;\, X,Y)$

We need the following results to establish our main results.

Lemma 3 ([20], Theorem VII). *Let $\chi(\zeta) = \sum_{k=1}^{\infty} a_k\zeta^k$ be analytic in Ω and $g(\zeta) = \sum_{k=1}^{\infty} b_k\zeta^k$ be analytic and convex in Ω. If $\chi(\zeta) \prec g(\zeta)$, then $|a_k| \le |b_1|$ for $k = 1,2,\ldots$.*

Lemma 4 (see [21] (Lemma 6))**.** *Let the function $\aleph_q(\zeta)$, defined as in the right hand side of* (34)*, be convex in Ω where the function Ψ is defined as in* (3)*. If $r(\zeta) = [p]_q + \sum_{k=1}^{\infty} r_k \zeta^k$ is analytic in Ω and satisfies the subordination condition*

$$r(\zeta) \prec \frac{\left[(1+X)[p]_q + \alpha(Y-X)\right]\Psi(\zeta) + \left[(1-X)[p]_q - \alpha(Y-X)\right]}{\left[(Y+1)\Psi(\zeta) + (1-Y)\right]}, \tag{35}$$

then

$$|r_k| \leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)}{2}, \; k \geq 1. \tag{36}$$

Proof. If the function Ψ has the power series expansion (3), then, from (1), we have

$$\aleph(\zeta) = [p]_q + \frac{(X-Y)([p]_q - \alpha)\ell_1}{2}\zeta + \ldots, \; \zeta \in \Omega.$$

Since the subordination relation is invariant under translation, the assumption (35) is equivalent to

$$r(\zeta) - [p]_q \prec \aleph(\zeta) - [p]_q.$$

Further, because the convexity of $\aleph$ implies the convexity of $\aleph(\zeta) - [p]_q$, from Lemma 3, the conclusion follows (36). □

Theorem 3. *Let $\chi \in \mathcal{Q}_p^m(u, v; b; \Psi; \alpha; \lambda, \delta; X, Y)$ and Ψ be chosen so that $\aleph_q(\zeta)$ is convex in Ω. If $|Y+1| < |Y-1|$, then, for $k \geq 1$,*

$$|a_{p+k}| \leq \frac{1}{\Upsilon_k^q[p,\, m,\, \delta]} \prod_{n=0}^{k-1} \frac{\left| \begin{array}{c} b(X-Y)([p]_q-\alpha)\ell_1\left(u^{p+n}-v^{p+n}\right) \\ -2\left([p+n]_q(u^p-v^p)-[p]_q(u^{p+n}-v^{p+n})\right)Y \end{array} \right|}{2\left|[p+n+1]_q(u^p - v^p) - [p]_q\left(u^{p+n+1} - v^{p+n+1}\right)\right|} \tag{37}$$

with $\Upsilon_k^q[p,\, m,\, \delta] = \left[(1-\lambda) + \frac{\lambda[p+k]_q}{[p]_q}\right]^m \frac{\prod_{i=2}^{k+1}([i]_q - [2]_q\delta)}{k!}, 0 \leq \delta < 1,\; p = 1, 2, 3, \ldots$.

Proof. Let us consider

$$[u^p - v^p]\zeta D_q\left[\mathcal{QT}_{\lambda,\delta}^m \chi(\zeta)\right]' = \left\{[p]_q + b\left[h(\zeta) - [p]_q\right]\right\}\left[\mathcal{QT}_{\lambda,\delta}^m \chi(u\zeta) - \mathcal{QT}_{\lambda,\delta}^m \chi(v\zeta)\right] \tag{38}$$

where $h(\zeta) = [p]_q + \sum_{k=1}^{\infty} r_k \zeta^k$ is analytic in Ω and satisfies the subordination condition $h(\zeta) \prec \aleph_q(\zeta)$.

Equivalently, (38) can be rewritten as

$$\begin{aligned} &(u^p - v^p)\left([p]_q\zeta^p + \sum_{k=1}^{\infty}[p+k]_q \Upsilon_k^q[p,\, m,\, \delta]\, a_{p+k}\zeta^{p+k}\right) \\ &= \left[(u^p - v^p)\zeta^p + \sum_{k=1}^{\infty}\left(u^{p+k} - v^{p+k}\right)\Upsilon_k^q[p,\, m,\, \delta]\, a_{p+k}\zeta^{p+k}\right]\left\{[p]_q + b\sum_{k=1}^{\infty} r_k\zeta^k\right\}, \end{aligned}$$

where $\Upsilon_k^q[p,\, m,\, \delta] = \left[(1-\lambda) + \frac{\lambda[p+k]_q}{[p]_q}\right]^m \Theta_\delta^k$. On equating the coefficient of ζ^{p+n}, we obtain

$$\begin{aligned} &(u^p - v^p)[p+n]_q \Upsilon_n^q[p,\, m,\, \delta]\, a_{p+n} = [p]_q\left[u^{p+n} - v^{p+n}\right]\Upsilon_n^q[p,\, m,\, \delta]a_{p+n} + \\ &b\sum_{i=0}^{n-1} r_{n-i}\left[u^{p+i} - v^{p+i}\right]\Upsilon_i^q[p,\, m,\, \delta]a_{p+i}, \quad (a_p = 1,\, \Upsilon_0[p,\, m,\, \delta] = 1). \end{aligned}$$

On computation, we have

$$|a_{p+n}| \leq \frac{|b|}{\big|[p+n]_q(u^p - v^p) - [p]_q(u^{p+n} - v^{p+n})\big|\Upsilon_n^q[p,\, m,\, \delta]} \times \left[\sum_{i=0}^{n-1} |r_{n-i}|\left|u^{p+i} - v^{p+i}\right|\Upsilon_i^q[p,\, m,\, \delta]|a_{p+i}|\right].$$

Using (36) in the above inequality, we have

$$|a_{p+n}| \leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)|b|}{2\big|[p+n]_q(u^p - v^p) - [p]_q(u^{p+n} - v^{p+n})\big|\Upsilon_n^q[p,\, m,\, \delta]} \sum_{i=0}^{n-1}\left|u^{p+i} - v^{p+i}\right|\Upsilon_i^q[p,\, m,\, \delta]|a_{p+i}|, \tag{39}$$

where $a_p = 1$, $\Upsilon_0[p,\, m,\, \delta] = 1$. Taking $n = 1$ in (39), we obtain

$$|a_{p+1}| \leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)|u^p - v^p||b|}{2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big|\Upsilon_1^q[p,\, m,\, \delta]}.$$

The hypothesis is true for $n = 1$. Now, let $n = 2$ in (39); we obtain

$$\begin{aligned}
|a_{p+2}| &\leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)|b|}{2\big|[p+2]_q(u^p - v^p) - [p]_q\big(u^{p+2} - v^{p+2}\big)\big|\Upsilon_2^q[p,\, m,\, \delta]} \times \left\{|u^p - v^p| + \Upsilon_1^q[p,\, m,\, \delta]\left|u^{p+1} - v^{p+1}\right||a_{p+1}|\right\} \\
&\leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)|b||u^p - v^p|}{2\big|[p+2]_q(u^p - v^p) - [p]_q\big(u^{p+2} - v^{p+2}\big)\big|\Upsilon_2^q[p,\, m,\, \delta]} \left\{1 + \frac{|\ell_1|(X-Y)([p]_q - \alpha)\big|u^{p+1} - v^{p+1}\big||b|}{2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big|}\right\}.
\end{aligned}$$

If we let $k = 2$ in (37), we have

$$\begin{aligned}
|a_{p+2}| &\leq \frac{1}{\Upsilon_2^q[p,\, m,\, \delta]}\left[\frac{|\ell_1|(X-Y)([p]_q - \alpha)|u^p - v^p||b|}{2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big|}\right. \\
&\quad \left. \times \frac{\big|b(X-Y)([p]_q - \alpha)\ell_1\big(u^{p+1} - v^{p+1}\big) - 2\big([p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big)Y\big|}{2\big|[p+2]_q(u^p - v^p) - [p]_q\big(u^{p+2} - v^{p+2}\big)\big|}\right] \\
&\leq \frac{1}{\Upsilon_2^q[p,\, m,\, \delta]}\left[\frac{|\ell_1|(X-Y)([p]_q - \alpha)|u^p - v^p||b|}{2\big([p+2]_q(u^p - v^p) - [p]_q\big(u^{p+2} - v^{p+2}\big)\big)}\right. \\
&\quad \left. \times \frac{|b|(X-Y)([p]_q - \alpha)|\ell_1|\big|u^{p+1} - v^{p+1}\big| + 2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big||Y|}{2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big|}\right] \\
&\leq \frac{|\ell_1|(X-Y)([p]_q - \alpha)|b||u^p - v^p|}{2\big|[p+2]_q(u^p - v^p) - [p]_q\big(u^{p+2} - v^{p+2}\big)\big|\Upsilon_2[p,\, m,\, \delta]} \left\{\frac{|\ell_1|(X-Y)([p]_q - \alpha)\big|u^{p+1} - v^{p+1}\big||b|}{2\big|[p+1]_q(u^p - v^p) - [p]_q\big(u^{p+1} - v^{p+1}\big)\big|} + 1\right\}.
\end{aligned}$$

Hence, the hypothesis of the theorem is true for $k = 2$. Following the steps as in [22] (Theorem 2), we can obtain the desired result using mathematical induction. □

If we let $u = 1$, $v = -1$ and $q \to 1-$ in Theorem 3, we obtain the coefficient estimate of the class $\Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$ (see Definition 1).

Corollary 1. *Let $\chi \in \Lambda_p^m(b;\, \Psi;\, \alpha;\, \lambda, \delta;\, X, Y)$ and Ψ be chosen so that $\aleph_q(\zeta)$ is convex in Ω. If $|Y+1| < |Y-1|$, then, for $k \geq 1$,*

$$|a_{p+k}| \leq \frac{1}{\Upsilon_k[p,\, m,\, \delta]} \prod_{n=0}^{k-1} \frac{\left| \begin{array}{c} b(X-Y)(p-\alpha)\ell_1\big(1-(-1)^{p+n}\big) \\ -2((p+n)(1-(-1)^{p})-p(1-(-1)^{p+n}))Y \end{array} \right|}{2\big|(p+n+1)[1-(-1)^p] - p\big[1-(-1)^{p+n+1}\big]\big|},$$

with $\Upsilon_k[p,\, m,\, \delta] = \left[\frac{p+\lambda k}{p}\right]^m \frac{\prod_{i=2}^{k+1}(i-2\delta)}{k!}, 0 \leq \delta < 1,\ p = 1,\, 2,\, 3, \ldots.$

Letting $q \to 1-$, $p = u = b = 1$, $\delta = 1/2$, $m = \alpha = 0$ and $\Psi(\zeta) = p_{\nu,\sigma}(\zeta)$ in Theorem 3, we obtain the following corollary.

Corollary 2 ([12], Theorem 2.3)**.** *Let $\chi \in \nu - \mathcal{US}(X, Y, \sigma, t)$ (see Remark 3), then for $k \geq 2$,*

$$|\ a_k\ | \leq \prod_{n=1}^{k-1} \frac{u_n|\tau_1|(X-Y) + 2(n-u_n)}{2(n+1-u_{n+1})},$$

where τ_1 is defined as in (33) *and $u_n = 1 + v + v^2 + \cdots + v^{n-1}$.*

If $p = u = b = 1$, $v = m = \alpha = 0$, $\delta = 1/2$ and $\Psi(\zeta) = p_{\nu,0}(\zeta)$ in Theorem 3, we obtain the following result.

Corollary 3 ([23], Theorem 2.3)**.** *Let $\chi \in \nu - \mathcal{SL}_q(X, Y)$, then*

$$|\ a_k\ | \leq \prod_{n=1}^{k-1} \frac{|\tau_1(X-Y) - 2([n]_q - 1)Y|}{2([n+1]_q - 1)} \quad (k \geq 2),$$

where τ_1 is defined as in (33) *with $\sigma = 0$.*

If we choose $p = u = b = 1$, $\delta = 1/2$, $v = m = \alpha = 0$, $\Psi(\zeta) = p_{\nu,0}(\zeta)$ and $q \to 1-$ in Theorem 3, we obtain the following corollary.

Corollary 4 ([24], Theorem 2.6)**.** *For a function $p_{\nu,0}(\zeta)$ defined as in* (31) *with $\sigma = 0$, let $\chi \in \Pi$ satisfy the condition*

$$\frac{\zeta\chi'(\zeta)}{\chi(\zeta)} \prec \frac{(X+1)p_{\nu,0}(\zeta) - (X-1)}{(Y+1)p_{\nu,0}(\zeta)(\zeta) - (Y-1)}$$

Then, for $k \geq 2$,

$$|a_k| \leq \prod_{j=0}^{k-2} \frac{|(X-Y)\tau_1 - 2jY|}{2(j+1)}.$$

where τ_1 is defined as in (33) *with $\sigma = 0$.*

Letting $p = u = b = 1$, $v = m = \alpha = \delta = 0$, $\Psi(\zeta) = p_{\nu,\sigma}(\zeta)$ and $q \to 1-$ in Theorem 3, we obtain the following corollary.

Corollary 5. *For a function $p_{\nu,\sigma}(\zeta)$ defined as in* (31)*. Let $\chi \in \Pi$ satisfy the condition*

$$1 + \frac{1}{b}\left(\frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}\right) \prec \frac{(X+1)p_{\nu,\sigma}(\zeta) - (X-1)}{(Y+1)p_{\nu,\sigma}(\zeta) - (Y-1)},$$

Then, for $k = 2,\, 3,\, \ldots,$

$$|a_k| \leq \frac{1}{k}\prod_{j=0}^{k-2}\frac{|(X-Y)b\tau_1 - 2jY|}{2(j+1)},$$

where τ_1 *is defined as in* (33).

By putting $p = u = b = 1$, $X = 1 - 2\eta$, $0 \leq \eta < 1$, $Y = -1$, $\delta = 1/2$ $v = m = \alpha = 0$ and $\Psi(\zeta) = p_{\nu,\sigma}(\zeta)$, as $q \to 1-$ in Theorem 3, we obtain the coefficient bounds for the class $\mathcal{SD}(k,\eta)$, defined by Shams et al. [25].

Corollary 6. *Let* $\chi \in \mathcal{SD}(\nu,\eta)$. *Then,*

$$|\ a_k\ | \leq \prod_{j=0}^{k-2}\frac{|\delta_1(1-\gamma)+j|}{j+1} \quad (k \geq 2). \tag{40}$$

The inequality (40) *is better than the result obtained by Owa et al.* [26].

Letting $X = 1$, $Y = -1$, $v = \alpha = 0$, $\delta = 1/2$, $p = b = u = 1$ and $\Psi(\zeta) = 1 + \frac{\gamma-\eta}{\pi} i \log\left(\frac{1-e^{2\pi i((1-\gamma)/(\gamma-\eta))}\zeta}{1-\zeta}\right)$.

Corollary 7 ([27], Theorem 2.1)**.** *Let the function* $\chi \in \Pi$ *satisfy the condition*

$$\gamma < \Re\left(\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}\right) < \eta,$$

Then, for $k = 2, 3, \ldots,$

$$|a_k| \leq \frac{1}{(k-1)!}\prod_{n=2}^{k}\left[n - 2 + \frac{2(\eta-\gamma)}{\pi}\sin\frac{\pi(1-\gamma)}{\eta-\gamma}\right].$$

4. Conclusions

Using the Hadamard product, we define a new family of multivalent differential operator involving the convex combinations of analytic functions. Using the newly defined operator, the family $\Lambda_p^m(b;\ \Psi;\ \alpha;\ \lambda,\delta;\ X,Y)$ of multivalent functions of complex order with respect to symmetric points is defined to unify the study of various classes of p-valent functions. Inclusion relationship and solution to the Fekete–Szegő problem for the defined function class are here established.

Further, a more comprehensive class of multivalent functions involving quantum calculus is introduced. Srivastava, in [15] (Equation (9.4)), showed that all the results investigated using quantum derivative (q-derivative) can be translated into the corresponding so called post-quantum analogues ((r, q)-derivative) using a straightforward parametric and argument variation of the following types:

$$D_{r,q}\chi(\zeta) = D_{\frac{q}{r}}\chi(r\zeta) \quad \text{and} \quad D_q\chi(\zeta) = D_{r,rq}\chi\left(\frac{\zeta}{r}\right), \quad (0 < q < r \leq 1).$$

Hence, the additional parameter r is unnecessary; therefore, here, we restrict our study with a q-derivative rather than extending it to a $(r,\ q)$-derivative. Numerous q-results obtained by various authors are shown as special cases of our main results.

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