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Harmonic Aggregation Operator with Trapezoidal Picture Fuzzy Numbers and Its Application in a Multiple-Attribute Decision-Making Problem

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Abstract: Picture fuzzy sets (PFSs) can be used to handle real-life problems with uncertainty and vagueness more effectively than intuitionistic fuzzy sets (IFSs). In the process of information aggregation, many aggregation operators under PFSs are used by different authors in different fields. In this article, a multi-attribute decision-making (MADM) problem is introduced utilizing harmonic mean aggregation operators with trapezoidal fuzzy number (TrFN) under picture fuzzy information. Three harmonic mean operators are developed namely trapezoidal picture fuzzy weighted harmonic mean (TrPFWHM) operator, trapezoidal picture fuzzy order weighted harmonic mean (TrPFOWHM) operator and trapezoidal picture fuzzy hybrid harmonic mean (TrPFHHM) operator. The related properties about these operators are also studied. At last, an MADM problem is considered to interrelate among these operators. Furthermore, a numerical instance is considered to explain the productivity of the proposed operators.

Keywords: trapezoidal picture fuzzy number (TrPFN); TrPFWHM operator; TrPFOWHM operator; TrPFHHM operator; MADM

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1. Introduction

MADM plays a vital role in decision-making science. It is important to various field such as economics, engineering and management. Due to the uncertainty and vagueness of data, it is very laborious to consider the attribute values as real numbers. FSs theory fixed this issue by considering the membership grades of the elements. In the present day, many researchers are interested about the subject. In an MADM problem information aggregation is a common process to ranking the alternatives. The main contribution of this study is development of harmonic aggregation operator under TrPFN. We propose three operators: TrPFWHM, TrPFOWHM and TrPFHHM operators and their related properties.

1.1. Research Background

In 1965, Zadeh [1] proposed FSs theory which is an augmentation of crisp set and can deal with uncertainty and vagueness. In FSs theory, a membership grade of the element is available which indicates the importance of the element. Attanassov [2] introduced IFSs theory which is an augmentation of FSs theory. In IFSs, there is a membership (μ) grade and a non-membership (ν) grade of the element, such that their sum does not exceed 1—that is, $\mu + \nu \leq 1$. It is seen that FSs are IFSs but IFSs are not necessarily FSs. IFSs theory has been applied by various researchers in different fields. Attanassov and Gargov [3] proposed the interval-valued intuitionistic fuzzy set (IVIFS) theory. However, the concept of neutrality was not present in the IFSs theory. As, for example, in the voting process, the voters are divided in three groups: yes, no, and refusal. The group "yes" means that the voters select the candidate; the group "no" means that the voters do not select the candidate; the group "refusal" means that the voters neither select nor reject the candidate. For this issue, Cuong [4,5] introduced PFSs in which positive membership grade (μ), negative membership grade (ν) as well as neutral membership grade (η) is present, such that their sum does not exceed 1—that is, $\mu + \eta + \nu \leq 1$.

1.2. Literature Review

Zadeh [1] replace ordinary set theory by FS theory to fix uncertainty and fuzziness in a real-life situation. In FSs theory, the sum of membership grades of belongingness and not belongingness of an element is exactly equal to 1. This problem motivates the researchers to extend the FSs theory. There are several extension of Zadeh's FSs theory, among which these extensions, IFSs, Pythagorean fuzzy set (PFSs), Fermatean fuzzy set (FFSs), Picture fuzzy set (PFSs), etc., are often used in the literature. In the year 1986, Attanassov [2,3] introduced IFSs theory, which is the generalization of ordinary FSs theory containing membership, non-membership and indeterminacy degree and also in the year 1989 expanded it in IVIFSs theory. Later in the year 2013, Cuong [4–6] introduced PFSs theory in place of Attanassov's IFSs theory. For the MADM problem, the aggregation of information in a real scenario may not always be easy. In this regard, various aggregation operators under IFSs, PFSs, FFSs are developed by various authors. Along with Xu, [7] proposed aggregation operators under IFSs. Xu and Yager [8] developed some geometric aggregation operators under IFSs. Harmonic mean reduces the effect of asymmetric distribution of data, which is the turning point of information aggregation. For instance, Xu [9] developed harmonic aggregation operators. Power Harmonic aggregation operators under Trapezoidal intuitionistic fuzzy sets (TrIFSs) environment and harmonic aggregation operators under IFSs are developed by Das and Guha [10,11]. For an issue in IFSs, aggregation operators are developed by using PFSs. Garg [12] proposed an MADM problem by using aggregation operators under PFSs. Jana and Pal [13] proposed a assessment of enterprise performance by hamacher aggregation operators under PFSs. Jana [14] and coworkers also proposed an MADM problem under PFSs utilizing Dombi operations. Later different aggregation operators, similarity measures, correlation coefficient, distance measure under PFSs are developed by various authors for smooth running of MADM problem [15–33]. Applications of hesitant fuzzy set (HFSs) are useful in MADM problem for information aggregation. Donyatalab, Farrokhizadeh, and Seyfi [34] proposed harmonic mean aggregation operators in spherical fuzzy environment. Zhao, Xu, and Cui [35] developed a group decision making under hesitant fuzzy harmonic mean operators. Zhou, Balezentis, and Streimikiene [36] proposed weighted Bonferroni harmonic mean operator. Lalotra and Sing [37] proposed an MADM problem under HFSs and applied it in knowledge measure. Saikia, Garg and Dutta [38] proposed an MADM problem with novel distance measure under HFSs. Rahman and Abdullah [39] presented Einstein hybrid aggregation operators under IFSs and applied them to the MADM problem. In the existing literature, aggregation was made simple by using TrFN under IFSs, PFSs, FFSs. For instance, Shaw and Roy [40] proposed some arithmetic mean operator under TrIFN. Aydin, Kahraman, and Kabak [41] developed an MADM method for harmonic mean operators under trapezoidal Pythagorean fuzzy number. Deli [42] proposed a TOPSIS method using TrFN under HFSs and applied it to the robot selection process. Many researchers have introduced aggregation operators under IFSs, PFSs, and FFSs. In this article, we have introduced the score and accuracy function to rank the TrPFN and developed TrPFWHM, TrPFOWHM, and TrPFHHM operators which are discussed in the upcoming section.

1.3. Motivation

Due to the presence of a neutrality degree in PFSs, it plays an important role in the selection process. Suppose in an area there are 1000 voters for a candidate, among which, 500 vote for one candidate, 300 vote for the other candidates, and the remaining 200 voters either bypass their vote to "NOTA" or elect not to vote. TrPFN is the hybridization of PFN and TrFN. There are various harmonic aggregation operators present, but only in the PFSs environment. In this paper, we propose a harmonic aggregation operator under TrPFN, due to the presence of neutral membership grades. The harmonic mean operator is the factually usable operator in the information aggregation. If, in the problem, there are exceptional alternatives, then it is very useful for the decision makers, because the information that the harmonic mean operator gives is less important to the exceptional case of preferences. We give an example to illustrate this: Suppose the marks of 10 students in mathematics of a class are given. The obtained marks are 35, 47, 43, 37, 99, 27, 29, 31, 30, and 41 out of 100. We see that 99 is very high mark compared to the other values. We calculate arithmetic mean and harmonic mean as follows:

$$\text{Arithmetic mean} = \frac{35+47+43+37+99+27+29+31+30+41}{10} = 41.90.$$

$$\text{Harmonic mean} = \frac{10}{\frac{1}{35}+\frac{1}{47}+\frac{1}{43}+\frac{1}{37}+\frac{1}{99}+\frac{1}{27}+\frac{1}{29}+\frac{1}{31}+\frac{1}{30}+\frac{1}{41}} = 36.80.$$

If 99 is replaced by 49 and we calculate the arithmetic mean and harmonic mean, then we have the result: respectively, 36.90 and 35.46. Therefore, the result obtained by harmonic mean is better than arithmetic mean. Harmonic mean reduced the effect of the abnormal value 99 to the mean. This motivated us to work with picture fuzzy harmonic mean operator.

1.4. Framework of This Study

The paper is arranged as follows: After the introduction, Section 2 contains some preliminary concepts of PFSs, TrPFNs, harmonic mean (HM), and weighted harmonic mean (WHM), and some useful operations related to this paper. Section 3 contains information regarding the TrPFWHM operator, TrPFOWHM operator, and TrPFHHM operator and their related properties. An MADM problem related to these operators is constructed using TrPFWHM and TrPFHHM operators in Section 4. A numerical instance and a comparative study of the proposed method is given in Section 5 to illustrate the advantage of the proposed method. All results and discussion are in Section 6. Section 7 contains the conclusions and future scope of the proposed work.

2. Preliminaries

2.1. Basics of Picture Fuzzy Number (PFN) and TrPFN

In this portion, we discuss some primary concepts about PFSs, PFN, and TrPFNs and some useful operations related to this paper.

Definition 1 ([4])**.** *Let a non-empty set, $\mathcal{X}$, be known as the universal set. A PFS $\mathcal{P}$ on $\mathcal{X}$ is defined as*

$$\mathcal{P} = \{(x, \mu(x), \eta(x), \nu(x)) : x \in \mathcal{X}\},$$

where $\mu(x) : \mathcal{X} \to [0,1]$ is called degree of positive membership, $\eta(x) : \mathcal{X} \to [0,1]$ is called degree of neutral membership, and $\nu(x) : \mathcal{X} \to [0,1]$ is called degree of negative membership. The membership function satisfying $0 \leq \mu(x) + \eta(x) + \nu(x) \leq 1$. Furthermore, $\pi(x) = 1 - \mu(x) - \eta(x) - \nu(x)$ is called degree of refusal. For simplicity we denote the PFSs $\mathcal{P} = \{(x, \mu(x), \eta(x), \nu(x)) : x \in \mathcal{X}\}$ as $\mathcal{P} = (\mu, \eta, \nu)$ and called PFNs.

Definition 2 ([14])**.** *Let $\mathcal{P}_1 = (\mu_1, \eta_1, \nu_1)$ and $\mathcal{P}_2 = (\mu_2, \eta_2, \nu_2)$ be two PFNs over the universal set $\mathcal{X}$. Then, the subsequent operations are as follows:*

1. $\bar{\mathcal{P}}_1 = (\nu_1, \eta_1, \mu_1)$.
2. $\mathcal{P}_1 \wedge \mathcal{P}_2 = (\min\{\mu_1, \mu_2\}, \max\{\eta_1, \eta_2\}, \max\{\nu_1, \nu_2\})$.
3. $\mathcal{P}_1 \vee \mathcal{P}_2 = (\max\{\mu_1, \mu_2\}, \min\{\eta_1, \eta_2\}, \min\{\nu_1, \nu_2\})$.
4. $\mathcal{P}_1 \oplus \mathcal{P}_2 = (\mu_1 + \mu_2 - \mu_1\mu_2, \eta_1\eta_2, \nu_1\nu_2)$.

5. $\mathcal{P}_1 \otimes \mathcal{P}_2 = (\mu_1\mu_2, \eta_1 + \eta_2 - \eta_1\eta_2, \nu_1 + \nu_2 - \nu_1\nu_2)$.
6. $\lambda\mathcal{P}_1 = (1 - (1-\mu_1)^\lambda, \eta_1^\lambda, \nu_1^\lambda)$.
7. $\mathcal{P}_1^\lambda = (\mu_1^\lambda, 1 - (1-\eta_1)^\lambda, 1 - (1-\nu_1)^\lambda)$

Definition 3 ([15]). *The membership function of the TrFN $\mathcal{T} = (\hat{f}, \hat{e}, \hat{h}, \hat{g})$ is given by*

$$\mu_{\mathcal{T}}(x) = \begin{cases} 0 & \text{when } x < \hat{f}, \\ \frac{x-\hat{f}}{\hat{e}-\hat{f}} & \text{when } \hat{f} \le x \le \hat{e}, \\ 1 & \text{when } \hat{e} \le x \le \hat{h}, \\ \frac{x-\hat{g}}{\hat{h}-\hat{g}} & \text{when } \hat{h} \le x \le \hat{g}, \\ 0 & \text{when } x > \hat{g}. \end{cases}$$

Definition 4 ([41]). *Let $\hat{f}$, $\hat{e}$, $\hat{h}$, $\hat{g}$ be non zero number in $[0,1]$. Then $\mathcal{P} = [(\hat{f}, \hat{e}, \hat{h}, \hat{g}); \mu, \eta, \nu]$ over the universal set $\mathcal{X}$ is called positive TrPFN.*

Definition 5 ([41]). *Let $\mathcal{P}_1 = [(\hat{f}_1, \hat{e}_1, \hat{h}_1, \hat{g}_1); \mu_1, \eta_1, \nu_1]$ and $\mathcal{P}_2 = [(\hat{f}_2, \hat{e}_2, \hat{h}_2, \hat{g}_2); \mu_2, \eta_2, \nu_2]$ be two positive TrPFNs. Then, the subsequent operations are as follows:*

1. $\mathcal{P}_1 + \mathcal{P}_2 = [(\hat{f}_1 + \hat{f}_2, \hat{e}_1 + \hat{e}_2, \hat{h}_1 + \hat{h}_2, \hat{g}_1 + \hat{g}_2); \mu_1 + \mu_2 - \mu_1\mu_2, \eta_1\eta_2, \nu_1\nu_2]$.
2. $\mathcal{P}_1 \times \mathcal{P}_2 = [(\hat{f}_1\hat{f}_2, \hat{e}_1\hat{e}_2, \hat{h}_1\hat{h}_2, t_1\hat{g}_2); \mu_1\mu_2, \eta_1 + \eta_2 - \eta_1\eta_2, \nu_1 + \nu_2 - \nu_1\nu_2]$
3. $\lambda\mathcal{P}_1 = [(\lambda\hat{f}_1, \lambda\hat{e}_1, \lambda\hat{h}_1, \lambda\hat{g}_1); 1 - (1-\mu_1)^\lambda, \eta_1^\lambda, \nu_1^\lambda]$, $\lambda \ge 0$.
4. $\mathcal{P}^\lambda = [(\hat{f}_1^\lambda, \hat{e}_1^\lambda, \hat{h}_1^\lambda, \hat{g}_1^\lambda); \mu_1^\lambda, 1 - (1-\eta_1)^\lambda, 1 - (1-\nu_1)^\lambda]$, $\lambda \ge 0$.

Definition 6 ([41]). *Let $\mathcal{P} = [(\hat{f}, \hat{e}, \hat{h}, \hat{g}); \mu, \eta, \nu]$ be a positive TrPFN, then*

$$\frac{1}{\mathcal{P}} = \mathcal{P}^{-1} = [(\frac{1}{\hat{g}}, \frac{1}{\hat{h}}, \frac{1}{\hat{e}}, \frac{1}{\hat{f}}); \mu, \eta, \nu]. \tag{1}$$

2.2. HM and WHM

Definition 7 ([41]). *Let $\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_k$ be k real numbers. Then the HM of k numbers is calculated as*

$$\mathcal{M}_{harmonic}(\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_k) = \frac{k}{\frac{1}{\tilde{a}_1} + \frac{1}{\tilde{a}_2} + \ldots + \frac{1}{\tilde{a}_k}} = \frac{k}{\sum\limits_{r=1}^{k} \frac{1}{\tilde{a}_r}}. \tag{2}$$

Definition 8 ([41]). *Let $\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_k$ be k real numbers. Then the WHM of k numbers is calculated as*

$$\mathcal{M}_{weighted\ harmonic}(\tilde{a}_1, \tilde{a}_2, \ldots \tilde{a}_k) = \frac{k}{\frac{\hat{\tau}_1}{\tilde{a}_1} + \frac{\hat{\tau}_2}{\tilde{a}_2} + \ldots + \frac{\hat{\tau}_k}{\tilde{a}_k}} = \frac{k}{\sum\limits_{r=1}^{k} \frac{\hat{\tau}_r}{\tilde{a}_r}} \tag{3}$$

where, the weight vector $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_k)^T$ of a_r for $r = 1, 2, \ldots, k$ and $\sum\limits_{r=1}^{k} \hat{\tau}_r = 1$.

3. Different WHM Operators for TrPFN

Definition 9. *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$. A TrPFWHM operator is a mapping $f^{\hat{\xi}}_{TrPFWHM} : \mathcal{P}^k \to \mathcal{P}$ and*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\mathcal{P}_r}} \tag{4}$$

where, $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ is the associated weight vector of $\mathcal{P}_r$ for $r = 1, 2, \ldots, k$ and $\sum\limits_{r=1}^{k} \hat{\xi}_r = 1$.

Theorem 1. *Let $\mathcal{P}_r = \left[(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r\right]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$ and the associated weight vector of $\mathcal{P}_r$ is $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ for $r = 1, 2, \ldots, k$ and $\sum_{r=1}^{k} \hat{\xi}_r = 1$ then*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = \left[\left(\frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{k} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{k} \nu_r^{\hat{\xi}_r}\right] \quad (5)$$

Proof. When $k = 2$, then

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\mathcal{P}_r}} = \frac{1}{\frac{\hat{\xi}_1}{\mathcal{P}_1} + \frac{\hat{\xi}_2}{\mathcal{P}_2}}$$

$$= \frac{1}{\frac{\hat{\xi}_1}{[(\hat{f}_1, \hat{e}_1, \hat{h}_1, \hat{g}_1); \mu_1, \eta_1, \nu_1]} + \frac{\hat{\xi}_2}{[(\hat{f}_2, \hat{e}_2, \hat{h}_2, \hat{g}_2); \mu_2, \eta_2, \nu_2]}}$$

$$= \frac{1}{\hat{\xi}_1\left[\left(\frac{1}{\hat{g}_1}, \frac{1}{\hat{h}_1}, \frac{1}{\hat{e}_1}, \frac{1}{\hat{f}_1}\right); \mu_1, \eta_1, \nu_1\right] + \hat{\xi}_2\left[\left(\frac{1}{\hat{g}_2}, \frac{1}{\hat{h}_2}, \frac{1}{\hat{e}_2}, \frac{1}{\hat{f}_2}\right); \mu_2, \eta_2, \nu_2\right]}$$

$$= \frac{1}{\left[\left(\frac{\hat{\xi}_1}{\hat{g}_1}, \frac{\hat{\xi}_1}{\hat{h}_1}, \frac{\hat{\xi}_1}{\hat{e}_1}, \frac{\hat{\xi}_1}{\hat{f}_1}\right); 1-(1-\mu_1)^{\hat{\xi}_1}, \eta_1^{\hat{\xi}_1}, \nu_1^{\hat{\xi}_1}\right] + \left[\left(\frac{\hat{\xi}_2}{t_2}, \frac{\hat{\xi}_2}{\hat{h}_2}, \frac{\hat{\xi}_2}{\hat{e}_2}, \frac{\hat{\xi}_2}{\hat{f}_2}\right); 1-(1-\mu_2)^{\hat{\xi}_2}, \eta_2^{\hat{\xi}_2}, \nu_2^{\hat{\xi}_2}\right]}$$

$$= \frac{1}{\left[\left(\frac{\hat{\xi}_1}{\hat{g}_1} + \frac{\hat{\xi}_2}{\hat{g}_2}, \frac{\hat{\xi}_1}{\hat{h}_1} + \frac{\hat{\xi}_2}{\hat{h}_2}, \frac{\hat{\xi}_1}{\hat{e}_1} + \frac{\hat{\xi}_2}{\hat{e}_2}, \frac{\hat{\xi}_1}{\hat{f}_1} + \frac{\hat{\xi}_2}{\hat{f}_2}\right); 1-(1-\mu_1)^{\hat{\xi}_1} + 1-(1-\mu_2)^{\hat{\xi}_2} - \left(1-(1-\mu_1)^{\hat{\xi}_1}\right)\left(1-(1-\mu_2)^{\hat{\xi}_2}\right), \eta_1^{\hat{\xi}_1}\eta_2^{\hat{\xi}_2}, \nu_1^{\hat{\xi}_1}\nu_2^{\hat{\xi}_2}\right]}$$

$$= \frac{1}{\left[\left(\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{g}_r}, \sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{h}_r}, \sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{e}_r}, \sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{f}_r}\right); 1 - \prod_{r=1}^{2}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{2} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{2} \nu_r^{\hat{\xi}_r}\right]}$$

$$= \left[\left(\frac{1}{\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{2}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{2} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{2} \nu_r^{\hat{\xi}_r}\right]$$

Assume that the Theorem 1 is true for $k = q$.

$\therefore f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_q)$

$$= \left[\left(\frac{1}{\sum_{r=1}^{q} \frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{q} \frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{q} \frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{q} \frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{q}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{q} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{q} \nu_r^{\hat{\xi}_r}\right]$$

$$\therefore f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_q, \mathcal{P}_{q+1}) = \frac{1}{\sum_{r=1}^{q} \frac{\hat{\xi}_r}{\mathcal{P}_r} + \frac{\hat{\xi}_{q+1}}{\mathcal{P}_{q+1}}}$$

$$= \left[\left(\frac{1}{\sum_{r=1}^{q+1} \frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{q+1} \frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{q+1} \frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{q+1} \frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{q+1}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{q+1} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{q+1} \nu_r^{\hat{\xi}_r}\right]$$

Hence the result. □

Theorem 2 (Idempotency property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of TrPFNs for $r = 1, 2, \ldots, k$. If $\mathcal{P}_r = \mathcal{P}$ for all r that is all are identical then,*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}, \mathcal{P}, \ldots, \mathcal{P}) = \mathcal{P}. \tag{6}$$

Proof. We know that

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$$

$$= \left[\left(\frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{k} \eta_r^{\hat{\xi}_r}, \prod_{r=1}^{k} \nu_r^{\hat{\xi}_r}\right]$$

$$= \left[\left(\frac{1}{\frac{\left(\sum_{r=1}^{k} \hat{\xi}_r\right)}{\hat{f}}}, \frac{1}{\frac{\left(\sum_{r=1}^{k} \hat{\xi}_r\right)}{\hat{e}}}, \frac{1}{\frac{\left(\sum_{r=1}^{k} \hat{\xi}_r\right)}{\hat{h}}}, \frac{1}{\frac{\left(\sum_{r=1}^{k} \hat{\xi}_r\right)}{\hat{g}}}\right); 1 - (1-\mu)^{\sum_{r=1}^{k} \hat{\xi}_r}, \eta^{\sum_{r=1}^{k} \hat{\xi}_r}, \nu^{\sum_{r=1}^{k} \hat{\xi}_r}\right]$$

$$= \left[\left(\frac{1}{\frac{1}{\hat{f}}}, \frac{1}{\frac{1}{\hat{e}}}, \frac{1}{\frac{1}{\hat{h}}}, \frac{1}{\frac{1}{\hat{g}}}\right); 1 - (1-\mu), \eta, \nu\right] = [(\hat{f}, \hat{e}, \hat{h}, \hat{g}); \mu, \eta, \nu] = \mathcal{P}.$$

Hence the result. □

Theorem 3 (Monotonicity property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ and $\mathcal{P}'_r = [(\hat{f}'_r, \hat{e}'_r, \hat{h}'_r, \hat{g}'_r); \mu'_r, \eta'_r, \nu'_r]$ be two collection of TrPFNs. If $\hat{f}_r \leq \hat{f}'_r$, $\hat{e}_r \leq \hat{e}'_r$, $\hat{h}_r \leq \hat{h}'_r$, $\hat{g}_r \leq \hat{g}'_r$, $\mu_r \leq \mu'_r$, $\eta_r \geq \eta'_r$, and $\nu_r \geq \nu'_r$. Then*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k) \tag{7}$$

Proof. Since $\hat{f}_r \leq \hat{f}'_r$ and $\hat{\xi}_r \geq 0$ for all r.

$$\therefore \frac{1}{\hat{f}_r} \geq \frac{1}{\hat{f}'_r} \Rightarrow \frac{\hat{\xi}_r}{\hat{f}_r} \geq \frac{\hat{\xi}_r}{\hat{f}'_r} \Rightarrow \sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}_r} \geq \sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}'_r} \Rightarrow \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}_r}} \leq \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}'_r}}$$

Similarly we have the other relations

$$\frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{e}_r}} \leq \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{e}'_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{h}_r}} \leq \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{h}'_r}}, \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{g}_r}} \leq \frac{1}{\sum_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{g}'_r}}$$

Again

$$\mu_r \leq \mu'_r \Rightarrow -\mu_r \geq -\mu'_r \Rightarrow (1-\mu_r) \geq (1-\mu'_r) \Rightarrow (1-\mu_r)^{\hat{\xi}_r} \geq (1-\mu'_r)^{\hat{\xi}_r}$$

$$\Rightarrow \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r} \geq \prod_{r=1}^{k}(1-\mu'_r)^{\hat{\xi}_r} \Rightarrow 1 - \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r} \leq 1 - \prod_{r=1}^{k}(1-\mu'_r)^{\hat{\xi}_r}, \hat{\xi}_r \geq 0$$

$$\eta_r \geq \eta'_r \Rightarrow \eta_r^{\hat{\xi}_r} \geq \eta_r'^{\hat{\xi}_r} \Rightarrow \prod_{r=1}^{k} \eta_r^{\hat{\xi}_r} \geq \prod_{r=1}^{k} \eta_r'^{\hat{\xi}_r}$$

Similarly

$$\prod_{r=1}^{k} \nu_r^{\hat{\xi}_r} \geq \prod_{r=1}^{k} \nu_r'^{\hat{\xi}_r}$$

$$\therefore \left[\left(\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}_r}}\right); \mu_r, \eta_r, \nu_r\right] \leq \left[\left(\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}'_r}}\right); \mu'_r, \eta'_r, \nu'_r\right]$$

$$\Rightarrow f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$$

Hence the result. □

Theorem 4 (Boundedness property). *Let* $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ *be a collection of positive TrPFNs. Let* $\mathcal{P}^- = [(\min\{\hat{f}_r\}, \min\{\hat{e}_r\}, \min\{\hat{h}_r\}, \min\{\hat{g}_r\}); \min\{\mu_r\}, \max\{\eta_r\}, \max\{\nu_r\}]$ *and* $\mathcal{P}^+ = [(\max\{\hat{f}_r\}, \max\{\hat{e}_r\}, \max\{\hat{h}_r\}, \max\{\hat{g}_r\}); \max\{\mu_r\}, \min\{\eta_r\}, \min\{\nu_r\}]$. *Then*

$$\mathcal{P}^- \leq f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq \mathcal{P}^+ \tag{8}$$

Proof. Since

$$\min\{\hat{f}_r\} \leq \hat{f}_r \leq \max\{\hat{f}_r\} \Rightarrow \frac{1}{\min\{\hat{f}_r\}} \geq \frac{1}{\hat{f}_r} \geq \frac{1}{\max\{\hat{f}_r\}}$$

$$\Rightarrow \frac{\hat{\xi}_r}{\min\{\hat{f}_r\}} \geq \frac{\hat{\xi}_r}{\hat{f}_r} \geq \frac{\hat{\xi}_r}{\max\{\hat{f}_r\}}, \hat{\xi}_r \geq 0 \forall r \Rightarrow \sum_{r=1}^{k}\frac{\hat{\xi}_r}{\min\{\hat{f}_r\}} \geq \sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}_r} \geq \sum_{r=1}^{k}\frac{\hat{\xi}_r}{\max\{\hat{f}_r\}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\min\{\hat{f}_r\}}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}_r}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\max\{\hat{f}_r\}}}$$

Similarly the other relations are as follows

$$\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\min\{\hat{e}_r\}}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}_r}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\max\{\hat{e}_r\}}}$$

$$\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\min\{\hat{h}_r\}}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}_r}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\max\{\hat{h}_r\}}}$$

$$\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\min\{\hat{g}_r\}}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}_r}} \leq \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\max\{\hat{g}_r\}}}$$

Again,

$$\min\{\mu_r\} \leq \mu_r \leq \max\{\mu_r\} \Rightarrow (1 - \min\{\mu_r\}) \geq (1 - \mu_r) \geq (1 - \max\{\mu_r\})$$

$$\Rightarrow (1 - \min\{\mu_r\})^{\hat{\xi}_r} \geq (1 - \mu_r)^{\hat{\xi}_r} \geq (1 - \max\{\mu_r\})^{\hat{\xi}_r}, \hat{\xi}_r \geq 0, \forall r$$

$$\Rightarrow \prod_{r=1}^{k}(1 - \min\{\mu_r\})^{\hat{\xi}_r} \geq \prod_{r=1}^{k}(1 - \mu_r)^{\hat{\xi}_r} \geq \prod_{r=1}^{k}(1 - \max\{\mu_r\})^{\hat{\xi}_r}$$

$$\Rightarrow 1 - \prod_{r=1}^{k}(1 - \min\{\mu_r\})^{\hat{\xi}_r} \leq 1 - \prod_{r=1}^{k}(1 - \mu_r)^{\hat{\xi}_r} \leq 1 - \prod_{r=1}^{k}(1 - \max\{\mu_r\})^{\hat{\xi}_r}$$

and

$$\min\{\eta_r\} \leq \eta_r \leq \max\{\eta_r\} \Rightarrow (\min\{\eta_r\})^{\hat{\xi}_r} \leq (\eta_r)^{\hat{\xi}_r} \leq (\max\{\eta_r\})^{\hat{\xi}_r}$$

$$\Rightarrow \prod_{r=1}^{k}(\min\{\eta_r\})^{\hat{\xi}_r} \leq \prod_{r=1}^{k}(\eta_r)^{\hat{\xi}_r} \leq \prod_{r=1}^{k}(\max\{\eta_r\})^{\hat{\xi}_r}$$

Similarly,

$$\prod_{r=1}^{k}(\min\{\nu_r\})^{\hat{\xi}_r} \leq \prod_{r=1}^{k}(\nu_r)^{\hat{\xi}_r} \leq \prod_{r=1}^{k}(\max\{\nu_r\})^{\hat{\xi}_r}$$

$$\therefore \mathcal{P}^- \leq f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq \mathcal{P}^+$$

Hence the result. □

Theorem 5 (Commutativity property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ and $\mathcal{P}'_r = [(\hat{f}'_r, \hat{e}'_r, \hat{h}'_r, \hat{g}'_r); \mu'_r, \eta'_r, \nu'_r]$ be two sets of positive trapezoidal picture number for $r = 1, 2, \ldots, k$. Then*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k). \tag{9}$$

where $\mathcal{P}'_r$ is any permutation of $\mathcal{P}_r$ for $r = 1, 2, \ldots, k$.

Proof. We have from Equation (5)

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$$

$$= \left[\left(\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{k}\eta_r^{\hat{\xi}_r}, \prod_{r=1}^{k}\nu_r^{\hat{\xi}_r}\right]$$

Since $(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k)$ is any permutation of $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$.
Therefore

$$\left[\left(\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1 - \prod_{r=1}^{k}(1-\mu_r)^{\hat{\xi}_r}, \prod_{r=1}^{k}\eta_r^{\hat{\xi}_r}, \prod_{r=1}^{k}\nu_r^{\hat{\xi}_r}\right]$$

$$= \left[\left(\frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{f}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{e}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{h}'_r}}, \frac{1}{\sum_{r=1}^{k}\frac{\hat{\xi}_r}{\hat{g}'_r}}\right); 1 - \prod_{r=1}^{k}(1-\mu'_r)^{\hat{\xi}_r}, \prod_{r=1}^{k}(\eta'_r)^{\hat{\xi}_r}, \prod_{r=1}^{k}(\nu'_r)^{\hat{\xi}_r}\right].$$

Thus

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k).$$

Hence the result. □

Example 1. *Evaluations given by three decision makers in form of TrFN under picture fuzzy information. Let $\mathcal{P}_1 = [(0.2, 0.4, 0.4, 0.5); 0.4, 0.2, 0.3]$, $\mathcal{P}_2 = [(0.7, 0.6, 0.6, 0.6); 0.5, 0.1, 0.4]$, $\mathcal{P}_3 = [(0.8, 0.7, 0.3, 0.6); 0.3, 0.2, 0.4]$ be three positive TrPFN. Let the decision makers weight vector is $\hat{\xi} = (0.3, 0.4, 0.3)^T$. Then the complete solution is calculated by using the TrPFWHM operator*

$$f^{\hat{\xi}}_{TrPFWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$$

$$= \Bigg[\left(\frac{1}{\frac{0.3}{0.2}+\frac{0.4}{0.7}+\frac{0.3}{0.8}}, \frac{1}{\frac{0.3}{0.4}+\frac{0.4}{0.6}+\frac{0.3}{0.7}}, \frac{1}{\frac{0.3}{0.4}+\frac{0.4}{0.6}+\frac{0.3}{0.3}}, \frac{1}{\frac{0.3}{0.5}+\frac{0.4}{0.4}+\frac{0.3}{0.6}}\right);$$

$$1-(1-0.4)^{0.3}\times(1-0.5)^{0.4}\times(1-0.3)^{0.3}, 0.2^{0.3}\times 0.1^{0.4}\times 0.2^{0.3}, 0.3^{0.3}\times 0.4^{0.4}\times 0.4^{0.3}\Bigg]$$

$$= [(0.41, 0.54, 0.41, 0.48); 0.42, 0.15, 0.37].$$

Now we define score functions and accuracy functions for TrPFN.

Definition 10. *Let $\mathcal{P} = [(\hat{f}, \hat{e}, \hat{h}, \hat{g}); \mu, \eta, \nu]$ be a positive TrPFN. Then*

$$S(\mathcal{P}) = \frac{\hat{f} + \hat{e} + \hat{h} + \hat{g}}{4} \times \frac{1 + \mu - \nu}{2} \tag{10}$$

is called score function and

$$A(\mathcal{P}) = \frac{\hat{f} + \hat{e} + \hat{h} + \hat{g}}{4} \times \frac{\mu - \nu}{2} \tag{11}$$

is called accuracy function.

Definition 11. *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$. A TrPFOWHM operator is a mapping $f^{\hat{\xi}}_{TrPFOWHM} : \mathcal{P}^k \to \mathcal{P}$ and*

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\mathcal{P}_{\sigma(r)}}}, \tag{12}$$

where the associated weight vector $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ of $\mathcal{P}_r$ such that $\hat{\xi}_r \in [0,1]$, $\sum\limits_{r=1}^{k} \hat{\xi}_r = 1$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be any permutation such that $\mathcal{P}_{\sigma(r)} \leq \mathcal{P}_{\sigma(r-1)}$ for $r = 2, 3, \ldots, k$.

Theorem 6. *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$ and the weight vector $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ of $\mathcal{P}_r$ such that $\hat{\xi}_r \in [0,1]$, $\sum\limits_{r=1}^{k} \hat{\xi}_r = 1$ then*

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$$

$$= \left[\left(\frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{f}_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{e}_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{h}_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\xi}_r}{\hat{g}_{\sigma(r)}}} \right); 1 - \prod_{r=1}^{k} \left(1 - \mu_{\sigma(r)}\right)^{\hat{\xi}_r}, \prod_{r=1}^{k} \eta_{\sigma(r)}^{\hat{\xi}_r}, \prod_{r=1}^{k} \nu_{\sigma(r)}^{\hat{\xi}_r} \right] \tag{13}$$

where $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be any permutation such that $\mathcal{P}_{\sigma(r)} \leq \mathcal{P}_{\sigma(r-1)}$ for $r = 2, 3, \ldots, k$.

Proof. Proof is same as Theorem 1. □

Theorem 7 (Idempotency property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of TrPFNs. If $\mathcal{P}_r = \mathcal{P}$ for all r that is all are identical then,*

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}, \mathcal{P}, \ldots, \mathcal{P}) = \mathcal{P}. \tag{14}$$

Proof. Proof is same as Theorem 2. □

Theorem 8 (Monotonicity property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ and $\mathcal{P}'_r = [(\hat{f}'_r, \hat{e}'_r, \hat{h}'_r, \hat{g}'_r); \mu'_r, \eta'_r, \nu'_r]$ be two collection of TrPFNs. If $\hat{f}_r \leq \hat{f}'_r$, $\hat{e}_r \leq \hat{e}'_r$, $\hat{h}_r \leq \hat{h}'_r$, $\hat{g}_r \leq \hat{g}'_r$, $\mu_r \leq \mu'_r$, $\eta_r \geq \eta'_r$, and $\nu_r \geq \nu'_r$. Then*

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k) \tag{15}$$

Proof. Proof is same as Theorem 3. □

Theorem 9. (*Boundedness property*) *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs. Let $\mathcal{P}^- = [(\min\{\hat{f}_r\}, \min\{\hat{e}_r\}, \min\{\hat{h}_r\}, \min\{\hat{g}_r\}); \min\{\mu_r\}, \max\{\eta_r\}, \max\{\nu_r\}]$ and $\mathcal{P}^+ = [(\max\{\hat{f}_r\}, \max\{\hat{e}_r\}, \max\{\hat{h}_r\}, \max\{\hat{g}_r\}); \max\{\mu_r\}, \min\{\eta_r\}, \min\{\nu_r\}]$. Then*

$$\mathcal{P}^- \leq f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) \leq \mathcal{P}^+ \tag{16}$$

Proof. Proof is same as Theorem 4. □

Theorem 10 (Commutativity property)**.** *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ and $\mathcal{P}'_r = [(\hat{f}'_r, \hat{e}'_r, \hat{h}'_r, \hat{g}'_r); \mu'_r, \eta'_r, \nu'_r]$ be two sets of positive TrPFN. Then*

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}'_1, \mathcal{P}'_2, \ldots, \mathcal{P}'_k). \tag{17}$$

where $\mathcal{P}'_r$ is any permutation of $\mathcal{P}_r$ for $r = 1, 2, \ldots, k$.

Proof. Proof is same as Theorem 5. □

Example 2. *Evaluations given by three decision makers in form of TrPFN picture fuzzy information. Let $\mathcal{P}_1 = [(0.2, 0.4, 0.4, 0.5); 0.4, 0.2, 0.3]$, $\mathcal{P}_2 = [(0.7, 0.6, 0.6, 0.6); 0.5, 0.1, 0.4]$, $\mathcal{P}_3 = [(0.8, 0.7, 0.3, 0.6); 0.3, 0.2, 0.4]$ be three positive TrPFN. Let the decision makers weight vector $\hat{\xi} = (0.3, 0.4, 0.3)^T$. Now we compute the score functions of the TrPFN as*

$$Sc(\mathcal{P}_1) = \frac{0.2 + 0.4 + 0.4 + 0.5}{4} . \frac{1 + 0.4 - 0.3}{2} = 0.2602.$$

$$Sc(\mathcal{P}_2) = \frac{0.7 + 0.6 + 0.6 + 0.4}{4} . \frac{1 + 0.5 - 0.4}{2} = 0.3162.$$

$$Sc(\mathcal{P}_3) = \frac{0.8 + 0.7 + 0.3 + 0.6}{4} . \frac{1 + 0.3 - 0.4}{2} = 0.2700.$$

Therefore, the order is $\mathcal{P}_2 \succ \mathcal{P}_3 \succ \mathcal{P}_1$. Then, the completed solution is calculated by using the TrPFOWHM operator

$$f^{\hat{\xi}}_{TrPFOWHM}(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$$

$$= \left[\left(\frac{1}{\frac{0.3}{0.7} + \frac{0.4}{0.8} + \frac{0.3}{0.2}}, \frac{1}{\frac{0.3}{0.6} + \frac{0.4}{0.7} + \frac{0.3}{0.4}}, \frac{1}{\frac{0.3}{0.4} + \frac{0.4}{0.3} + \frac{0.3}{0.4}}, \frac{1}{\frac{0.3}{0.4} + \frac{0.4}{0.6} + \frac{0.3}{0.5}} \right); \right.$$

$$\left. 1 - (1 - 0.5)^{0.3} \times (1 - 0.3)^{0.4} \times (1 - 0.4)^{0.3}, 0.1^{0.3} \times 0.2^{0.4} \times 0.2^{0.3}, 0.4^{0.3} \times 0.4^{0.4} \times 0.3^{0.3} \right]$$

$$= [(0.41, 0.55, 0.39, 0.50); 0.40, 0.16, 0.37].$$

Definition 12. *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$. A TrPFHHM operator is a mapping $f^{\hat{\xi}, \hat{\chi}}_{TrPFHHM} : \mathcal{P}^k \to \mathcal{P}$ then*

$$f^{\hat{\xi}, \hat{\chi}}_{TrPFHHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k) = \frac{1}{\sum\limits_{r=1}^{k} \frac{\hat{\chi}_r}{\mathcal{P}'_{\sigma(r)}}} \tag{18}$$

where, the rth largest TrPFN $\mathcal{P}'_{\sigma(r)}$ is calculated by $\mathcal{P}'_r = k\hat{\xi}_r \mathcal{P}_r$ for r=1,2,...,k. Here k is called balancing factor. The weight vector $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ of $\mathcal{P}_r$, $\hat{\xi}_r \in [0, 1]$ such that $\sum\limits_{r=1}^{k} \hat{\xi}_r = 1$. $\hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_k)^T$ be the position vector.

Theorem 11. *Let $\mathcal{P}_r = [(\hat{f}_r, \hat{e}_r, \hat{h}_r, \hat{g}_r); \mu_r, \eta_r, \nu_r]$ be a collection of positive TrPFNs for $r = 1, 2, \ldots, k$. Then*

$$f^{\hat{\xi},\hat{\chi}}_{TrPFHHM}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$$

$$= \left[\left(\frac{1}{\sum\limits_{r=1}^{k}\frac{\hat{\chi}_r}{\hat{f}'_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k}\frac{\hat{\chi}_r}{\hat{e}'_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k}\frac{\hat{\chi}_r}{\hat{h}'_{\sigma(r)}}}, \frac{1}{\sum\limits_{r=1}^{k}\frac{\hat{\chi}_r}{\hat{g}'_{\sigma(r)}}}\right); 1 - \prod_{r=1}^{k}(1 - \mu'_{\sigma(r)})^{\hat{\chi}_r}, \prod_{r=1}^{k}(\eta'_{\sigma(r)})^{\hat{\chi}_r}, \prod_{r=1}^{k}(\nu'_{\sigma(r)})^{\hat{\chi}_r}\right] \tag{19}$$

where $\hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_k)^T$ *be an associated weight vector such that* $\sum\limits_{r=1}^{k} \hat{\chi}_r = 1$.

Proof. Proof is same as Theorem 1. □

Specially, if $\hat{\xi} = (\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})^T$ then TrPFHHM operators becomes TrPFOWHM operators and if $\hat{\chi} = (\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})^T$ then TrPFHHM operators becomes TrPFWHM operators. Thus, we say that TrPFHHM operator is a generalization of TrPFWHM and TrPFOWHM operators.

Example 3. *Evaluations given by three decision makers in form of TrPFN under picture fuzzy information. Let* $\mathcal{P}_1 = [(0.2, 0.4, 0.4, 0.5); 0.4, 0.2, 0.3]$, $\mathcal{P}_2 = [(0.7, 0.6, 0.6, 0.4); 0.5, 0.1, 0.4]$, $\mathcal{P}_3 = [(0.8, 0.7, 0.3, 0.6); 0.3, 0.2, 0.4]$ *be three positive TrPFN. Let* $\hat{\xi} = (0.3, 0.4, 0.3)^T$ *be the weighting vector of the decision makers and* $\hat{\chi} = (0.4, 0.2, 0.4)^T$ *be the position vector.*

The hybrid TrPFN are given by

$$\tilde{\mathcal{P}'}_1 = 3 \times 0.3 \times [(0.2, 0.4, 0.4, 0.5); 0.4, 0.2, 0.3]$$
$$= [(0.9 \times 0.2, 0.9 \times 0.4, 0.9 \times 0.4, 0.9 \times 0.5); 1 - (1 - 0.4)^{0.9}, 0.2^{0.9}, 0.3^{0.9}]$$
$$= [(0.18, 0.36, 0.36, 0.45); 0.37, 0.23, 0.34],$$
$$\tilde{\mathcal{P}'}_2 = 3 \times 0.4 \times [(0.7, 0.6, 0.6, 0.4); 0.5, 0.1, 0.4]$$
$$= [(1.2 \times 0.7, 1.2 \times 0.6, 1.2 \times 0.6, 1.2 \times 0.4); 1 - (1 - 0.5)^{1.2}, 0.1^{1.2}, 0.4^{1.2}]$$
$$= [(0.84, 0.72, 0.72, 0.48); 0.56, 0.06, 0.33],$$

and

$$\tilde{\mathcal{P}'}_3 = 3 \times 0.3 \times [(0.8, 0.7, 0.3, 0.6); 0.3, 0.2, 0.4]$$
$$= [(0.9 \times 0.8, 0.9 \times 0.7, 0.9 \times 0.3, 0.9 \times 0.6); 1 - (1 - 0.3)^{0.9}, 0.2^{0.9}, 0.4^{0.9}]$$
$$= [(0.72, 0.63, 0.27, 0.54); 0.27, 0.23, 0.44].$$

Now we can calculate the score of this hybrid TrPFN using score functions

$$Sc(\tilde{\mathcal{P}'}_1) = \frac{0.18 + 0.36 + 0.36 + 0.45}{4} \times \frac{1 + 0.37 - 0.34}{2} = 0.1738.$$
$$Sc(\tilde{\mathcal{P}'}_2) = \frac{0.84 + 0.72 + 0.72 + 0.48}{4} \times \frac{1 + 0.56 - 0.33}{2} = 0.4243.$$
$$Sc(\tilde{\mathcal{P}'}_3) = \frac{0.72 + 0.63 + 0.27 + 0.54}{4} \times \frac{1 + 0.27 - 0.44}{2} = 0.2241.$$

Therefore the order of the hybrid TrPFN is $\tilde{\mathcal{P}'}_2 \succ \tilde{\mathcal{P}'}_3 \succ \tilde{\mathcal{P}'}_1$. *Then the completed solution is calculated by TrPFHHM operator*

$$f^{\hat{\xi},\hat{\chi}}_{TrPFHHM}(\tilde{\mathcal{P}'}_1, \tilde{\mathcal{P}'}_2, \tilde{\mathcal{P}'}_3)$$

$$= \left[\left(\frac{1}{\frac{0.4}{0.84} + \frac{0.2}{0.72} + \frac{0.4}{0.18}}, \frac{1}{\frac{0.4}{0.72} + \frac{0.2}{0.63} + \frac{0.4}{0.36}}, \frac{1}{\frac{0.4}{0.72} + \frac{0.2}{0.27} + \frac{0.4}{0.36}}, \frac{1}{\frac{0.4}{0.48} + \frac{0.2}{0.54} + \frac{0.4}{0.45}} \right); \right.$$
$$\left. 1 - (1-0.56)^{0.4} \times (1-0.27)^{0.2} \times (1-0.37)^{0.4}, 0.06^{0.4} \times 0.23^{0.2} \times 0.23^{0.4}, 0.33^{0.4} \times 0.44^{0.2} \times 0.34^{0.4} \right]$$

$$= [(0.34, 0.50, 0.41, 0.48); 0.44, 0.13, 0.35].$$

4. Materials and Methods

In this portion, we shall present an MADM problem with TrFN under picture fuzzy information using TrPFHHM operator. Let $\mathcal{B}_r$ be the set of discrete alternatives for $r = 1, 2, \ldots, p$. $\mathcal{E}_s$ be the set of attributes for $s = 1, 2, \ldots, k$ and $\tilde{D}_l$ be the decision makers for $(l = 1, 2, \ldots, q)$. The matrix representation of the MADM problem is given below where the element $\mathcal{B}_{rs}$ means that r th alternative satisfies s th attribute.

$$\tilde{D} = [\tilde{\mathcal{B}}^l_{rs}]p \times k = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \ldots & \mathcal{B}_{1k} \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \ldots & \mathcal{B}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{p1} & \mathcal{B}_{p2} & \ldots & \mathcal{B}_{pk}. \end{pmatrix} \tag{20}$$

Let the weight vector $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ of the attributes be such that $\sum_{r=1}^{k} \hat{\xi}_r = 1$ and the weight vector $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_q)^T$ of the q decision makers with $\sum_{l=1}^{q} \hat{\tau}_l = 1$. Let the associated weight vector be $\hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_k)^T$ with $\sum_{r=1}^{k} \hat{\chi}_r = 1$. In the Algorithm 1, utilizing TrPFHHM operator we solve the MADM problem.

Algorithm 1:

Input: To the selection of best possible alternative.
Output: Best alternative.

1. Identify the alternatives and attributes and then construct the decision matrix $\tilde{D} = [\tilde{\mathcal{B}}^l_{rs}]_{p \times k}$, where $\tilde{\mathcal{B}}^l_{rs}$ is the trapezoidal picture fuzzy number of the form $[(\hat{f}, \hat{e}, \hat{h}, \hat{g}); \mu, \eta, \nu]$ given by the q decision maker.
2. Convert the matrix $\tilde{D} = [\tilde{\mathcal{B}}^l_{rs}]_{p \times k}$ to the normalize matrix $\tilde{D}' = [\tilde{\mathcal{B}}'^l_{rs}]_{p \times k}$ by the following property:
$$\tilde{\mathcal{B}}'^l_{rs} = \begin{cases} \left[\left(\frac{\hat{f}^l_{rs}}{v_{\max}}, \frac{\hat{e}^l_{rs}}{v_{\max}}, \frac{\hat{h}^l_{rs}}{v_{\max}}, \frac{\hat{g}^l_{rs}}{v_{\max}} \right); \mu^l_{rs}, \eta^l_{rs}, \nu^l_{rs} \right] & \text{for benefit attributes.} \\ \left[\left(\frac{\hat{f}^l_{rs}}{v_{\max}}, \frac{\hat{e}^l_{rs}}{v_{\max}}, \frac{\hat{h}^l_{rs}}{v_{\max}}, \frac{\hat{g}^l_{rs}}{v_{\max}} \right); \nu^l_{rs}, \eta^l_{rs}, \mu^l_{rs} \right] & \text{for cost attributes.} \end{cases}$$
where $v_{\max} = \max\{\hat{f}^l_{rs}, \hat{e}^l_{rs}, \hat{h}^l_{rs}, \hat{g}^l_{rs}\}$, $r = 1, 2, \ldots, p$, $s = 1, 2, \ldots, k$, and $l = 1, 2, \ldots, q$.
3. Consider the weight vector $\hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k)^T$ of the attributes. Furthermore, identify
the weight vector $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_p)^T$ of the decision makers to their background, knowledge, and experience. Determine the associated position vector $\hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2, \ldots, \hat{\chi}_k)^T$.
4. Calculate the individual ratings of each alternative for each decision maker from normalized decision matrix by TrPFWHM operator.
5. Calculate the final ratings of the alternative by TrPFHHM operator and compute their score by score functions (10) to ranking.

5. Numerical Example

In this portion, we shall present a numerical instance to illustrate the flexibility of the proposed method. A telecom company has decided that they will setup a tower at a particular place in Midnapore town. The company always wants to put the tower in the right place, because once installed it is very expansive to move. Furthermore, they have to keep in mind that people from all corners of the town will get all kinds of facilities. The opinion of the person in which place the tower will be erected has to be taken. The company employ three experts to fix the place in the Midnapore town. The expert will evaluate four places: Keranitola ($\mathcal{B}_1$), Sepoi Bazar ($\mathcal{B}_2$), Rangamati ($\mathcal{B}_3$), and Ashokenagar ($\mathcal{B}_4$) according to four attributes. The attributes are as follows:

1. Population of locality ($\mathcal{E}_1$): A telecom company expects more customers all the time, because more customers equals greater profit. So, the experts will wish to choose a locality with a larger population. Further, it does not happen that all the people of a locality will be the customers of that telecom company. They may be the customers of another company. So, more population of a locality is important to install a tower.
2. Commercial environment ($\mathcal{E}_2$): If the commercial environment of a locality is good then it is convenient to do business. Commercial environment means that there is school, college, hospital, shopping mall, etc., around the locality.
3. Eco-friendly ($\mathcal{E}_3$): The telecom company wants to install the tower without any harm to the environment. The company always takes care of the beauty of the environment so that the tower can be installed. If there are some large trees next to it, they take care of the trees. So, it is important that the locality will be eco-friendly to the company.
4. Cost ($\mathcal{E}_4$): Before choosing the place, the company should decide how much it will cost and they should fix their maximum budget, including management cost.

6. Result and Discussion

The weight vector for the attribute set by the telecom company as $\hat{\xi} = (0.20, 0.15, 0.45, 0.20)^T$. Here, three attributes $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ are benefit attributes and $\mathcal{E}_4$ is cost attribute. The weight vector of the expert is $\hat{\tau} = (0.40, 0.35, 0.25)^T$ and the associated weight vector is $\hat{\chi} = (0.35, 0.40, 0.25)^T$. The three experts give their ratings which are displayed in Tables 1–3. The normalized decision matrix is given by Tables 4–6.

Table 1. Decision maker 1 responses.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^1$	$[(0.6, 0.4, 0.7, 0.9); 0.55, 0.34, 0.11]$	$[(0.5, 0.5, 0.7, 0.6); 0.60, 0.25, 0.10]$	$[(0.6, 0.4, 0.7, 0.8); 0.61, 0.19, 0.18]$	$[(0.7, 0.8, 0.5, 0.6); 0.57, 0.21, 0.17]$
$\mathcal{B}_2^1$	$[(0.2, 0.3, 0.5, 0.7); 0.65, 0.15, 0.15]$	$[(0.6, 0.5, 0.7, 0.3); 0.54, 0.25, 0.18]$	$[(0.5, 0.4, 0.6, 0.9); 0.59, 0.19, 0.18]$	$[(0.7, 0.6, 0.2, 0.3); 0.75, 0.09, 0.15]$
$\mathcal{B}_3^1$	$[(0.5, 0.3, 0.6, 0.2); 0.25, 0.45, 0.25]$	$[(0.3, 0.2, 0.7, 0.3); 0.45, 0.35, 0.14]$	$[(0.5, 0.4, 0.7, 0.8); 0.51, 0.17, 0.27]$	$[(0.3, 0.2, 0.7, 0.8); 0.49, 0.27, 0.19]$
$\mathcal{B}_4^1$	$[(0.4, 0.6, 0.7, 0.2); 0.31, 0.43, 0.25]$	$[(0.5, 0.2, 0.7, 0.9); 0.58, 0.13, 0.19]$	$[(0.6, 0.7, 0.3, 0.4); 0.56, 0.24, 0.20]$	$[(0.3, 0.4, 0.2, 0.1); 0.48, 0.26, 0.21]$

Table 2. Decision maker 2 responses.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^2$	$[(0.4, 0.7, 0.8, 0.2); 0.53, 0.29, 0.11]$	$[(0.4, 0.4, 0.7, 0.3); 0.47, 0.12, 0.34]$	$[(0.3, 0.5, 0.4, 0.8); 0.35, 0.19, 0.43]$	$[(0.2, 0.7, 0.4, 0.5); 0.65, 0.13, 0.17]$
$\mathcal{B}_2^2$	$[(0.3, 0.5, 0.4, 0.7); 0.42, 0.28, 0.19]$	$[(0.2, 0.4, 0.2, 0.7); 0.49, 0.23, 0.28]$	$[(0.5, 0.3, 0.7, 0.9); 0.59, 0.09, 0.27]$	$[(0.4, 0.3, 0.2, 0.8); 0.61, 0.09, 0.29]$
$\mathcal{B}_3^2$	$[(0.4, 0.7, 0.2, 0.5); 0.31, 0.11, 0.51]$	$[(0.5, 0.2, 0.9, 0.6); 0.63, 0.08, 0.21]$	$[(0.3, 0.4, 0.7, 0.5); 0.59, 0.13, 0.22]$	$[(0.2, 0.5, 0.7, 0.2); 0.58, 0.15, 0.23]$
$\mathcal{B}_4^2$	$[(0.5, 0.3, 0.6, 0.8); 0.72, 0.05, 0.19]$	$[(0.6, 0.4, 0.5, 0.2); 0.71, 0.18, 0.06]$	$[(0.3, 0.2, 0.5, 0.6); 0.65, 0.15, 0.18]$	$[(0.4, 0.2, 0.8, 0.7); 0.61, 0.14, 0.21]$

Table 3. Decision maker 3 responses.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^3$	$[(0.4, 0.2, 0.7, 0.5); 0.52, 0.22, 0.21]$	$[(0.5, 0.2, 0.7, 0.4); 0.09, 0.78, 0.08]$	$[(0.4, 0.3, 0.5, 0.7); 0.07, 0.82, 0.07]$	$[(0.4, 0.1, 0.5, 0.8); 0.12, 0.76, 0.09]$
$\mathcal{B}_2^3$	$[(0.1, 0.5, 0.7, 0.2); 0.11, 0.75, 0.10]$	$[(0.2, 0.4, 0.5, 0.3); 0.63, 0.12, 0.17]$	$[(0.3, 0.5, 0.7, 0.3); 0.52, 0.22, 0.17]$	$[(0.3, 0.5, 0.8, 0.1); 0.11, 0.59, 0.09]$
$\mathcal{B}_3^3$	$[(0.4, 0.2, 0.5, 0.7); 0.71, 0.09, 0.12]$	$[(0.3, 0.7, 0.5, 0.1); 0.49, 0.21, 0.22]$	$[(0.4, 0.5, 0.3, 0.8); 0.17, 0.58, 0.08]$	$[(0.5, 0.7, 0.4, 0.8); 0.53, 0.22, 0.24]$
$\mathcal{B}_4^3$	$[(0.1, 0.7, 0.5, 0.9); 0.12, 0.75, 0.06]$	$[(0.2, 0.8, 0.6, 0.8); 0.50, 0.20, 0.20]$	$[(0.3, 0.2, 0.5, 0.1); 0.22, 0.51, 0.10]$	$[(0.2, 0.4, 0.8, 0.1); 0.58, 0.09, 0.08]$

Table 4. Normalized responses by decision maker 1.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^1$	$[(0.67,0.44,0.78,1.00);0.55,0.34,0.11]$	$[(0.71,0.71,1.00,0.86);0.60,0.25,0.10]$	$[(0.75,0.50,0.87,1.00);0.61,0.19,0.18]$	$[(0.87,1.00,0.62,0.75);0.17,0.21,0.57]$
$\mathcal{B}_2^1$	$[(0.29,0.43,0.71,1.00);0.65,0.15,0.15]$	$[(0.86,0.71,1.00,0.43);0.54,0.25,0.18]$	$[(0.56,0.44,0.67,1.00);0.59,0.19,0.18]$	$[(1.00,0.86,0.29,0.43);0.15,0.09,0.75]$
$\mathcal{B}_3^1$	$[(0.83,0.50,1.00,0.33);0.25,0.45,0.25]$	$[(0.43,0.29,1.00,0.43);0.45,0.35,0.14]$	$[(0.63,0.50,0.88,1.00);0.51,0.17,0.27]$	$[(0.38,0.25,0.88,1.00);0.19,0.27,0.49]$
$\mathcal{B}_4^1$	$[(0.57,0.86,1.00,0.29);0.31,0.43,0.25]$	$[(0.56,0.22,0.78,1.00);0.58,0.13,0.19]$	$[(0.86,1.00,0.43,0.57);0.56,0.24,0.20]$	$[(0.75,1.00,0.50,0.25);0.21,0.26,0.48]$

Table 5. Normalized responses by decision maker 2.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^2$	$[(0.50,0.88,1.00,0.25);0.53,0.29,0.11]$	$[(0.57,0.57,1.00,0.43);0.47,0.12,0.34]$	$[(0.38,0.63,0.50,1.00);0.35,0.19,0.43]$	$[(0.29,1.00,0.57,0.71);0.17,0.13,0.65]$
$\mathcal{B}_2^2$	$[(0.43,0.71,0.57,1.00);0.42,0.28,0.19]$	$[(0.29,0.57,0.29,1.00);0.49,0.23,0.28]$	$[(0.56,0.33,0.78,0.90);0.59,0.09,0.27]$	$[(0.50,0.38,0.25,1.00);0.29,0.09,0.61]$
$\mathcal{B}_3^2$	$[(0.57,1.00,0.29,0.71);0.31,0.11,0.51]$	$[(0.56,0.22,1.00,0.67);0.63,0.08,0.21]$	$[(0.43,0.57,1.00,0.71);0.59,0.13,0.22]$	$[(0.29,0.71,1.00,0.29);0.23,0.15,0.58]$
$\mathcal{B}_4^2$	$[(0.63,0.38,0.75,1.00);0.72,0.05,0.19]$	$[(1.00,0.67,0.83,0.33);0.71,0.18,0.06]$	$[(0.50,0.33,0.83,1.00);0.65,0.15,0.18]$	$[(0.50,0.25,1.00,0.88);0.21,0.14,0.61]$

Table 6. Normalized responses by decision maker 3.

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$
$\mathcal{B}_1^3$	$[(0.57,0.29,1.00,0.71);0.52,0.22,0.21]$	$[(0.71,0.29,1.00,0.57);0.09,0.78,0.08]$	$[(0.57,0.43,0.71,1.00);0.07,0.82,0.07]$	$[(0.50,0.13,0.63,1.00);0.09,0.76,0.12]$
$\mathcal{B}_2^3$	$[(0.14,0.71,1.00,0.29);0.11,0.75,0.10]$	$[(0.40,0.80,1.00,0.60);0.63,0.12,0.17]$	$[(0.43,0.71,1.00,0.43);0.52,0.22,0.17]$	$[(0.38,0.63,1.00,0.12);0.09,0.59,0.11]$
$\mathcal{B}_3^3$	$[(0.57,0.29,0.71,1.00);0.71,0.09,0.12]$	$[(0.43,1.00,0.71,0.14);0.49,0.21,0.22]$	$[(0.50,0.63,0.38,1.00);0.17,0.58,0.08]$	$[(0.63,0.88,0.25,1.00);0.24,0.22,0.53]$
$\mathcal{B}_4^3$	$[(0.11,0.78,0.56,1.00);0.12,0.75,0.06]$	$[(0.25,1.00,0.75,1.00);0.50,0.20,0.20]$	$[(0.60,0.40,1.00,0.20);0.22,0.51,0.10]$	$[(0.25,0.50,1.00,0.12);0.08,0.09,0.58]$

6.1. Decision Process

In this portion, we shall talk about the decision process step by step.

1. At first we calculate the individual ratings of each alternatives by utilizing the TrPFWHM operator given by Equation (5), as follows:

$$\tilde{\mathcal{B}'}^l_r = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^l_{rs}, \tilde{\mathcal{B}'}^l_{rs}, \tilde{\mathcal{B}'}^l_{rs}, \tilde{\mathcal{B}'}^l_{rs}) =$$

$$\left[\left(\frac{1}{\sum_{r=1}^{4}\frac{\hat{\xi}_r}{\hat{f}_r}}, \frac{1}{\sum_{r=1}^{4}\frac{\hat{\xi}_r}{\hat{e}_r}}, \frac{1}{\sum_{r=1}^{4}\frac{\hat{\xi}_r}{\hat{h}_r}}, \frac{1}{\sum_{r=1}^{4}\frac{\hat{\xi}_r}{\hat{g}_r}}\right); 1-\prod_{r=1}^{4}(1-\mu_r), \prod_{r=1}^{4}\eta_r, \prod_{r=1}^{4}\nu_r\right].$$

Therefore we have from Table 4,

$$\tilde{\mathcal{B}'}^1_1 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^1_{11}, \tilde{\mathcal{B}'}^1_{12}, \tilde{\mathcal{B}'}^1_{13}, \tilde{\mathcal{B}'}^1_{14}) = [(0.75,0.57,0.80,0.92);0.53,0.23,0.19].$$

$$\tilde{\mathcal{B}'}^1_2 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^1_{21}, \tilde{\mathcal{B}'}^1_{22}, \tilde{\mathcal{B}'}^1_{23}, \tilde{\mathcal{B}'}^1_{24}) = [(0.54,0.65,0.56,0.68);0.53,0.16,0.23].$$

$$\tilde{\mathcal{B}'}^1_3 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^1_{31}, \tilde{\mathcal{B}'}^1_{32}, \tilde{\mathcal{B}'}^1_{33}, \tilde{\mathcal{B}'}^1_{34}) = [(0.55,0.38,0.92,0.62);0.40,0.21,0.27].$$

$$\tilde{\mathcal{B}'}^1_4 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^1_{41}, \tilde{\mathcal{B}'}^1_{42}, \tilde{\mathcal{B}'}^1_{43}, \tilde{\mathcal{B}'}^1_{44}) = [(0.71,0.64,0.54,0.41);0.46,0.25,0.25].$$

Similarly we have from Tables 5 and 6,

$$\tilde{\mathcal{B}'}^2_1 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^2_{11}, \tilde{\mathcal{B}'}^2_{12}, \tilde{\mathcal{B}'}^2_{13}, \tilde{\mathcal{B}'}^2_{14}) = [(0.39,0.71,0.62,0.53);0.38,0.18,0.34].$$

$$\tilde{\mathcal{B}'}^2_2 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^2_{21}, \tilde{\mathcal{B}'}^2_{22}, \tilde{\mathcal{B}'}^2_{23}, \tilde{\mathcal{B}'}^2_{24}) = [(0.46,0.41,0.44,0.95);0.49,0.13,0.30].$$

$$\tilde{\mathcal{B}'}^2_3 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^2_{31}, \tilde{\mathcal{B}'}^2_{32}, \tilde{\mathcal{B}'}^2_{33}, \tilde{\mathcal{B}'}^2_{34}) = [(0.42,0.51,0.67,0.55);0.49,0.12,0.31].$$

$$\tilde{\mathcal{B}'}^2_4 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^2_{41}, \tilde{\mathcal{B}'}^1_{42}, \tilde{\mathcal{B}'}^2_{43}, \tilde{\mathcal{B}'}^2_{44}) = [(0.56,0.34,0.84,0.75);0.62,0.12,0.20].$$

$$\tilde{\mathcal{B}'}^3_1 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^3_{11}, \tilde{\mathcal{B}'}^3_{12}, \tilde{\mathcal{B}'}^3_{13}, \tilde{\mathcal{B}'}^3_{14}) = [(0.57,0.26,0.77,0.84);0.19,0.61,0.10].$$

$$\tilde{\mathcal{B}'}^3_2 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^3_{21}, \tilde{\mathcal{B}'}^3_{22}, \tilde{\mathcal{B}'}^3_{23}, \tilde{\mathcal{B}'}^3_{24}) = [(0.30, 0.70, 1.00, 0.27); 0.41, 0.31, 0.14].$$

$$\tilde{\mathcal{B}'}^3_3 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^3_{31}, \tilde{\mathcal{B}'}^3_{32}, \tilde{\mathcal{B}'}^3_{33}, \tilde{\mathcal{B}'}^3_{34}) = [(0.52, 0.56, 0.40, 0.52); 0.38, 0.28, 0.14].$$

$$\tilde{\mathcal{B}'}^3_4 = f^{\hat{\xi}}_{TrPFWHM}(\tilde{\mathcal{B}'}^3_{41}, \tilde{\mathcal{B}'}^3_{42}, \tilde{\mathcal{B}'}^3_{43}, \tilde{\mathcal{B}'}^3_{44}) = [(0.25, 0.52, 0.83, 0.23); 0.23, 0.34, 0.14].$$

2. In this step, we calculate $\tilde{\mathcal{B}''}^l_i = n\hat{\tau}_l \tilde{\mathcal{B}'}^l_i$ as follows:

$$\tilde{\mathcal{B}''}^1_1 = 4 \times 0.40 \times [(0.75, 0.57, 0.80, 0.92); 0.53, 0.23, 0.19]$$

$$= [(0.75 \times 1.6, 0.57 \times 1.6, 0.80 \times 1.6, 0.92 \times 1.6); 1 - (1 - 0.53)^{1.6}, 0.23^{1.6}, 0.19^{1.6}]$$

$$= [(1.20, 0.91, 1.28, 1.47); 0.70, 0.09, 0.07].$$

Similarly the other values are

$$\tilde{\mathcal{B}''}^2_1 = [(0.86, 1.04, 0.90, 1.09); 0.70, 0.05, 0.09].$$

$$\tilde{\mathcal{B}''}^3_1 = [(0.88, 0.61, 1.47, 0.99); 0.56, 0.08, 0.12].$$

$$\tilde{\mathcal{B}''}^4_1 = [(1.14, 1.02, 0.86, 0.66); 0.63, 0.11, 0.11].$$

$$\tilde{\mathcal{B}''}^1_2 = [(0.55, 0.99, 0.87, 0.74); 0.49, 0.09, 0.22].$$

$$\tilde{\mathcal{B}''}^2_2 = [(0.64, 0.57, 0.62, 1.33); 0.61, 0.06, 0.18].$$

$$\tilde{\mathcal{B}''}^3_2 = [(0.59, 0.71, 0.94, 0.77); 0.61, 0.05, 0.19].$$

$$\tilde{\mathcal{B}''}^4_2 = [(0.78, 0.48, 1.18, 1.05); 0.79, 0.05, 0.10].$$

$$\tilde{\mathcal{B}''}^1_3 = [(0.57, 0.26, 0.77, 0.84); 0.19, 0.61, 0.10].$$

$$\tilde{\mathcal{B}''}^2_3 = [(0.30, 0.70, 1.00, 0.27); 0.41, 0.31, 0.14].$$

$$\tilde{\mathcal{B}''}^3_3 = [(0.52, 0.56, 0.40, 0.52); 0.38, 0.28, 0.14].$$

$$\tilde{\mathcal{B}''}^4_3 = [(0.25, 0.52, 0.83, 0.23); 0.23, 0.34, 0.14].$$

3. Next the score functions of each $\tilde{\mathcal{B}''}^l_i$ is calculated as follows:

$$Sc(\tilde{\mathcal{B}''}^1_1) = \frac{1.20 + 0.91 + 1.28 + 1.47}{4} \times \frac{1 + 0.70 - 0.07}{2}$$

$$= 1.215 \times 0.815 = 0.9902.$$

Similarly the other values are obtained. So $Sc(\tilde{\mathcal{B}''}^1_2) = 0.7829$, $Sc(\tilde{\mathcal{B}''}^1_3) = 0.7110$, $Sc(\tilde{\mathcal{B}''}^1_4) = 0.6992$, $Sc(\tilde{\mathcal{B}''}^2_1) = 0.5001$, $Sc(\tilde{\mathcal{B}''}^2_2) = 0.5648$, $Sc(\tilde{\mathcal{B}''}^2_3) = 0.5343$, $Sc(\tilde{\mathcal{B}''}^2_4) = 0.7373$, $Sc(\tilde{\mathcal{B}''}^3_1) = 0.3324$, $Sc(\tilde{\mathcal{B}''}^3_2) = 0.3604$, $Sc(\tilde{\mathcal{B}''}^3_3) = 0.3100$, $Sc(\tilde{\mathcal{B}''}^3_4) = 0.2493$.

4. In this step, we arrange $\tilde{\mathcal{B}''}^l_i$ with respect to each decision makers using their score functions. The arrangement are as follows: $\tilde{\mathcal{B}''}^1_1 \succ \tilde{\mathcal{B}''}^2_1 \succ \tilde{\mathcal{B}''}^3_1$, $\tilde{\mathcal{B}''}^1_2 \succ \tilde{\mathcal{B}''}^2_2 \succ \tilde{\mathcal{B}''}^3_2$, $\tilde{\mathcal{B}''}^1_3 \succ \tilde{\mathcal{B}''}^2_3 \succ \tilde{\mathcal{B}''}^3_3$, $\tilde{\mathcal{B}''}^2_4 \succ \tilde{\mathcal{B}''}^1_4 \succ \tilde{\mathcal{B}''}^3_4$.
5. In this step, the overall ratings of the decision makers utilizing TrPFHHM operator given by Equation (19) is calculated as follows:

$$\mathcal{B}_1 = f_{TrPFHHM}^{\hat{\xi},\hat{\omega}}(\tilde{\mathcal{B}''}^1_1, \tilde{\mathcal{B}''}^2_1, \tilde{\mathcal{B}''}^3_1)$$
$$= \Bigg[\Bigg(\frac{1}{\frac{0.35}{1.20}+\frac{0.40}{0.55}+\frac{0.25}{0.57}}, \frac{1}{\frac{0.35}{0.91}+\frac{0.40}{0.99}+\frac{0.25}{0.26}}, \frac{1}{\frac{0.35}{1.28}+\frac{0.40}{0.87}+\frac{0.25}{0.77}}, \frac{1}{\frac{0.35}{1.47}+\frac{0.40}{0.74}+\frac{0.25}{0.84}}\Bigg);$$
$$1-(1-0.70)^{0.35}\times(1-0.49)^{0.40}\times(1-0.19)^{0.25}, 0.09^{0.35}\times 0.09^{0.40}\times 0.61^{0.25}, 0.07^{0.35}\times 0.22^{0.40}\times 0.10^{0.25}\Bigg]$$
$$= [(0.69, 0.57, 0.94, 0.93); 0.52, 0.14, 0.12].$$

Similarly, the other aggregated values are
$\mathcal{B}_2 = [(0.54, 0.72, 0.78, 0.65); 0.60, 0.08, 0.13]$,
$\mathcal{B}_3 = [(0.64, 0.63, 0.78, 0.74); 0.54, 0.09, 0.15]$,
and $\mathcal{B}_4 = [(0.56, 0.62, 0.94, 0.49); 0.64, 0.11, 0.11]$.

6. In the last step, the score functions of the alternatives is calculated using Equation (10) $Sc(\mathcal{B}_1) = 0.5478$, $Sc(\mathcal{B}_2) = 0.4943$, $Sc(\mathcal{B}_3) = 0.4848$, and $Sc(\mathcal{B}_4) = 0.4992$. The arrangement of the alternative is $\mathcal{B}_1 \succ \mathcal{B}_4 \succ \mathcal{B}_2 \succ \mathcal{B}_3$.

6.2. Comparative Study

In this article, the proposed method has been studied under picture fuzzy information with TrFN. We have utilized TrPFWHM and TrPFHHM operators to aggregate the information. PFNs play a major part in ranking of the alternatives due to present of its neutral membership degree. Aydin et al. [41] has studied MADM problem under pythagorean fuzzy number. We have compared our method to Aydin's [41] method, taking the neutral membership degree equal to 0. Furthermore, we have compared our proposed operator to Garg's [12] method and it is seen that the ranking of the alternative is $\mathcal{B}_4 \approx \mathcal{B}_2 \succ \mathcal{B}_3 \succ \mathcal{B}_1$. Two alternatives decide their first position by score function in [12]. However, the ranking of the alternative is $\mathcal{B}_4 \succ \mathcal{B}_2 \succ \mathcal{B}_3 \succ \mathcal{B}_1$ by accuracy function in [12]. So, the most desirable alternative is $\mathcal{B}_4$ by Garg's method. The compared results are shown in Table 7. It is evident that the most desirable alternative is $\mathcal{B}_2$ in Aydin's method and it is $\mathcal{B}_4$ in Garg's method, but in the proposed method, the most desirable alternative is $\mathcal{B}_1$.

Table 7. Comparative table.

Method	$Sc(\mathcal{B}_1)$	$Sc(\mathcal{B}_2)$	$Sc(\mathcal{B}_3)$	$Sc(\mathcal{B}_4)$	Ranking Order
Aydin [41]	0.2692	0.2845	0.2761	0.2821	$\mathcal{B}_2 \succ \mathcal{B}_4 \succ \mathcal{B}_3 \succ \mathcal{B}_1$
Garg [12]	0.2600	0.3900	0.2900	0.3900	$\mathcal{B}_4 \approx \mathcal{B}_2 \succ \mathcal{B}_3 \succ \mathcal{B}_1$
Proposed method	0.5478	0.4943	0.4848	0.4992	$\mathcal{B}_1 \succ \mathcal{B}_4 \succ \mathcal{B}_2 \succ \mathcal{B}_3$

6.3. Discussion: Advantages and Disadvantages

Some of the advantages of this study are given below.

- The main advantage of the proposed operators is that the presence of neutral membership grades.
- If there is a situation where the object (element) requires a neutrality degree, then Aydin, Kahraman, and Kabak's [41] method fails.
- A real-life instance of mobile tower site selection is presented utilizing a TrPFWHM and TrPFHHM operator.

The disadvantages of this study are as follows:

- If the values of membership grade, neutral membership grade, and non-membership grade are high in terms of the importance of alternatives, then these operators may not be applicable.

- All the data must be given. However, collection of membership values may not be easy.

6.4. Limitations

Some of the limitations of the proposed work are

- The result of the proposed work are made by using only TrPFWHM and TrPFHHM operators.
- Data collection in real environment may not be easy always.
- If the membership values of the attributes are taken in different environment then this method failed.

7. Conclusions

Information aggregation plays a major part in decision-making process. In the existing papers, the authors have studied aggregation operators under IFSs. In this article, we have introduced some aggregation operators, including TrPFWHM, TrPFOWHM, and TrPFHHM operators. We have introduced a score and accuracy function for TrPFNs. We have studied idempotency, monotonicity, boundedness, and commutativity properties of these operators. We have developed an MADM problem, utilizing the proposed operators. A real-life instance has been considered to illustrate the productivity of the proposed operators. Finally, we have compared our proposed method to the existing method to illustrate the advantages of the proposed operators. In future, we will extend the method to other FSs and apply it for image processing and pattern recognition.

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