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Subclasses of Uniform Univalent Functions Associated with Srivastava and Attiya Operator

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Abstract: In this paper, we introduce new subclasses $k-ST_s(p,\beta)$ and $k-UK_s(p,\beta)$ of analytic and univalent functions in the canonical domain associated with the Srivastava and Attiya operator. The radius problems of these subclasses regarding symmetrical points are investigated and compared with previous known results. Certain properties and conditions of these subclasses such as integral representation are also discussed in this work.

Keywords: Srivastava and Attiya operator; canonical domain; symmetrical points; integral representation; radius of convexity

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1. Introduction

Suppose $f \in A$, where A is the set of analytic functions having the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m,$$

in $E = \{z : |z| < 1\}$. Then, in geometric function theory, E is replaced with an arbitrary domain by Riemann mapping theorem [1].

Let us consider P as the class of the positive real part given by

$$p(z) = 1 + \sum_{m=1}^{\infty} a_m z^m,$$

such that $\Re(p(z)) > 0$.

According to [2], the definition of UCV is

$$\left|\frac{zf''(z)}{f'(z)}\right| < \Re\left(1 + \frac{zf''(z)}{f'(z)}\right), \qquad z \in E.$$

Similarly, any convex function having the property that the image curve of any circular arc γ given by $f(\gamma)$ is a convex arc; then, for every circular arc, γ, which belongs to E with center ξ also in E is called uniformly convex.

According to [2], the definition of UST is

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \Re\left(\frac{zf'(z)}{f(z)}\right), \qquad z \in E. \tag{1}$$

We can also define the class of UST by the Alexander relation if $zf' \in UST$, then $f \in UCV$. Goodman [3] introduced these classes, and several other researchers have also worked on these classes in various repects.

If any function w with conditions $w(0) = 0$ and $|w(z)| < 1$ exists, then it is called a Schwartz function. We can relate any two functions f and g using the Schwartz function w such that $f(z) = g(w(z))$; in this case, it is called "f is subordinate to g" and can be written as $f \prec g$. Similarly, if g is univalent in E, then in particular $f(0) = g(0)$ and $f(E) \subset g(E)$.

The conic region Ω_k with $k \in [0, \infty)$ is studied by [4]:

$$\Omega_k = \left[u + iv : k\sqrt{(u-1)^2 + v^2} < u\right]. \tag{2}$$

For any fixed k, Ω_k denotes the set of conic regions successively bounded by the imaginary axis ($k = 0$), a parabola $v^2 = u - 1$ ($k = 1$), and the right branch of a hyperbolic ($0 < k < 1$). For $k > 1$, it represents the interior of the ellipse, where the domain becomes a bounded domain.

In our condition, we taking $k \in [0, 1]$. Then, using Ω_k, our functions are $p_k(z)$, where k belongs to the closed interval $[0, 1]$, which plays the role of extremal functions mapping in E onto Ω_k:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & (k = 0) \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k = 1) \\ 1 + \frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & (0 < k < 1) \end{cases}. \tag{3}$$

These functions are in class P and univalent in E. Using the subordination technique, the class $P(p_k)$ was introduced in the following form.

Suppose $p(z) \in A$ with the condition $p(0) = 1$. Then, $p(z)$ belongs to $P(p_k)$ iff $p \prec p_k$ in E. Furthermore, $p_k(z)$ is represented by Equation (3).

The generalized conic domain $\Omega_{k,\beta}$ is given by

$$\Omega_{k,\beta} = (1-\beta)\Omega_k + \beta,$$

with the extremal function

$$p_{k,\beta}(z) = (1-\beta)p_k + \beta, \quad \text{with} \quad (0 \le \beta < 1, k \in [0, 1]).$$

The function $p \in P(p_{k,\beta})$ if $p(z) \prec p_{k,\beta}(z)$ in E.

Similarly, it is known from [5] that $P(p_{k,\beta})$ is a convex set. So,

$$P(p_k) \subset P\left(\frac{k}{k+1}\right) \subset P.$$

For $p \in P(p_k)$, we also know that

$$|\arg p(z)| \le \sigma\frac{\pi}{2},$$

with

$$\sigma = \frac{2}{\pi}\arctan\frac{1}{k}. \tag{4}$$

Thus, $p(z) = h^\sigma(z)$, with $h \in P$. Similarly,

$$P(p_{k,\beta}) \subset P\left(\frac{k+\beta}{k+1}\right) \subset P.$$

The starlike functions w.r.t. symmetrical points S_s^* was introduced by Sakaguchi [6]. A necessary and sufficient condition of this class was studied in [6] and is represented as

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) \in P, \qquad z \in E.$$

The convex functions w.r.t. symmetrical points C_s was introduced by Das and Singh [7]. A necessary and sufficient condition of this class was studied in [7] and is represented as

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \in P, \qquad z \in E,$$

and we know that $f \in C_s \Leftrightarrow zf' \in S_s^*$ [7].

According to [8], suppose that $f \in A$. Then f may be in the class $k - ST_s(\beta)$ iff,

$$\frac{2zf'(z)}{f(z)-f(-z)} \in P(p_{k,\beta}), \qquad z \in E.$$

Moreover, an integral operator $\Im_{c,a}$, defined by [9], is

$$\Im_{c,a}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{1+a}{k+a}\right)^c b_k z^k, z \in E.$$

2. Definitions

In this section, we introduce the following new subclasses of univalent function $k - ST_s(p,\beta)$ and $k - UK_s(p,\beta)$.

Definition 1. *Consider $f \in A$. Then f belongs to class of $k - ST_c(p,\beta)$ iff*

$$\frac{(2+c)z(\Im_{c,a}f)'(z)}{\Im_{c,a}(f)(z)-\Im_{c,a}(f)(-z)} - \frac{c}{2} \in P(p_{k,\beta}),$$

where z belongs to E.

Definition 2. *Suppose a function f belongs from the class of analytic functions A. Then f is in the class $k - UCV_c(p,\beta)$ iff zf' belongs to $k - ST_c(p,\beta)$.*

Definition 3. *Let f be an analytic function of class A. Then f belongs to $k - UK_c(p,\beta)$ iff there exists g which is in class $k - ST_c(p,\beta)$. Thus,*

$$\frac{(2+c)z(\Im_{c,a}f)'(z)}{\Im_{c,a}g(z)-\Im_{c,a}g(-z)} - \frac{c}{2} \in P(p_{k,\beta}),$$

where z is in E.

3. Preliminary Results

Our main results depend on the following lemmas:

Lemma 1. *[10] Consider any two functions. Let $q(z)$ and $p(z)$ be convex and analytic functions, respectively, in E with $q(0) = 1 = p(0)$, and function $h_* : E \to C$ for $\Re(fh_*(z)) > 0$, whenever*

$$(h_*(z)zp'(z) + p(z)) \prec q(z), \qquad z \in E.$$

Then, $p(z) \prec q(z)$, where $z \in E$.

Lemma 2. *[8] Consider two analytic functions $N(z)$, $D(z)$ in E such that $N(0) = 0 = D(0)$. Suppose that D is in the class of starlike functions, that is, S^* for $z \in E$, then $\frac{N^*(z)}{D^*(z)} \in P(p_{k,\beta})$ implies that $\frac{N(z)}{D(z)} \in P(p_{k,\beta})$ for $z \in E$.*

Lemma 3. *[4] For any two complex numbers γ_2, δ_2 with $\gamma_2 \neq 0$ and $\Re\left(\frac{\gamma_2 k}{k+1}\right) + \delta_2 > \beta$, the analytic function $h_*(z) \in E$, we have*

$$\left(h_*(z) + \frac{zh_*'(z)}{\gamma_2 h_*(z) + \delta_2}\right) \prec p_{k,\beta}(z). \tag{5}$$

If $q_{k,\beta}$ is the analytic solution of equation

$$p_{k,\beta}(z) = \left(\frac{zq_{k,\beta}'(z)}{\gamma_2 q_{k,\beta}(z) + \delta_2} q_{k,\beta}(z)\right),$$

then $q_{k,\beta}$ is a univalent function whenever

$$h_* \prec q_{k,\beta} \prec p_{k,\beta}.$$

Hence, $q_{k,\beta}(z)$ is said to be the best dominant of Equation (5).

4. Main Results

In this section, we study certain properties of our newly defined subclasses of univalent function $k - ST_s(p,\beta)$ and $k - UK_s(p,\beta)$. The desired results are also compared with existing results.

Theorem 1. *If*

$$\Psi(z) = \frac{1}{2}[-\Im_{c,a} f(-z) + \Im_{c,a} f(z)] \tag{6}$$

is an odd S^* function of order $\beta_1 = \frac{k+\beta}{k+1}$ in E, where $\Im_{c,a} f(z)$ is in the class of k-starlike related with symmetrical points of (p,β), then $\Psi(z) \in k - ST(p,\beta)$.

Proof. Let,

$$\Psi(z) = \frac{1}{2}[-\Im_{c,a} f(-z) + \Im_{c,a} f(z)].$$

Then, after simplification

$$\frac{z\Psi'(z)}{\Psi(z)} = \frac{1}{2+c}[p_2(z) + p_1(z)] + \frac{c}{2+c} \in P(p_{k,\beta}).$$

Here, $\frac{z\Psi'(z)}{\Psi(z)} \in P(p_{k,\beta})$ because $P(p_{k,\beta})$ is a convex set. Therefore,

$$\Psi(z) \in k - ST(\beta).$$

□

Theorem 2. *Let $\Im_{c,a} f(z) \in k - ST_s(p,\beta)$. Then, with $z = e^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi, 1 > \beta > 0$ and $1 \ge k \ge 0$, we can say*

$$\int_{\theta_1}^{\theta_2} \Re\left[\frac{(z(\Im_{c,a} f)'(z))'}{(\Im_{c,a} f)'(z)}\right] d\theta > -\sigma\pi + 2\cos^{-1}\left[\frac{2(1-\beta)}{1-(1-2\beta)r^2}\right] + \beta_1(\theta_2 - \theta_1),$$

where $\sigma = \frac{\pi}{2}\arctan(\frac{1}{k})$ and $\beta_1 = \frac{k+\beta}{k+1}$.

Proof. Suppose,

$$\frac{[\Im_{c,a} f]'}{\Psi'} \in P(p_{\beta,k}),$$

$$\Psi(z) = \frac{1}{2}(-\Im_{c,a} f(-z) + \Im_{c,a} f(z)),$$

where $\Psi \in k - UCV(p,\beta)$ and $C(\beta,p) \supset k - UCV(\beta,p)$.
Therefore,

$$[\Im_{c,a} f]' = (\Psi')^{(-\beta_1+1)} h^{\sigma}(z),$$

with

$$h \in P(p,\beta), \quad \Psi_1 \in C, \quad z = e^{i\theta}, \quad 0 \le \theta_1 < \theta_2 \le 2\pi \quad \text{and} \quad 0 \le r < 1,$$

takes the form

$$\int_{\theta_1}^{\theta_2} \Re\left[\frac{(z(\Im_{c,a} f)'(z))'}{(\Im_{c,a} f)'(z)}\right] d\theta = (1-\beta_1)\int_{\theta_1}^{\theta_2} \Re\left[\frac{(z\Psi'(z))'}{\Psi'(z)}\right] d\theta$$
$$+\sigma \int_{\theta_1}^{\theta_2} \Re\left[\frac{2h'(z)}{h(z)}\right] d\theta + \beta_1(\theta_2 - \theta_1).$$

Let us consider, for $h \in P(p,\beta)$,

$$\frac{\partial}{\partial\theta} \arg h(re^{i\theta}) = \frac{\partial}{\partial\theta} \Re\Big(-i \ln(re^{i\theta})\Big),$$
$$\frac{\partial}{\partial\theta} \arg h(re^{i\theta}) = \Re\left(re^{i\theta} \frac{h' re^{i\theta}}{hre^{i\theta}}\right).$$

Therefore,

$$\int_{\theta_1}^{\theta_2} \Re\left[\frac{re^{i\theta} h' re^{i\theta}}{hre^{i\theta}}\right] d\theta = \arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1}),$$

and

$$\max_{h \in P(p,\beta)} \left| \int_{\theta_1}^{\theta_2} \Re\left[\frac{re^{i\theta} h' re^{i\theta}}{hre^{i\theta}}\right] d\theta \right| = \max_{h \in P(p,\beta)} \left| \arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1}) \right|.$$

So, from above equations

$$p(z) = \frac{1}{-\beta+1}(-\beta + h(z)), \qquad as \qquad p \in P(p_{k,\beta}).$$

With known results $|z| = r < 1$ and

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| \le \frac{2r}{1-r^2},$$

we can write

$$\frac{2(1-\beta)r}{1-r^2} \ge \left| h(z) - \frac{1+(1-2\beta)r^2}{1-r^2} \right|.$$

An Apollonius circle encloses all the values of h. Its diameter is a line-segment which is the combination of points from $\frac{1+(-2\beta+1)r}{1+r}$ to $\frac{1+(1-2\beta)r}{1-r}$, and its radius is $\frac{2(1-\beta)r}{1-r^2}$. Therefore, $|\arg h(z)|$ approaches its max. value wherever a ray passing through the origin is tangent to the circle, i.e.,

$$\arg h(z) = \pm \sin^{-1}\left[\frac{2(-\beta+1)r}{1-(-2\beta+1)r^2}\right]. \tag{7}$$

We can observe from Equation (7) that

$$\max_{h\in P(\beta,p)}\left|\int_{\theta_1}^{\theta_2}\Re\left[\frac{re^{i\theta}h're^{i\theta}}{hre^{i\theta}}\right]d\theta\right|\leq 2\sin^{-1}\left(\frac{2(1-\beta)r}{1-(1-2\beta)r^2}\right),$$

$$\max_{h\in P(p,\beta)}\left|\int_{\theta_1}^{\theta_2}\Re\left[\frac{re^{i\theta}h're^{i\theta}}{hre^{i\theta}}\right]d\theta\right|=\pi-2\cos^{-1}\left(\frac{2(1-\beta)r}{1-(1-2\beta)r^2}\right), \tag{8}$$

and for $\Psi_1\in C$,

$$\int_{\theta_1}^{\theta_2}\Re\left[1+re^{i\theta}\frac{\Psi_1''(re^{i\theta})}{\Psi_1'(re^{i\theta})}\right]d\theta\geq 0. \tag{9}$$

Using Equations (7)–(9), we obtain

$$\int_{\theta_1}^{\theta_2}\Re\left[\frac{(z(\Im_{c,a}f)'(z))'}{(\Im_{c,a}f)'(z)}\right]d\theta>-\sigma\pi+2\cos^{-1}\left[\frac{2(1-\beta)}{1-(1-2\beta)r^2}\right]+\beta_1(\theta_2-\theta_1).$$

□

Theorem 3. *Let $\Im_{c,a}f(z)\in k-ST_s(p,\beta)$; then its integral representation is*

$$(\Im_{c,a}f(z))'=\frac{1}{2+c}p(z)\exp\int_0^z\frac{1}{t(2+c)}(-p(-t)+p(t)-(2+c))dt,$$

where $z\in E$ and $p\in P(p_{k,\beta})$.

Proof. Let $\Im_{c,a}f$ be taken from $k-ST_s(p,\beta)$; then

$$p(z)=\frac{(2+c)z(\Im_{c,a}f(z))'}{\Im_{c,a}f(z)-\Im_{c,a}f(-z)}-\frac{c}{2},\qquad p\in P(p,\beta).$$

$$\frac{[\Im_{c,a}f(z)-\Im_{c,a}f(-z)]'}{-\Im_{c,a}f(-z)+\Im_{c,a}f(z)}-\frac{1}{z}=\frac{1}{(2+c)z}[p(z)-p(-z)-(2+c)].$$

After simplification, we can get

$$(2+c)(\Im_{c,a}f(z))'\frac{1}{p(z)}=\exp\int_0^z\frac{1}{(2+c)t}[p(t)-p(-t)-(2+c)]dt.$$

By putting $F'(z)=(\Im_{c,a}f(z))'$, we obtain

$$F'(z)=\frac{p(z)}{2+s}\exp\int_0^z\frac{1}{(2+c)t}[p(t)-p(-t)-(2+c)]dt.$$

□

Special case:
If $c=0$, one can get the result in the form

$$\frac{p(z)}{2}\exp\int_0^z\frac{1}{2t}[p(t)-p(-t)-2]dt=f'(z),$$

proved by K. I. Noor [8].

Theorem 4. *Let $\Im_{c,a}g(z)\in k-ST_s(p,\beta)$ and $m=1,2,3,4,\ldots,G$, where*

$$G(z)=\frac{m+c+1}{2z^{(m+c)}}\int_0^z t^{(m+c-1)}[\Im_{c,a}g(t)-\Im_{c,a}g(-t)]dt. \tag{10}$$

Then $G(z)\in k-ST(p,\beta)$.

Proof. Let

$$J(z) = \int_0^z t^{(m+c-1)} \left[\frac{1}{2}[\Im_{c,a}g(t) - \Im_{c,a}g(-t)]\right] dt.$$

since $\Im_{c,a}g(z) \in k - ST_s(p,\beta)$, $\frac{1}{2}[\Im_{c,a}g(t) - \Im_{c,a}g(-t)] \in k - ST(p,\beta) \subset S^*(\beta_1) \subset S^*$ and $\beta_1 = \frac{k+\beta}{k+1}$. Therefore, using [8], we can say that $J(z)$ is a function and $(1+m)$-valently starlike (S^*) in E. So Equation (10) can be written as,

$$z^{(m+c)}G(z) = (m+c+1)\int_0^z t^{(m+c-1)} \left[\frac{1}{2}[\Im_{c,a}g(t) - \Im_{c,a}g(-t)]\right] dt,$$

or

$$z^{(m+c)}G(z) = (m+c+1)J(z).$$

After simplification,

$$z\frac{G'(z)}{G(z)} = \frac{N(z)}{D(z)} = \frac{zJ'(z) - (m+c)J(z)}{J(z)}. \tag{11}$$

As $D(0) = 0$ and $N(0) = 0$. Furthermore, $D(z)$ is $(1+m)$-valently S^*. Let $\frac{N(z)}{D(z)} = h(z)$, then

$$h'(z)D(z) + h(z)D'(z) = N'(z).$$

Therefore,

$$h(z) + \frac{zh^z}{h_\circ(z)} = \frac{N'(z)}{D'(z)}.$$

Let, $h_\circ(z) = \frac{zD'(z)}{D(z)} \in P(p_{k,\beta})$ and $H_\circ(z) = \frac{1}{h_\circ(z)} \in P(p_{k,\beta})$. Then,

$$\frac{N'(z)}{D'(z)} = h(z) + H_\circ(z)(zh^z).$$

From Equation (11), we have

$$\frac{N(z)}{D(z)} = \frac{zJ'(z) - (m+s)J(z)}{J(z)}.$$

This implies

$$\frac{N'(z)}{D'(z)} = \left[\frac{(zJ')'}{J'(z)} - (m+c)\right] \in P(p_{k,\beta}).$$

Using Lemma 2, we can say

$$\frac{N(z)}{D(z)} = \frac{zG(z)}{G(z)} \in P(p_{p,\beta}), \qquad z \in E.$$

Therefore, $G \in k - ST(p,\beta)$ in E. □

Theorem 5. *Let $\Im_{c,a}f, \Im_{c,a}g \in k - ST_s(p,\beta)$, and $\Im_{c,a}F(z)$ be defined as*

$$\Im_{c,a}F(z) = \left(\gamma + s + \frac{1}{\delta}\right) z^{(1-c-\frac{1}{\delta})} \int_0^z t^{(\frac{1}{\delta}+c-2)} \left[\frac{\Im_{c,a}f(t) - \Im_{c,a}f(-t)}{2}\right]^{\frac{1}{1+\gamma}} \left[\frac{\Im_{c,a}g(t) - \Im_{c,a}g(-t)}{2}\right] dt, \tag{12}$$

where $z \in E$, $0 < \delta$, $c \geq 0$, $\gamma = 0$ and $\frac{k(1+\gamma)}{k+1} + (c + \frac{1}{\delta} - 1) > \beta$. Then $\Im_{c,a}F(z) \in k -$ starlike with (p, β) where $z \in E$. If $z = \Im_{c,a}g(z)$ and $\gamma = c = 0$, then we can get the Bernardi operator in its generalized form [11]. For $\Im_{c,a}g(z) = z$, $\gamma = 0$, $\delta = \frac{1}{2}$ and $c = 0$, we can get an integral operator introduced by Libra that preserves geometric properties of close-to-convexity, convexity, and starlikeness [12,13].

Proof. Let, $\dfrac{\Im_{c,a}f(z) - \Im_{c,a}f(-z)}{2} = \Psi_1(z)$ and $\dfrac{\Im_{c,a}g(z) - \Im_{c,a}g(-z)}{2} = \Psi_2(z)$. Then, $\Psi_1, \Psi_2 \in k - ST(p, \beta)$ in E, and we can write Equation (12) as

$$F_1 = \Im_{c,a}F(z) = \left(\gamma + c + \frac{1}{\delta}\right) z^{(1-c-\frac{1}{\delta})} \int_0^z t^{(\frac{1}{\delta}+c-2)} [\Psi_1(t)]^{\frac{1}{1+\gamma}} [\Psi_2(t)] dt. \tag{13}$$

If $p(z) = \frac{zF_1'(z)}{F_1(z)}$, then after simplification, we have

$$\frac{\gamma}{1+\gamma} \frac{z\Psi_1'}{\Psi_1(z)} + \frac{1}{1+\gamma} \frac{z\Psi_2'}{\Psi_2(z)} = p(z) + \frac{zp'(z)}{(1+\gamma)p(z) + (c + \frac{1}{\delta} - 1)}. \tag{14}$$

Since Ψ_1 and $\Psi_2 \in k - ST(p, \beta)$, which implies that $\dfrac{z\Psi_1'}{\Psi_1}$ and $\dfrac{z\Psi_2'}{\Psi_2} \in P(p_{k,\beta})$ in E, and $P(p_{k,\beta})$ also belongs to convex set. It follows that

$$\left(\frac{\gamma}{1+\gamma} \frac{z\Psi_1'}{\Psi_1(z)} + \frac{1}{1+\gamma} \frac{z\Psi_2'}{\Psi_2(z)}\right) \in P(p_{k,\beta}), \qquad z \in E. \tag{15}$$

Similarly, Equations (14) and (15) give

$$\left(p(z) + \frac{zp'(z)}{(1+\gamma)p(z) + (c + \frac{1}{\delta} - 1)}\right) \prec p_{k,\beta}(z).$$

Using Lemma 3, we can also say

$$p(z) \prec q_{k,\beta}(z) \prec p_{k,\beta}(z).$$

Hence, $F_1 \in k - ST(p, \beta)$. $\square$

5. The Class $k - UK_s(p, \beta)$

In this work, we study certain properties of $k - UK_s(p, \beta)$, which consists of $k - UK$ functions with symmetrical points of order β [14].

Theorem 6. *Let $\Im_{c,a}f \in k - UK_s(k, \beta)$ and $F_1(z) = \Im_{c,a}F$ be defined by*

$$F_1(z) = \frac{m + c + 1}{2z^{(m+c)}} \int_0^z t^{(m+c-1)} [\Im_{c,a}f(t) - \Im_{c,a}f(-t)] dt. \tag{16}$$

Then $F_1(z)$ belongs to the class $k - UK_s(p, \beta)$ in E.

Proof. Since $\Im_{c,a}f \in k - UK_s(k, \beta)$, we have

$$\frac{2z\Im_{c,a}f'(z)}{\Im_{c,a}g(z) - \Im_{c,a}g(-z)} \in P(p_{k,\beta}),$$

and

$$\Im_{c,a}g \in k - ST_s(k, \beta) \subset S_s^*(k\beta_1).$$

Let $G_1 = \dfrac{\Im_{c,a}g_1(z) - \Im_{c,a}g_1(-z)}{2}$ be defined by Theorem 4, also $g_1 \in k - ST(p, \beta)$ and $G_1 \in k - ST_s(p, \beta) \subset S_s^*(p, \beta_1)$. Similarly, $G = zG_1'$, then

$$G' = \frac{1}{2}[zg_1(z) - (-z)g(-z)]',$$

$$G \in k - UCV_s(p, \beta),$$

$$g = z\Im_{c,a} g_1',$$

and

$$g \in C_s(p, \beta_1).$$

From Equation (16), we have

$$\frac{2}{m+c+1}\left[(m+c)z^{(m+c-1)}F_1 + z^{(m+c)}F_1'\right] = z^{(m+c-1)}[\Im_{c,a}f(z) - \Im_{c,a}f(-z)],$$

which implies that

$$\frac{2F_1'}{G_1'} = \frac{N(z)}{D(z)}.$$

We can conclude that $D(0) = N(0) = 0$, also $g \in C_s(p, \beta_1)$,

$$\frac{(zD'(z))'}{D'(z)} = (m+c) + \frac{[z\Im_{c,a}g_1(z) - \Im_{c,a}g_1(-z)']'}{[\Im_{c,a}g_1(z) - \Im_{c,a}g_1(-z)]'},$$

and

$$\frac{(zD'(z))'}{D'(z)} = (m+c) + h_1, \qquad h_1 \in P(p, \beta_1).$$

Since $P(p, \beta_1)$ belongs to the convex set, where $D \in C_s(p, \beta_1) \subset S^*(p, \beta_1)$ in E [8]. Therefore,

$$\frac{N(z)}{D(z)} = \frac{2F_1'}{G_1'} \in P(p_{k,\beta}), \qquad for \quad z \in E.$$

Hence, $F_1(z) \in k - UK_s(p, \beta)$ in E. $\square$

Theorem 7. *Let us consider,*

$$\left(\frac{(2+c)z\Im_{c,a}f'(z)}{\Im_{c,a}g(z) - \Im_{c,a}g(-z)} - \frac{c}{2}\right) \prec p_{k,\beta}(z)$$

in E, and

$$F_1(z) = \frac{1}{1+m}z^{(1-m)}[z^m \Im_{c,a}f(z)]', \tag{17}$$

where $m = 1, 2, 3, 4, \ldots$. Thus, $F_1 \in K_s(p, \beta_1)$ for $|z| < r_1$, with

$$r_1 = \frac{1+m-\frac{c}{2}}{(2-\beta) + \sqrt{(2-\beta)^2 - (1+m-\frac{c}{2})(-m-2\beta_1+1+\frac{c}{2})}}, \tag{18}$$

where

$$\beta_1 = \frac{k+\beta}{k+1}.$$

Proof. For $p \in P(p, \alpha)$ with $1 > \alpha \geq 0$, we require the following results [15]:

$$\frac{1+(1-2\alpha)r}{1-r} \geq |p(z)| \geq \frac{1-(1-2\alpha)r}{1+r}, \tag{19}$$

with

$$\frac{2[\Re(p(z)) - \alpha]r}{1-r^2} \geq |p'(z)|. \tag{20}$$

Since $\Im_{c,a}f \in k - UK_s(p, \beta)$, $\exists\ \Im_{c,a}g \in S_s^*(p, \beta_1)$, such that, for $z \in E$

$$\left[\frac{(2+c)z\Im_{c,a}f'(z)}{\Im_{c,a}g(z)-\Im_{c,a}g(-z)}-\frac{c}{2}\right]=p(z),$$

where

$$p \in P_{k,\beta} \subset P(\alpha), \qquad \text{and} \qquad \alpha = \frac{k}{k+1}.$$

From Equation (17), we have

$$F_1(z)=\frac{1}{m+1}z^{(1-m)}\left[mz^{(m-1)}\Im_{c,a}f(z)+z^m\Im_{c,a}f'(z)\right].$$

After simplification, we can write

$$\frac{(2+c)zF_1'(z)}{\Im_{c,a}g(z)-\Im_{c,a}g(-z)}-\frac{c}{2}=\frac{1}{1+m}\left[mp(z)+zp'(z)+\left(p(z)+\frac{c}{2}\right)h(z)-\frac{c}{2}\right],$$

with

$$h(z)=\frac{z\Psi'(z)}{\Psi(z)} \in P(p,\beta_1), \qquad \text{and} \qquad \Psi(z)=\Im_{c,a}g(z)-\Im_{c,a}g(-z).$$

By the use of Equations (19) and (20), we have

$$\Re\left[\frac{(2+c)zF_1'}{\Im_{c,a}g(z)-\Im_{c,a}g(-z)}-\frac{c}{2}\right] \geq \frac{\Re[p(z)-\alpha]}{1+m}$$
$$\left[\frac{2m(1-r^2)-4r+2(1-(1-2\beta_1 r))(1-r)}{2(1-r^2)}-\frac{c(1-r^2)}{2(1-r^2)}\right], \tag{21}$$

where

$$T(r)=2m(1-r^2)-4r+2(1-(1-2\beta_1 r))(1-r)-c(1-r^2),$$

or

$$T(r)=(-2m-4\beta_1+c+2)r^2-4(2-\beta_1)r+(2m-c+2).$$

So,

$$r_1=\frac{1+m-\frac{c}{2}}{(2-\beta)+\sqrt{(2-\beta)^2-(1+m-\frac{c}{2})(-m-2\beta_1+1+\frac{c}{2})}}.$$

Therefore, $F_1 \in K_s(p,\beta_1)$ for $|z_1|<r_1$. $\square$

Special cases:

1. For $c=0$, we have the result obtained by [8].
2. For $\beta=k=0$, $f \in K_s$ and $c=0$. Then, $F_1 \in K_s$ for $r_\circ = \frac{1+m}{2+\sqrt{3+m^2}} > |z|$, defined by Equation (17).
3. When $\beta_1=0$ $(\beta=0=k)$, $c=0$ and $m=1$; then $F_1(z)=\frac{[zf(z)]'}{2}$ is in the class for $|z|<\frac{1}{2}$, which is proved by Livingston [16] for S^* and C functions.

6. Conclusions

New subclasses $k-ST_s(p,\beta)$ and $k-UK_s(p,\beta)$ of analytic and univalent functions have been defined in canonical domain associated with the Srivastava and Attiya operator. Furthermore, several results including integral representation and radius problems of these subclasses have been derived and compared with different known results in this work.

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