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Linear Diophantine Fuzzy Relations and Their Algebraic Properties with Decision Making

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Abstract: Binary relations are most important in various fields of pure and applied sciences. The concept of linear Diophantine fuzzy sets (LDFSs) proposed by Riaz and Hashmi is a novel mathematical approach to model vagueness and uncertainty in decision-making problems. In LDFS theory, the use of reference or control parameters corresponding to membership and non-membership grades makes it most accommodating towards modeling uncertainties in real-life problems. The main purpose of this paper is to establish a robust fusion of binary relations and LDFSs, and to introduce the concept of linear Diophantine fuzzy relation (LDF-relation) by making the use of reference parameters corresponding to the membership and non-membership fuzzy relations. The novel concept of LDF-relation is more flexible to discuss the symmetry between two or more objects that is superior to the prevailing notion of intuitionistic fuzzy relation (IF-relation). Certain basic operations are defined to investigate some significant results which are very useful in solving real-life problems. Based on these operations and their related results, it is analyzed that the collection of all LDF-relations gives rise to some algebraic structures such as semi-group, semi-ring and hemi-ring. Furthermore, the notion of score function of LDF-relations is introduced to analyze the symmetry of the optimal decision and ranking of feasible alternatives. Additionally, a new algorithm for modeling uncertainty in decision-making problems is proposed based on LDFSs and LDF-relations. A practical application of proposed decision-making approach is illustrated by a numerical example. Proposed LDF-relations, their operations, and related results may serve as a foundation for computational intelligence and modeling uncertainties in decision-making problems.

Keywords: linear Diophantine fuzzy sets; linear Diophantine fuzzy relations; equivalence linear Diophantine fuzzy relations; symmetry; decision-making

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1. Introduction

In this modern age of technology, modeling uncertainties in engineering, computer sciences, social sciences, medical sciences and economics is growing widely. To tackle such types of problems, classical mathematical methods are not always useful. In 1965, Zadeh [1], coined the notion of fuzzy set (FS) to handel the uncertainties in every day language. In FS theory, a person who is very sick could have the degree of sickness near to 0.89. On contrary, a person who is having degree of sickness 0.12 indicates that he has nearly recovered from illness. In decision making (DM) and operational research the concept of FS theory was broadly studied since 1965 (see [2–4]).

However, only the membership function is not always sufficient to describe the complexities in real life problems. In [5–7], Atanassov proposed the concept of intuitionistic fuzzy set (IFS) as an extension of FS. Atanassov's IFS enhanced the idea of FS by allowing the objects non-membership degrees along with the existing membership degrees satisfying the condition that their sum will not exceed 1. Atanassov [8] introduced the concept of IF-relations on IFSs. Since then, there is a lot of research on IFSs and they have many real-life applications such as optimization in IF-environment [9], multi-attribute decision making (MADM) [10–12]. Feng et al. [13] introduced lexicographic orders of IF-values and their relationships.

Although, in some real life problems, the sum of membership and non-membership grades of the objects may exceed 1. To eradicate this problem, Yager [14,15] extended the concept of IFS to Pythagorean fuzzy set (PFS), where the sum of the squares of its membership and non-membership degrees must not exceed 1. Yager's concept of PFS is also familiar as Atanassov's IFS of type 2 [16]. Many scientists have been studied PFS in various aspects (see [17–19]). Furthermore, Yager in [20] generalized the notion of PFS and defined a new concept of q-rung orthopair fuzzy set (q-ROFS). Further research on q-ROFS with significant advances were made in [21,22]. Zhang [23] introduced bipolar FS and relations, and Chen [24] proposed m-polar FS. Akram [25] studied m-polar F graphs, theory, methods and applications in DM.

The concepts of FSs, IFSs, PFSs, and q-ROFSs, have a lot of research and applications in real-life. However, these sets have some strict conditions on membership and non-membership grades. In order to relax these strict conditions, Riaz and Hashmi [26], introduced the novel concept of linear Diophantine fuzzy set (LDFS). A LDFS extended the space of above mentioned sets by adding reference parameters corresponding to the membership and non-membership grades. LDFSs are most suitable mathematical structure in MADM where the decision makers can freely select the grades [26]. The study of LDFSs is growing rapidly, in recent days. Riaz et al. [27] established the notions of hybrid models namely, linear Diophantine fuzzy soft rough sets (LDFSRSs), and soft rough linear Diophantine fuzzy sets (SRLDFSs). They have also applied these notions on robust MCDM problem for the selection of sustainable material handling equipment. Recently, Kamaci [28] extended LDFSs towards various algebraic structures such as groups, rings and fields, and studied some related important properties. Almagrabi [29] introduced a new approach to q-linear Diophantine fuzzy emergency decision support system for COVID-19.

Binary relations are important and fundamental in different fields of pure and applied sciences to describe the correspondences among various objects. Since there are many real life objects which may not satisfy the bivalent condition and may be related to each other to a certain degree. In 1971, Zadeh [2] fuzzified the notion of binary relation and introduced the concept of fuzzy relation (F-relation). F-relations play an important role in FS theory and its applications. To model the situations where interactions between elements are more or less strong, F-relations are very useful. FS and FS-relations have a lot of important applications in diverse type of areas, for instance in F modeling, F control and uncertainty reasoning, neural network, data bases, pattern recognition, artificial intelligence (AI), clustering, medicine, economy and MCDM. A detailed study on FSs and F-relations is presented in "Mathematics of fuzziness—Basic issues" by Wang et al. in [30].

Zadeh's F-relation provide the degree to which two objects are related to each other. However, in real life, there may some objects which are related to each other to a certain degree but may not. That is, there may be some hesitation or uncertainty about the degree that is assigned to the relationship of the objects. This problem is addressed by Atanassov in [8] and introduced the notion of intuitionistic fuzzy relation (IF-relation) as an extension of F-relation. IF-relation is basically a pair of F-relations, named as membership and non-membership F-relations, which represents both the negative and positive aspects of the given information. IF-relation is the most effective approach to deal with DM in medical diagnosis. Composition of two IF-relations and some of its properties were studied by [31–33]. Further, the concept of IF-equivalence relations waas studied by Hur et al. in [34]. Naeem et al. presented the novel concepts of Pythagorean m-polar FSs with applications in MCDM. Molodtsov [35] introduced soft set theory to deal vague and uncertain real-life problems with the help of parameterizations. Riaz et al., introduced the idea of m-polar neutrosophic sets and m-polar neutrosophic topology with applications to MADM [36].

The main objective of this paper is to introduce the concept of linear Diophantine fuzzy relation (LDF-relation) as an extension of IF-relation. A second objective of LDF-relation is to address modeling uncertainties in MCDM. Because LDF-relation is more efficient to relax the strict restrictions of IF-relation regarding membership and non-membership grades. Some operations on LDF-relation and their properties are investigated. Additionally, algebraic structures such as semigroup, semiring and hemiring are studied in the set of all LDF-relations. A third objective is to introduce notion of score function of LDF-relation and to analyze the symmetry of the optimal decision and ranking of feasible alternatives. A fourth objective is to develop a new algorithm and present its practical application to MCDM problems based on LDFSs and LDF-relations.

This manuscript is composed in the following order: Section 2 contains some basic concepts of FSs, IFSs, LDFSs, F-relation, IF-relations, semigroup, semiring and hemiring. Section 3 introduces the concept of LDF-relation and some fundamental operations with some significant properties. With the help of these operations and properties, some algebraic structures such as semigroup, semiring and hemiring, in the set of all LDF-relations are introduced. Section 4 is devoted to constructing an algorithm for DM with a numerical example. Finally, Section 5 presents the conclusion of this research paper.

2. Preliminaries

This section includes some essential concepts which are useful in the remaining sections of the manuscript. For detailed study, we refer the reader to [1,2,8,26,32,37]. In the whole manuscript, $\mathcal{Q}$ will be supposed to be a universal set.

Definition 1. *[1] A FS δ on $\mathcal{Q}$ is a mapping*

$$\delta : \mathcal{Q} \rightarrow [0,1]$$

known as membership function which assigns the grade of membership to each object $v \in \mathcal{Q}$ in δ. The set of all FSs on $\mathcal{Q}$ is denoted by $\mathcal{F}(\mathcal{Q})$.

A binary relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is a subset of the cartesian product $\mathcal{Q}_1 \times \mathcal{Q}_2$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two universes. In 1971, Zadeh [2] fuzzified the structure of binary relation and introduced a new concept, known as F-relation.

Definition 2. *[2] A F-relation $\mathcal{R}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is simply a F-subset of $\mathcal{Q}_1 \times \mathcal{Q}_2$. That is, a F-relation or a F-binary relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is a membership function*

$$\mathcal{R} : \mathcal{Q}_1 \times \mathcal{Q}_2 \rightarrow [0,1]$$

which assigns the grade of membership to each pair $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ in $\mathcal{R}$. The set of all F-relations from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is represented by $\mathcal{F}(\mathcal{Q}_1 \times \mathcal{Q}_2)$.

Definition 3. *[5] An IFS in $\mathcal{Q}$ is an object of the following form:*

$$\mathcal{I} = \{(v, < \delta^{\mathcal{M}}(v), \delta^{\mathcal{N}}(v) >) : v \in \mathcal{Q}\}$$

where the mappings

$$\delta^{\mathcal{M}}, \delta^{\mathcal{N}} : \mathcal{Q} \rightarrow [0,1]$$

represent the membership and non-membership functions, respectively, satisfying the following condition:

$$0 \leq \delta^{\mathcal{M}}(v) + \delta^{\mathcal{N}}(v) \leq 1.$$

for all $v \in \mathcal{Q}$. Hesitation part is defined by $\lambda(v) = 1 - (\delta^{\mathcal{M}}(v) + \delta^{\mathcal{N}}(v))$ for each $v \in \mathcal{Q}$. The set of all IFSs is denoted by $\mathcal{IF}(\mathcal{Q})$.

In 1984, Atanassov [8] also generalized the concept of F-relation [2] and introduced the concept of IF-relation.

Definition 4. *[8] An IF-relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is an IF-subset of $\mathcal{Q}_1 \times \mathcal{Q}_2$, that is an expression of the following form:*

$$\mathcal{R}_{\mathcal{I}} = \{((v_1, v_2), < \delta^{\mathcal{M}}_{\mathcal{R}_{\mathcal{I}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathcal{I}}}(v_1, v_2) >) : v_1 \in \mathcal{Q}_1, v_2 \in \mathcal{Q}_2\}$$

where the membership and non-membership F-relations

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathcal{I}}}, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathcal{I}}} : \mathcal{Q}_1 \times \mathcal{Q}_2 \to [0, 1]$$

satisfy the condition $0 \leq \delta^{\mathcal{M}}_{\mathcal{R}_{\mathcal{I}}}(v_1, v_2) + \delta^{\mathcal{N}}_{\mathcal{R}_{\mathcal{I}}}(v_1, v_2) \leq 1$ for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$. The set of all IF-relations from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ is denoted by $\mathcal{IFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$.

Definition 5. *[26] A LDFS on $\mathcal{Q}$ is an object defined as follows:*

$$\pounds_{\mathfrak{D}} = \{(v, < \delta^{\mathcal{M}}_{\mathfrak{D}}(v), \delta^{\mathcal{N}}_{\mathfrak{D}}(v) >, < \alpha(v), \beta(v) >) : v \in \mathcal{Q}\}$$

where

$$\delta^{\mathcal{M}}_{\mathfrak{D}}, \delta^{\mathcal{N}}_{\mathfrak{D}} : \mathcal{Q} \to [0, 1]$$

are membership and non-membership functions, and $\alpha(v), \beta(v) \in [0, 1]$ are the reference parameters of $\delta^{\mathcal{M}}_{\mathfrak{D}}(v), \delta^{\mathcal{N}}_{\mathfrak{D}}(v)$ respectively, with $0 \leq \alpha(v)\delta^{\mathcal{M}}_{\mathfrak{D}}(v) + \beta(v)\delta^{\mathcal{N}}_{\mathfrak{D}}(v) \leq 1$ satisfying $0 \leq \alpha(v) + \beta(v) \leq 1$ for all $v \in \mathcal{Q}$. The hesitation part is defined as $\xi(v)\pi_{\mathfrak{D}}(v) = 1 - (\alpha(v)\delta^{\mathcal{M}}_{\mathfrak{D}}(v) + \beta(v)\delta^{\mathcal{N}}_{\mathfrak{D}}(v))$, where $\pi_{\mathfrak{D}}(v)$ is known to be the degree of indeterminacy of v to $\pounds_{\mathfrak{D}}$, and $\xi(v)$ is reference parameter related to the degree of indeterminacy. We shall denote the collection of all LDFSs on $\mathcal{Q}$ by $\mathcal{LDF}(\mathcal{Q})$.

In the following of this section, we shall recall some definitions of semigroup, semiring and hemiring.

Definition 6. *A non-empty set $\mathcal{S}$ together with an associative binary operation $*$ defined on $\mathcal{S}$ is called a semigroup. It is usually denoted by the pair $(\mathcal{S}, *)$.*

Definition 7. *A semigroup $(\mathcal{S}, *)$ is called:*

(1) monoid, if there exists an element $e \in \mathcal{S}$ such that $e * a = a * e = a$ for all $a \in \mathcal{S}$.
(2) idempotent, if $a * a = a$ for all $a \in \mathcal{S}$.

Definition 8. *A non-empty set $\mathcal{R}$ with two binary operations $+$, and $\cdot$ is called a semiring, if*

(1) $(\mathcal{R}, +)$ is semigroup.
(2) $(\mathcal{R}, \cdot)$ is semigroup.
(3) Multiplication is distributive over addition from both sides, that is,
  (i) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
  (ii) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

  for all $a, b, c \in \mathcal{R}$. We shall denote a semiring with two binary operations $+, \cdot$ by $(\mathcal{R}, +, \cdot)$.
(4) A semiring $(\mathcal{R}, +, \cdot)$ is called commutative, if $(\mathcal{R}, \cdot)$ is commutative semigroup, that is, $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{R}$.
(5) A semiring $(\mathcal{R}, +, \cdot)$ is said to have an identity element e, if for any $a \in \mathcal{R}$ $a \cdot e = e \cdot a$ for some $e \in \mathcal{R}$.
(6) A semiring $(\mathcal{R}, +, \cdot)$ is said to have a zero element 0, if for any $a \in \mathcal{R}$
  (i) $a + 0 = 0 + a = a$.

(ii) $a \cdot 0 = 0 \cdot a = 0$.

for some $0 \in \mathcal{R}$.

Definition 9. *A hemiring is a semiring* $(\mathcal{R}, +, \cdot)$ *such that*

(1) $(\mathcal{R}, +)$ *is commutative semigroup.*
(2) $(\mathcal{R}, +, \cdot)$ *have zero element* 0.

3. Linear Diophantine Fuzzy Relation (Ldf-Relation)

We know that the binary relations are just the subsets of the cartesian product of two universes and they play a vital role in both pure and applied sciences. To extend the existing notion of IF-relation, we applied the notion of LDFS [26] to binary relations which removes the restrictions of IF-relations on membership and non-membership F-relations. In this regard, a new concept of LDF-relation is introduced in the motivation of Riaz and Hashmi's work [26] only with the addition of reference parameters corresponding to membership and non-membership F-relations respectively.

Definition 10. *A LDF-relation* $\mathcal{R}_{\mathfrak{D}}$ *from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$ *is an expression of the following form:*

$$\mathcal{R}_{\mathfrak{D}} = \{((v_1, v_2), < \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) >, < \alpha(v_1, v_2), \beta(v_1, v_2) >) : v_1 \in \mathcal{Q}_1, v_2 \in \mathcal{Q}_2\}$$

where the mappings

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} : \mathcal{Q}_1 \times \mathcal{Q}_2 \to [0, 1]$$

are denoting the membership, and non-membership F-relations from $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$*, respectively, and* $\alpha(v_1, v_2), \beta(v_1, v_2) \in [0, 1]$ *are the corresponding reference parameters to* $\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2)$ *and* $\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2)$ *respectively. These membership and non-membership F-relations satisfy the condition*

$$0 \leq \alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \leq 1$$

for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ *with* $0 \leq \alpha(v_1, v_2) + \beta(v_1, v_2) \leq 1$*. For an LDF-relation from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$*, we shall use*

$$\mathcal{R}_{\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) >, < \alpha(v_1, v_2), \beta(v_1, v_2) >) \tag{1}$$

for the sake of simplicity. The F-relation $\pi_{\mathfrak{D}} : \mathcal{Q}_1 \times \mathcal{Q}_2 \to [0, 1]$ *associated with each LDF-relation 1, where*

$$\gamma_{\mathfrak{D}}(v_1, v_2)\pi_{\mathfrak{D}}(v_1, v_2) = 1 - (\alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2))$$

The number $\pi_{\mathfrak{D}}(v_1, v_2)$ *is an index (a degree) of hesitation wether* v_1 *and* v_2 *are the relation* $\mathcal{R}_{\mathfrak{D}}$ *or not, and* $\gamma_{\mathfrak{D}}(v_1, v_2)$ *is the reference parameter of degree of hesitation. We shall denote the set of all LDF-relations from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$ *by* $\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$*.*

By Definition 10, an LDF-relation $\mathcal{R}_{\mathfrak{D}}$ is simply an LDFS on $\mathcal{Q}_1 \times \mathcal{Q}_2$.

Remark 1.

(i) *Since every binary relation is a F-relation and every F-relation is an IF-relation with non-zero membership grade and zero non-membership grade. For parametric values* $\alpha(v_1, v_2) \neq 0$, $\beta(v_1, v_2) = 0$*, and* $\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \neq 0$*, an LDF-relation* $\mathcal{R}_{\mathfrak{D}}$ *is a F-relation. For any reference parameters* $\alpha(v_1, v_2), \beta(v_1, v_2) \in [0, 1]$ *with* $0 \leq \alpha(v_1, v_2) + \beta(v_1, v_2) \leq 1$*, an IF-relation satisfies the condition*

$$0 \leq \alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \leq 1$$

Hence, every IF-relation is also an LDF-relation. However, the converse is not true in general as it is proved in case of LDFSs [26], page 5423.

(ii) *If the reference parameters* $\alpha(v_1, v_2), \beta(v_1, v_2) \in [0, 1]$ *do not satisfy the condition* $0 \leq \alpha(v_1, v_2) + \beta(v_1, v_2) \leq 1$*, then* $\alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2)$ *may exceed* 1*. For instance, if* $\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) = 0.88$*,* $\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) = 0.91$*, and* $\alpha(v_1, v_2) = 0.55$*,*

$\beta(v_1, v_2) = 0.71$. *It is clear that* $\alpha(v_1, v_2) + \beta(v_1, v_2) > 1$, *and hence* $\alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) = 1.130 \nleq 1$.

(iii) *If* $\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \neq 0$, $\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) = 1$, *and* $\alpha(v_1, v_2) \neq 0$, *then* $\beta(v_1, v_2) \neq 1$. *Because, then* $\alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \geq 1$.

(iv) *The Definition 10 of LDF-relation can be extended to* $n-$*universal sets* $\mathcal{Q}_1 \times \mathcal{Q}_2 \times ... \times \mathcal{Q}_n$ *in similar manners.*

In the motivation of the matrix notation of F-relations defined in [30], the matrix notation of LDF-relations is defined below.

Definition 11. *Let* $\mathcal{R}_{\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j) >, < \alpha(\mathfrak{x}_i, \mathfrak{x}_j), (\mathfrak{x}_i, \mathfrak{x}_j) >)$ *be an LDF-relation from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$, *where* $\mathcal{Q}_1 = \{\mathfrak{x}_1, \mathfrak{x}_2, ..., \mathfrak{x}_m\}$, *and* $\mathcal{Q}_2 = \{\mathfrak{y}_1, \mathfrak{y}_2, ..., \mathfrak{y}_n\}$ *are finite universes. Consider* $\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j) = (\mathfrak{a}_{ij})_{m\times n}$, $\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j) = (\mathfrak{b}_{ij})_{m\times n}$, *and* $\alpha(\mathfrak{x}_i, \mathfrak{x}_j) = (\alpha_{ij})_{m\times n}$, $\beta(\mathfrak{x}_i, \mathfrak{x}_j) = (\beta_{ij})_{m\times n}$, *with* $0 \leq \alpha_{ij} + \beta_{ij} \leq 1$ *satisfying* $0 \leq \alpha_{ij}\mathfrak{a}_{ij} + \beta_{ij}\mathfrak{b}_{ij} \leq 1$ *for all* i, j, *where* $1 \leq i \leq |\mathcal{Q}_1|$ *and* $1 \leq j \leq |\mathcal{Q}_2|$. *Then, an LDF-relation* $\mathcal{R}_{\mathfrak{D}}$ *can be represented in the form of four matrices as follows:*

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} = (\mathfrak{a}_{ij})_{m\times n} = \begin{pmatrix} \mathfrak{a}_{11} & \mathfrak{a}_{12} & ... & \mathfrak{a}_{1n} \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & ... & \mathfrak{a}_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathfrak{a}_{m1} & \mathfrak{a}_{m2} & ... & \mathfrak{a}_{mn} \end{pmatrix}, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} = (\mathfrak{b}_{ij})_{m\times n} = \begin{pmatrix} \mathfrak{b}_{11} & \mathfrak{b}_{12} & ... & \mathfrak{b}_{1n} \\ \mathfrak{b}_{21} & \mathfrak{b}_{22} & ... & \mathfrak{b}_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathfrak{b}_{m1} & \mathfrak{b}_{m2} & ... & \mathfrak{b}_{mn} \end{pmatrix}$$

In addition,

$$\alpha = (\alpha_{ij})_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \alpha_{m1} & \alpha_{m2} & ... & \alpha_{mn} \end{pmatrix}, \beta = (_{ij})_{m\times n} = \begin{pmatrix} \beta_{11} & \beta_{12} & ... & \beta_{1n} \\ \beta_{21} & \beta_{22} & ... & \beta_{2n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \beta_{m1} & \beta_{m2} & ... & \beta_{mn} \end{pmatrix}$$

Or in the form of one matrix as follows:

$$\mathcal{R}_D = \begin{pmatrix} ((\mathfrak{a}_{11}, \mathfrak{b}_{11}), (\alpha_{11}, \beta_{11})) & ((\mathfrak{a}_{12}, \mathfrak{b}_{12}), (\alpha_{12}, \beta_{12})) & ... & ((\mathfrak{a}_{1n}, \mathfrak{b}_{1n}), (\alpha_{1n}, \beta_{1n})) \\ ((\mathfrak{a}_{21}, \mathfrak{b}_{21}), (\alpha_{21}, \beta_{21})) & ((\mathfrak{a}_{22}, \mathfrak{b}_{22}), (\alpha_{22}, \beta_{22})) & ... & ((\mathfrak{a}_{2n}, \mathfrak{b}_{2n}), (\alpha_{2n}, \beta_{2n})) \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ ((\mathfrak{a}_{m1}, \mathfrak{b}_{m1}), (\alpha_{m1}, \beta_{m1})) & ((\mathfrak{a}_{m2}, \mathfrak{b}_{m2}), (\alpha_{m2}, \beta_{m2})) & ... & ((a_{mn}, \mathfrak{b}_{mn}), (\alpha_{mn}, \beta_{mn})) \end{pmatrix},$$

where $\mathcal{R}_D = (< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\mathfrak{x}_i, \mathfrak{x}_j) >, < \alpha(\mathfrak{x}_i, \mathfrak{x}_j), \beta(\mathfrak{x}_i, \mathfrak{x}_j) >) = (< \mathfrak{a}_{ij}, \mathfrak{b}_{ij} >, < \alpha_{ij}, \beta_{ij} >)_{m\times n}$.

Since an LDF-relation is an LDFS on $\mathcal{Q}_1 \times \mathcal{Q}_2$, they have the same set-theoretic operations as LDFSs.

Definition 12. *Let* $\mathcal{R}_{1_{\mathfrak{D}}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{1_{\mathfrak{D}}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{1_{\mathfrak{D}}}}(v_1, v_2) >, < \alpha_1(v_1, v_2), \beta_1(v_1, v_2) >)$, *and* $\mathcal{R}_{2_{\mathfrak{D}}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{2_{\mathfrak{D}}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{2_{\mathfrak{D}}}}(v_1, v_2) >, < \alpha_2(v_1, v_2), \beta_2(v_1, v_2) >)$ *be two LDF-relations from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$. *Then,*

(1) $\mathcal{R}_{1_{\mathfrak{D}}} \subseteq \mathcal{R}_{2_{\mathfrak{D}}}$ *if and only if*

$$\delta^{\mathcal{M}}_{\mathcal{R}_{1_{\mathfrak{D}}}}(v_1, v_2) \leq \delta^{\mathcal{M}}_{\mathcal{R}_{2_{\mathfrak{D}}}}(v_1, v_2) \text{ and } \quad \delta^{\mathcal{N}}_{\mathcal{R}_{1_{\mathfrak{D}}}}(v_1, v_2) \geq \delta^{\mathcal{N}}_{\mathcal{R}_{2_{\mathfrak{D}}}}(v_1, v_2),$$

$$\alpha_1(v_1, v_2) \leq \alpha_2(v_1, v_2), \text{ and } \beta_1(v_1, v_2) \geq \beta_2(v_1, v_2)$$

for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

(2) $\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{2_\mathfrak{D}} = (< (\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2), (\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) >, < \alpha_1(v_1, v_2) \vee \alpha_2(v_1, v_2), \beta_1(v_1, v_2) \wedge \beta_2(v_1, v_2) >)$, *where*

$$(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) = \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2), \text{ and}$$

$$(\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) = \delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2)$$

for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

(3) $\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{2_\mathfrak{D}} = (< (\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2), (\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) >, < \alpha_1(v_1, v_2) \wedge \alpha_2(v_1, v_2), \beta_1(v_1, v_2) \vee \beta_2(v_1, v_2) >)$, *where*

$$(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) = \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2), \text{ and}$$

$$(\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2) = \delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2)$$

for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$.

(4) $\mathcal{R}^{-1}_{1_\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{R}^{-1}_{1_\mathfrak{D}}}(v_2, v_1), \delta^{\mathcal{N}}_{\mathcal{R}^{-1}_{1_\mathfrak{D}}}(v_2, v_1) >, < \alpha^{-1}_1(v_2, v_1), \beta^{-1}_1(v_2, v_1) >)$ *is an LDF-relation from* $\mathcal{Q}_2$ *to* $\mathcal{Q}_1$, *where*

$$\delta^{\mathcal{M}}_{\mathcal{R}^{-1}_{1_\mathfrak{D}}}(v_2, v_1) = \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2), \text{ and } \delta^{\mathcal{N}}_{\mathcal{R}^{-1}_{1_\mathfrak{D}}}(v_2, v_1) = \delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2)$$

$$\alpha^{-1}_1(v_2, v_1) = \alpha_1(v_1, v_2), \text{ and } \beta^{-1}_1(v_2, v_1) = \beta_1(v_1, v_2)$$

for all $(v_2, v_1) \in \mathcal{Q}_2 \times \mathcal{Q}_1$.

(5) $\mathcal{R}^c_{1_\mathfrak{D}} = (< \delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2), \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) >, < \beta_1(v_1, v_2), \alpha_1(v_1, v_2) >)$.

Proposition 1. *If* $\mathcal{R}_{1_\mathfrak{D}}, \mathcal{R}_{2_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, *then:*

(i) $\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{2_\mathfrak{D}}, \mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{2_\mathfrak{D}}, \mathcal{R}^c_{1_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$.

(ii) $\mathcal{R}^{-1}_{1_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_1)$.

Proof. The proof is straightforward in view of Definition 12. □

As an illustration of the Definition 12, we present the following example.

Example 1. *Let* $\mathcal{Q}_1 = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$, *and* $\mathcal{Q}_2 = \{\mathfrak{y}_1, \mathfrak{y}_2\}$. *The LDF-relations* $\mathcal{R}_\mathfrak{D}$ *and* $\mathcal{P}_\mathfrak{D}$ *from* $\mathcal{Q}_1$ *to* $\mathcal{Q}_2$ *are defined in Table 1 and Table 2, respectively.*

Table 1. LDF-relation $\mathcal{R}_\mathfrak{D}$.

$\mathcal{R}_\mathfrak{D}$	$\mathfrak{y}_1$	$\mathfrak{y}_2$
$\mathfrak{x}_1$	$((0.71, 0.21), (0.42, 0.58))$	$((0.95, 0.41), (0.74, 0.25))$
$\mathfrak{x}_2$	$((0.93, 0.52), (0.51, 0.47))$	$((0.87, 0.83), (0.64, 0.36))$
$\mathfrak{x}_3$	$((0.37, 0.61), (0.61, 0.33))$	$((0.68, 0.71), (0.49, 0.48))$

Table 2. LDF-relation $\mathcal{P}_\mathfrak{D}$.

$\mathcal{P}_\mathfrak{D}$	$\mathfrak{y}_1$	$\mathfrak{y}_2$
$\mathfrak{x}_1$	$((0.42, 0.65), (0.46, 0.52))$	$((0.46, 0.39), (0.22, 0.86))$
$\mathfrak{x}_2$	$((0.63, 0.99), (0.34, 0.64))$	$((0.56, 0.75), (0.75, 0.23))$
$\mathfrak{x}_3$	$((0.75, 0.71), (0.45, 0.47))$	$((0.95, 0.35), (0.43, 0.59))$

After simple calculations, the union $\mathcal{R}_\mathfrak{D} \cup \mathcal{P}_\mathfrak{D}$ *is obtained in the Table 3.*

Table 3. Union $\mathcal{R}_{\mathfrak{D}} \cup \mathcal{P}_{\mathfrak{D}}$.

$\mathcal{R}_{\mathfrak{D}} \cup \mathcal{P}_{\mathfrak{D}}$	$\mathfrak{y}_1$	$\mathfrak{y}_2$
$\mathfrak{x}_1$	$((0.71, 0.21), (0.46, 0.52))$	$((0.95, 0.39), (0.74, 0.25))$
$\mathfrak{x}_2$	$((0.93, 0.52), (0.51, 0.47))$	$((0.87, 0.75), (0.75, 0.23))$
$\mathfrak{x}_3$	$((0.75, 0.61), (0.61, 0.33))$	$((0.95, 0.35), (0.49, 0.48))$

Their intersection $\mathcal{R}_{\mathfrak{D}} \cap \mathcal{P}_{\mathfrak{D}}$ is given in Table 4.

Table 4. Intersection $\mathcal{R}_{\mathfrak{D}} \cap \mathcal{P}_{\mathfrak{D}}$.

$\mathcal{R}_{\mathfrak{D}} \cap \mathcal{P}_{\mathfrak{D}}$	$\mathfrak{y}_1$	$\mathfrak{y}_2$
$\mathfrak{x}_1$	$((0.42, 0.65), (0.42, 0.58))$	$((0.46, 0.41), (0.22, 0.86))$
$\mathfrak{x}_2$	$((0.63, 0.99), (0.34, 0.64))$	$((0.56, 0.83), (0.64, 0.36))$
$\mathfrak{x}_3$	$((0.37, 0.71), (0.45, 0.47))$	$((0.68, 0.71), (0.43, 0.59))$

Further, LDF-relation $\mathcal{P}_{\mathfrak{D}}^{-1}$ from $\mathcal{Q}_2$ to $\mathcal{Q}_1$ is calculated in Table 5.

Table 5. LDF-relation $\mathcal{P}_{\mathfrak{D}}^{-1}$ from $\mathcal{Q}_2$ to $\mathcal{Q}_1$.

$\mathcal{P}_{\mathfrak{D}}^{-1}$	$\mathfrak{x}_1$	$\mathfrak{x}_2$	$\mathfrak{x}_3$
$\mathfrak{y}_1$	$((0.42, 0.65), (0.46, 0.52))$	$((0.63, 0.99), (0.34, 0.64))$	$((0.75, 0.71), (0.45, 0.47))$
$\mathfrak{y}_2$	$((0.46, 0.39), (0.22, 0.86))$	$((0.56, 0.75), (0.75, 0.23))$	$((0.95, 0.35), (0.43, 0.59))$

In addition, $\mathcal{R}_{\mathfrak{D}}^c$ is presented in Table 6.

Table 6. $\mathcal{R}_{\mathfrak{D}}^c$.

$\mathcal{R}_{\mathfrak{D}}$	$\mathfrak{y}_1$	$\mathfrak{y}_2$
$\mathfrak{x}_1$	$((0.21, 0.71), (0.58, 0.42))$	$((0.41, 0.95), (0.25, 0.74))$
$\mathfrak{x}_2$	$((0.52, 0.93), (0.47, 0.51))$	$((0.83, 0.87), (0.36, 0.64))$
$\mathfrak{x}_3$	$((0.61, 0.37), (0.33, 0.61))$	$((0.71, 0.68), (0.48, 0.49))$

Proposition 2. *With the same notations as in Definition 12, the following properties hold:*

(1) $\mathcal{R}_{1_{\mathfrak{D}}} \subseteq \mathcal{R}_{2_{\mathfrak{D}}}$ *implies that* $\mathcal{R}_{1_{\mathfrak{D}}}^{-1} \subseteq \mathcal{R}_{2_{\mathfrak{D}}}^{-1}$.

(2) $(\mathcal{R}_{1_{\mathfrak{D}}} \cup \mathcal{R}_{2_{\mathfrak{D}}})^{-1} = \mathcal{R}_{1_{\mathfrak{D}}}^{-1} \cup \mathcal{R}_{2_{\mathfrak{D}}}^{-1}$.

(3) $(\mathcal{R}_{1_{\mathfrak{D}}} \cap \mathcal{R}_{2_{\mathfrak{D}}})^{-1} = \mathcal{R}_{1_{\mathfrak{D}}}^{-1} \cap \mathcal{R}_{2_{\mathfrak{D}}}^{-1}$.

(4) $(\mathcal{R}_{1_{\mathfrak{D}}}^{-1})^{-1} = \mathcal{R}_{1_{\mathfrak{D}}}$.

Proof. The proof is very easy in view of Definition 12. □

Definition 13. *In $\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, we denote and define full LDF-relation, and null LDF-relation as follows:*

$$\hat{1}_{\mathfrak{D}} = \{((v_1, v_2), < \hat{1}_D(v_1, v_2), \hat{0}_D(v_1, v_2) >, < \hat{1}(v_1, v_2), \hat{0}(v_1, v_2) >) : v_1 \in \mathcal{Q}_1, v_2 \in \mathcal{Q}_2\},$$

$$\hat{0}_{\mathfrak{D}} = \{((v_1, v_2), < \hat{0}_D(v_1, v_2), \hat{1}_D(v_1, v_2) >, < \hat{0}(v_1, v_2), \hat{1}(v_1, v_2) >) : v_1 \in \mathcal{Q}_1, v_2 \in \mathcal{Q}_2\}$$

where,

$$\hat{1}_D(v_1, v_2) = \hat{1}(v_1, v_2) = 1, \textit{ for all } (v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$$

$$\hat{0}_D(v_1, v_2) = \hat{0}(v_1, v_2) = 0, \textit{ for all } (v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$$

As a direct consequence of the Definition 12 (2) and (3), and Definition 13, we get the following result.

Proposition 3. *Let $\mathcal{R}_{1_\mathfrak{D}}, \mathcal{R}_{2_\mathfrak{D}}, \mathcal{R}_{3_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$. Then, the following properties hold:*

(1) $\mathcal{R}_{1_\mathfrak{D}} \cup \hat{0}_\mathfrak{D} = \mathcal{R}_{1_\mathfrak{D}}$.
(2) $\mathcal{R}_{1_\mathfrak{D}} \cap \hat{0}_\mathfrak{D} = \hat{0}_\mathfrak{D}$.
(3) $\mathcal{R}_{1_\mathfrak{D}} \cup \hat{1}_\mathfrak{D} = \hat{1}_\mathfrak{D}$.
(4) $\mathcal{R}_{1_\mathfrak{D}} \cap \hat{1}_\mathfrak{D} = \mathcal{R}_{1_\mathfrak{D}}$.
(5) $\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{1_\mathfrak{D}} = \mathcal{R}_{1_\mathfrak{D}}$.
(6) $\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{1_\mathfrak{D}} = \mathcal{R}_{1_\mathfrak{D}}$.
(7) $\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{2_\mathfrak{D}} = \mathcal{R}_{2_\mathfrak{D}} \cup \mathcal{R}_{1_\mathfrak{D}}$.
(8) $\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{2_\mathfrak{D}} = \mathcal{R}_{2_\mathfrak{D}} \cap \mathcal{R}_{1_\mathfrak{D}}$.
(9) $(\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{2_\mathfrak{D}}) \cup \mathcal{R}_{3_\mathfrak{D}} = \mathcal{R}_{1_\mathfrak{D}}(\cup \mathcal{R}_{2_\mathfrak{D}} \cup \mathcal{R}_{3_\mathfrak{D}})$.
(10) $(\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{2_\mathfrak{D}}) \cap \mathcal{R}_{3_\mathfrak{D}} = \mathcal{R}_{1_\mathfrak{D}}(\cap \mathcal{R}_{2_\mathfrak{D}} \cap \mathcal{R}_{3_\mathfrak{D}})$.

The above Proposition 3 is very important which yields to the following algebraic structure (see Corollary 1).

Corollary 1. *The pairs $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2), \cup)$ and $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2, \cap))$ are idempotent, commutative monoids with identity elements $\hat{0}_\mathfrak{D}$, and $\hat{1}_\mathfrak{D}$, respectively.*

The next result is very important which gives rise to some other algebraic structures.

Proposition 4. *With the same notations as in above Proposition 3, the following assertions hold:*

(1) $\mathcal{R}_{1_\mathfrak{D}} \cup (\mathcal{R}_{2_\mathfrak{D}} \cap \mathcal{R}_{3_\mathfrak{D}}) = (\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{2_\mathfrak{D}}) \cap (\mathcal{R}_{1_\mathfrak{D}} \cup \mathcal{R}_{3_\mathfrak{D}})$.
(2) $\mathcal{R}_{1_\mathfrak{D}} \cap (\mathcal{R}_{2_\mathfrak{D}} \cup \mathcal{R}_{3_\mathfrak{D}}) = (\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{2_\mathfrak{D}}) \cup (\mathcal{R}_{1_\mathfrak{D}} \cap \mathcal{R}_{3_\mathfrak{D}})$.
(3) *If $\mathcal{R}_{2_\mathfrak{D}} \subseteq \mathcal{R}_{1_\mathfrak{D}}$, and $\mathcal{R}_{3_\mathfrak{D}} \subseteq \mathcal{R}_{1_\mathfrak{D}}$, imply that $\mathcal{R}_{2_\mathfrak{D}} \cup \mathcal{R}_{3_\mathfrak{D}} \subseteq \mathcal{R}_{1_\mathfrak{D}}$.*
(4) *If $\mathcal{R}_{1_\mathfrak{D}} \subseteq \mathcal{R}_{2_\mathfrak{D}}$, and $\mathcal{R}_{1_\mathfrak{D}} \subseteq \mathcal{R}_{3_\mathfrak{D}}$, then $\mathcal{R}_{1_\mathfrak{D}} \subseteq \mathcal{R}_{2_\mathfrak{D}} \cap \mathcal{R}_{3_\mathfrak{D}}$.*

Proof. (1) Let $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$. Then,

$$\begin{aligned}
[\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup (\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}} \cap \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_2) &= \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee [(\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}} \cap \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})(v_1, v_2)] \\
&= \delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee [\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_1, v_2)] \\
&= [\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_1, v_2)] \wedge [\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \vee \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_1, v_2)] \\
&= [(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_2)] \wedge [(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})(v_1, v_2)] \\
&= [(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}) \cap (\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \cup \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_2)
\end{aligned}$$

In the similar manners, it can be prove that

$$[\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap (\delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}} \cup \delta^{\mathcal{N}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_2) = [(\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}}) \cup (\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \cap \delta^{\mathcal{N}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_2).$$

Moreover,

$$\alpha_1(v_1, v_2) \vee (\alpha_2(v_1, v_2) \wedge \alpha_3(v_1, v_2)) = (\alpha_1(v_1, v_2) \vee \alpha_2(v_1, v_2)) \wedge (\alpha_1(v_1, v_2) \vee \alpha_3(v_1, v_2))$$

In addition,

$$\beta_1(v_1, v_2) \wedge (\beta_2(v_1, v_2) \vee \beta_3(v_1, v_2)) = (\beta_1(v_1, v_2) \wedge \beta_2(v_1, v_2)) \vee (\beta_1(v_1, v_2) \wedge \beta_3(v_1, v_2))$$

(since $\alpha_i(v_1, v_2), \beta_i(v_1, v_2) \in [0,1]$, and $([0,1], \vee, \wedge)$ is a distributive lattice [38], where $i = 1, 2, 3$).

(2) The proof is similar to the proof of (1). (3) and (4) can easily be proved by using Definition 12 (1), (2). □

From Proposition 4, we have the following corollary.

Corollary 2. *The set $\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$ is the following algebraic structures:*

(1) Commutative semiring $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2), \cup, \cap)$ with identity element $\hat{1}_{\mathfrak{D}}$, and zero element $\hat{0}_{\mathfrak{D}}$.

(2) Commutative semiring $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2), \cap, \cup)$ with identity element $\hat{0}_{\mathfrak{D}}$, and zero element $\hat{1}_{\mathfrak{D}}$.

The above Corollary 2 gives rise to the following result.

Corollary 3. *The set $\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$ is hemiring $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2), \cup, \cap)$ with zero element $\hat{0}_{\mathfrak{D}}$.*

In the motivation of the composition of F-relations [2,30], we define the composition of two LDF-relations and study some of its important properties in the sequel of this manuscript.

Definition 14. *Let $\mathcal{R}_{\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) >, < \alpha(v_1, v_2), \beta(v_1, v_2) >)$ be an LDF-relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$, and $\mathcal{P}_{\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3), \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3) >, < \alpha'(v_2, v_3), \beta'(v_2, v_3) >)$ be an LDF-relation from $\mathcal{Q}_2$ to $\mathcal{Q}_3$. We denote and define their composition as follows:*

$$\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{P}_{\mathfrak{D}} = (< (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3), (\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3) >, < (\alpha \hat{\circ} \alpha')(v_1, v_3), (\beta \hat{\circ} \beta')(v_1, v_3) >)$$

where,

$$(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3) = \vee_{v_2 \in \mathcal{Q}_2} (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3))$$
$$(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3) = \wedge_{v_2 \in \mathcal{Q}_2} (\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \vee \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3))$$

and

$$(\alpha \hat{\circ} \alpha')(v_1, v_3) = \vee_{v_2 \in \mathcal{Q}_2} (\alpha(v_1, v_2) \wedge \alpha'(v_2, v_3)),$$
$$(\beta \hat{\circ} \beta')(v_1, v_3) = \wedge_{v_2 \in \mathcal{Q}_2} (\beta(v_1, v_2) \vee \beta'(v_2, v_3)).$$

for all $(v_1, v_3) \in \mathcal{Q}_1 \times \mathcal{Q}_3$.

Proposition 5. *With the same notations as in Definition 14, we have $\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{P}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_3)$.*

Proof. First, we prove that $0 \leq (\alpha \hat{\circ} \alpha')(v_1, v_3) + (\beta \hat{\circ} \beta')(v_1, v_3) \leq 1$. Since $0 \leq \alpha(v_1, v_2) + \beta(v_1, v_2) \leq 1$ and $0 \leq \alpha'(v_2, v_3) + \beta'(v_2, v_3) \leq 1$, then $\alpha(v_1, v_2) \leq 1 - \beta(v_1, v_2)$ and $\alpha'(v_2, v_3) \leq 1 - \beta'(v_2, v_3)$. So,

$$\begin{aligned}
(\alpha \hat{\circ} \alpha')(v_1, v_3) &= \vee_{v_2 \in \mathcal{Q}_2} (\alpha(v_1, v_2) \wedge \alpha'(v_2, v_3)) \\
&\leq \vee_{v_2 \in \mathcal{Q}_2} ((1 - \beta(v_1, v_2)) \wedge (1 - \beta'(v_2, v_3))) \\
&= \vee_{v_2 \in \mathcal{Q}_2} (1 - (\beta(v_1, v_2) \vee \beta'(v_2, v_3))) \\
&= 1 - \wedge_{v_2 \in \mathcal{Q}_2} (\beta(v_1, v_2) \vee \beta'(v_2, v_3)) \\
&= 1 - (\beta \hat{\circ} \beta')(v_1, v_3)
\end{aligned}$$

This proves that $0 \leq (\alpha \hat{\circ} \alpha')(v_1, v_3) + (\beta \hat{\circ} \beta')(v_1, v_3) \leq 1$. Now, to prove that

$$0 \leq (\alpha \hat{\circ} \alpha')(v_1, v_3)(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3) + (\beta \hat{\circ} \beta')(v_1, v_2)(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}})(v_1, v_3) \leq 1. \quad (2)$$

for all $(v_1, v_3) \in \mathcal{Q}_1 \times \mathcal{Q}_3$. Since $0 \leq \alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) + \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \leq 1$, for all $(v_1, v_2) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, and $0 \leq \alpha'(v_2, v_3)\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3) + \beta'(v_2, v_3)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_2, v_3) \leq 1$ for all $(v_2, v_3) \in \mathcal{Q}_2 \times \mathcal{Q}_3$. It follows that:

$$\alpha(v_1, v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \leq 1 - \beta(v_1, v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2), \text{ and} \quad (3)$$

$$\alpha'(v_2, v_3)\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3) \leq 1 - \beta'(v_2, v_3)\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2, v_3) \quad (4)$$

Let $(v_1, v_3) \in \mathcal{Q}_1 \times \mathcal{Q}_3$. Then, by using the Definition 14,

$$\begin{aligned}
(\alpha\hat{\circ}\alpha)(v_1,v_3)(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})(v_1,v_3) &= [\vee_{v_2\in\mathcal{Q}_2}(\alpha(v_1,v_2)\wedge\alpha'(v_2,v_3))][\vee_{v_2\in\mathcal{Q}_2}(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&= \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[(\alpha(v_1,v_2)\wedge\alpha'(v_2,v_3))(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&= \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[[\alpha(v_1,v_2)(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&\wedge[\alpha'(v_2,v_3)(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]]\\
&= \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[[\alpha(v_1,v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\alpha(v_1,v_2)\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3)]\\
&\wedge[\alpha'(v_2,v_3)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\alpha'(v_2,v_3)\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3)]]\\
&\leq \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[\alpha(v_1,v_2)\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\alpha'(v_2,v_3)\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3)]\\
&\leq \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[(1-\beta(v_1,v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2))\\
&\wedge(1-\beta'(v_2,v_3)\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&= \vee_{v_2\in\mathcal{Q}_2}\vee_{v_2\in\mathcal{Q}_2}[1-(\beta(v_1,v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\vee\beta'(v_2,v_3)\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&= 1-\wedge_{v_2\in\mathcal{Q}_2}\wedge_{v_2\in\mathcal{Q}_2}[\beta(v_1,v_2)\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\vee\beta'(v_2,v_3)\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3)]\\
&\leq 1-[\wedge_{v_2\in\mathcal{Q}_2}(\beta(v_1,v_2)\vee\beta'(v_2,v_3))][\wedge_{v_2\in\mathcal{Q}_2}(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\vee\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))]\\
&= 1-(\beta\hat{\circ}\beta')(v_1,v_3)(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}})(v_1,v_3))
\end{aligned}$$

(see Equations (3) and (4)). This proves Equation (2). Hence, $\mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_3)$. □

Theorem 1. *With the same assumptions as in the above Proposition 5, the following assertion hold:*

$$(\mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{\mathfrak{D}})^{-1} = \mathcal{P}^{-1}_{\mathfrak{D}}\hat{\circ}\mathcal{R}^{-1}_{\mathfrak{D}}$$

Proof. Let $(v_3, v_1) \in \mathcal{Q}_3 \times \mathcal{Q}_1$. According to the Definition 12 (4), and Definition 14,

$$\begin{aligned}
(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})^{-1}(v_3,v_1) &= (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}})(v_1,v_3)\\
&= \vee_{v_2\in\mathcal{Q}_2}(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{P}_{\mathfrak{D}}}(v_2,v_3))\\
&= \vee_{v_2\in\mathcal{Q}_2}(\delta^{\mathcal{M}}_{\mathcal{R}^{-1}_{\mathfrak{D}}}(v_2,v_1)\wedge\delta^{\mathcal{M}}_{\mathcal{P}^{-1}_{\mathfrak{D}}}(v_3,v_2))\\
&= \vee_{v_2\in\mathcal{Q}_2}(\delta^{\mathcal{M}}_{\mathcal{P}^{-1}_{\mathfrak{D}}}(v_3,v_2)\wedge\delta^{\mathcal{M}}_{\mathcal{R}^{-1}_{\mathfrak{D}}}(v_2,v_1))\\
&= (\delta^{\mathcal{M}}_{\mathcal{P}^{-1}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{M}}_{\mathcal{R}^{-1}_{\mathfrak{D}}})(v_3,v_1)
\end{aligned}$$

Similarly, it can be proved that $(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{N}}_{\mathcal{P}_{\mathfrak{D}}})^{-1}(v_3,v_1) = (\delta^{\mathcal{N}}_{\mathcal{P}^{-1}_{\mathfrak{D}}}\hat{\circ}\delta^{\mathcal{N}}_{\mathcal{R}^{-1}_{\mathfrak{D}}})(v_3,v_1)$. In addition,

$$\begin{aligned}
(\alpha\hat{\circ}\alpha')^{-1}(v_3,v_1) &= (\alpha\hat{\circ}\alpha')(v_1,v_3)\\
&= \vee_{v_2\in\mathcal{Q}_2}(\alpha(v_1,v_2)\wedge\alpha'(v_2,v_3))\\
&= \vee_{v_2\in\mathcal{Q}_2}(\alpha^{-1}(v_2,v_1)\wedge\alpha'^{-1}(v_3,v_2))\\
&= \vee_{v_2\in\mathcal{Q}_2}(\alpha'^{-1}(v_3,v_2)\wedge\alpha^{-1}(v_2,v_1))\\
&= (\alpha'^{-1}\hat{\circ}\alpha^{-1})(v_3,v_1)
\end{aligned}$$

Similar proof for $(\beta\hat{\circ}\beta')^{-1}(v_3,v_1) = (\beta'^{-1}\hat{\circ}\beta^{-1})(v_3,v_1)$. This completes the proof. □

Theorem 2. *If* $\mathcal{R}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, *and* $\mathcal{P}_{1_{\mathfrak{D}}}, \mathcal{P}_{2_{\mathfrak{D}}} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_3)$ *such that* $\mathcal{P}_{1_{\mathfrak{D}}} \subseteq \mathcal{P}_{2_{\mathfrak{D}}}$. *Then,*

(1) $\mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{1_{\mathfrak{D}}} \subseteq \mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{2_{\mathfrak{D}}}$.
(2) $\mathcal{P}_{1_{\mathfrak{D}}}\hat{\circ}\mathcal{P}_{1_{\mathfrak{D}}} \subseteq \mathcal{P}_{2_{\mathfrak{D}}}\hat{\circ}\mathcal{P}_{2_{\mathfrak{D}}}$.

Proof. (1)The proof is straightforward in view of Definition 12 (1) and 14.

(2) From (1) we have, $\mathcal{P}_{1_{\mathfrak{D}}}\hat{\circ}\mathcal{P}_{1_{\mathfrak{D}}} \subseteq \mathcal{P}_{2_{\mathfrak{D}}}\hat{\circ}\mathcal{P}_{1_{\mathfrak{D}}} \subseteq \mathcal{P}_{2_{\mathfrak{D}}}\hat{\circ}\mathcal{P}_{2_{\mathfrak{D}}}$. □

Theorem 3. *If* $\mathcal{P}_{1_\mathfrak{D}}, \mathcal{P}_{2_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, *and* $\mathcal{R}_\mathfrak{D} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_3)$ *with* $\mathcal{P}_{1_\mathfrak{D}} \subseteq \mathcal{P}_{2_\mathfrak{D}}$, *then:*

$$\mathcal{P}_{1_\mathfrak{D}} \hat{\circ} \mathcal{R}_\mathfrak{D} \subseteq \mathcal{P}_{2_\mathfrak{D}} \hat{\circ} \mathcal{R}_\mathfrak{D}.$$

Proof. This proof is similar to the proof of Theorem 2 (1). □

The following Theorem 4, informs us that LDF-relations satisfies the associative laws with respect to the composition defined in Definition 14.

Theorem 4. *Let* $\mathcal{R}_{1_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, $\mathcal{R}_{2_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_3)$, *and* $\mathcal{R}_{3_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_3 \times \mathcal{Q}_4)$. *Then:*

$$\mathcal{R}_{1_\mathfrak{D}} \hat{\circ} (\mathcal{R}_{2_\mathfrak{D}} \hat{\circ} \mathcal{R}_{3_\mathfrak{D}}) = (\mathcal{R}_{1_\mathfrak{D}} \hat{\circ} \mathcal{R}_{2_\mathfrak{D}}) \hat{\circ} \mathcal{R}_{3_\mathfrak{D}}.$$

Proof. Let $v_1 \in \mathcal{Q}_1, v_4 \in \mathcal{Q}_4$. Then, by Definition 14

$$\begin{aligned}
[\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \hat{\circ} (\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_4) &= \vee_{v_3 \in \mathcal{Q}_3} [(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}})(v_1, v_3) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_3, v_4)] \\
&= \vee_{v_3 \in \mathcal{Q}_3} \vee_{v_2 \in \mathcal{Q}_2} [(\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_2, v_3)) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_3, v_4)] \\
&= \vee_{v_2 \in \mathcal{Q}_2} \vee_{v_3 \in \mathcal{Q}_3} [\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \wedge (\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_2, v_3) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_3, v_4))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_3) \wedge (\vee_{v_3 \in \mathcal{Q}_3} (\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}(v_2, v_3) \wedge \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_3, v_4)))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}}(v_1, v_2) \wedge (\delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}})((v_2, v_4))] \\
&= (\delta^{\mathcal{M}}_{\mathcal{R}_{1_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{2_\mathfrak{D}}}) \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{3_\mathfrak{D}}}(v_1, v_4)
\end{aligned}$$

Similarly, $[\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \hat{\circ} (\delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{R}_{3_\mathfrak{D}}})](v_1, v_4) = [(\delta^{\mathcal{N}}_{\mathcal{R}_{1_\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{R}_{2_\mathfrak{D}}}) \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{R}_{3_\mathfrak{D}}}](v_1, v_4)$. Now, let $v_1 \in \mathcal{Q}_1$, $v_4 \in \mathcal{Q}_4$. According to the Definition 14,

$$\begin{aligned}
[\alpha_1 \hat{\circ} (\alpha_2 \hat{\circ} \alpha_3)](v_1, v_4) &= \vee_{v_3 \in \mathcal{Q}_3} [(\alpha_1 \hat{\circ} \alpha_2)(v_1, v_3) \wedge \alpha_3(v_3, v_4)] \\
&= \vee_{v_3 \in \mathcal{Q}_3} [(\vee_{v_2 \in \mathcal{Q}_2} (\alpha_1(v_1, v_2) \wedge \alpha_2(v_2, v_3))) \wedge \alpha_3(v_3, v_4)] \\
&= \vee_{v_3 \in \mathcal{Q}_3} \vee_{v_2 \in \mathcal{Q}_2} [(\alpha_1(v_1, v_2) \wedge \alpha_2(v_2, v_3)) \wedge \alpha_3(v_3, v_4)] \\
&= \vee_{v_3 \in \mathcal{Q}_3} \vee_{v_2 \in \mathcal{Q}_2} [\alpha_1(v_1, v_2) \wedge (\alpha_2(v_2, v_3) \wedge \alpha_3(v_3, v_4))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} \vee_{v_3 \in \mathcal{Q}_3} [\alpha_1(v_1, v_2) \wedge (\alpha_2(v_2, v_3) \wedge \alpha_3(v_3, v_4))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [\alpha_1(v_1, v_2) \wedge (\vee_{v_3 \in \mathcal{Q}_3} (\alpha_2(v_2, v_3) \wedge \alpha_3(v_3, v_4)))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [\alpha_1(v_1, v_2) \wedge (\alpha_2(\hat{\circ} \alpha_3)(v_2, v_4))] \\
&= [(\alpha_1 \hat{\circ} \alpha_2) \hat{\circ} \alpha_3](v_1, v_4)
\end{aligned}$$

Similar proof for $[\beta_1 \hat{\circ} (\beta_2 \hat{\circ} \beta_3)](v_1, v_4) = [(\beta_1 \hat{\circ} \beta_2) \hat{\circ} \beta_3](v_1, v_4)$. Thus proof is complete. □

In the following two results, the distributive laws of union and intersection over composition are proved.

Theorem 5. *Let* $\mathcal{R}_\mathfrak{D} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, *and* $\mathcal{P}_{1_\mathfrak{D}}, \mathcal{P}_{2_\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_3)$. *Then, the following properties hold:*

(1) $\mathcal{R}_\mathfrak{D} \hat{\circ} (\mathcal{P}_{1_\mathfrak{D}} \cup \mathcal{P}_{2_\mathfrak{D}}) = (\mathcal{R}_\mathfrak{D} \hat{\circ} \mathcal{P}_{1_\mathfrak{D}}) \cup (\mathcal{R}_\mathfrak{D} \hat{\circ} \mathcal{P}_{2_\mathfrak{D}})$.

(2) $\mathcal{R}_\mathfrak{D} \hat{\circ} (\mathcal{P}_{1_\mathfrak{D}} \cap \mathcal{P}_{2_\mathfrak{D}}) = (\mathcal{R}_\mathfrak{D} \hat{\circ} \mathcal{P}_{1_\mathfrak{D}}) \cap (\mathcal{R}_\mathfrak{D} \hat{\circ} \mathcal{P}_{2_\mathfrak{D}})$.

Proof. (1) Let $(v_1, v_3) \in \mathcal{Q}_1 \times \mathcal{Q}_3$. From Definition 14 and 12,

$$\begin{aligned}
[\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} (\delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}} \cup \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}})](v_1, v_3) &= \vee_{v_2 \in \mathcal{Q}_2} [\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge (\delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}} \cup \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}})(v_2, v_3)] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge (\delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}}(v_2, v_3) \vee \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}}(v_2, v_3))] \\
&= \vee_{v_2 \in \mathcal{Q}_2} [(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}}(v_2, v_3)) \\
&\vee (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}}(v_2, v_3))] \\
&= [\vee_{v_2 \in \mathcal{Q}_2} (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}}(v_2, v_3))] \\
&\vee [\vee_{v_2 \in \mathcal{Q}_2} (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) \wedge \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}}(v_2, v_3))] \\
&= (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}})(v_1, v_3) \vee (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}})(v_1, v_3) \\
&= ((\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{1_{\mathfrak{D}}}}) \cup (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{P}_{2_{\mathfrak{D}}}}))(v_1, v_3)
\end{aligned}$$

In a similar way, it can be proved that $[\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} (\delta^{\mathcal{N}}_{\mathcal{P}_{1_{\mathfrak{D}}}} \cap \delta^{\mathcal{N}}_{\mathcal{P}_{2_{\mathfrak{D}}}})](v_1, v_3) = [(\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{1_{\mathfrak{D}}}}) \cap (\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{P}_{2_{\mathfrak{D}}}})](v_1, v_3)$. Furthermore, since $\alpha(v_1, v_2), \alpha_1(v_2, v_3), \alpha_2(v_2, v_3) \in [0, 1]$ and $([0, 1], \vee, \wedge)$ is a distributive lattice. Therefore, $[\alpha \hat{\circ} (\alpha_1 \vee \alpha_2)](v_1, v_3) = [(\alpha \hat{\circ} \alpha_1) \vee (\alpha \hat{\circ} \alpha_2)](v_1, v_3)$ and $[\beta \hat{\circ} (\beta_1 \vee \beta_2)](v_1, v_3) = [(\beta \hat{\circ} \beta_1) \vee (\beta \hat{\circ} \beta_2)](v_1, v_3)$. (2) can be proved by following the same pattern. This completes the proof. □

Theorem 6. *Let $\mathcal{P}_{1_{\mathfrak{D}}}, \mathcal{P}_{2_{\mathfrak{D}}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2)$, and $\mathcal{R}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_2 \times \mathcal{Q}_3)$. Then, the following properties hold:*

(1) $(\mathcal{P}_{1_{\mathfrak{D}}} \cup \mathcal{P}_{2_{\mathfrak{D}}}) \hat{\circ} \mathcal{R}_{\mathfrak{D}} = (\mathcal{P}_{1_{\mathfrak{D}}} \hat{\circ} \mathcal{R}_{\mathfrak{D}}) \cup (\mathcal{P}_{2_{\mathfrak{D}}} \hat{\circ} \mathcal{R}_{\mathfrak{D}})$.
(2) $(\mathcal{P}_{1_{\mathfrak{D}}} \cap \mathcal{P}_{2_{\mathfrak{D}}}) \hat{\circ} \mathcal{R}_{\mathfrak{D}} = (\mathcal{P}_{1_{\mathfrak{D}}} \hat{\circ} \mathcal{R}_{\mathfrak{D}}) \cap (\mathcal{P}_{2_{\mathfrak{D}}} \hat{\circ} \mathcal{R}_{\mathfrak{D}})$.

Proof. The proof is similar to the proof of Theorem 5. □

Theorem 4, 5, and 5 giving rise the following algebraic structures.

Corollary 4. *The triplet $(\mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_1), \cup, \hat{\circ})$ is:*

(1) *semiring with identity element $\hat{1}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_1)$ and zero element $\hat{0}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_1)$.*
(2) *hemiring with zero element $\hat{0}_{\mathfrak{D}} \in \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_1)$.*

Now we define the concept of an equivalence LDF-relation. Let us assume that

$$\mathcal{R}_{\mathfrak{D}} = \Big(< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1, v_2) >, < \alpha(v_1, v_2), \beta(v_1, v_2) > \Big)$$

is an LDF-relation on a $\mathcal{Q}$.

Definition 15. *The LDF-relation $\mathcal{R}_{\mathfrak{D}}$ is called reflexive, if:*

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon, \epsilon) = 1, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon, \epsilon) = 0, \textit{ and } \alpha(\epsilon, \epsilon) = 1, \beta(\epsilon, \epsilon) = 0$$

for all $\epsilon \in \mathcal{Q}$.

If $|\mathcal{Q}| = n$, where $|.|$ denotes the number of elements, and

$$\begin{aligned}
\mathcal{R}_{\mathfrak{D}} &= \Big(< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_i, \epsilon_j), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_i, \epsilon_j) >, < \alpha(\epsilon_i, \epsilon_j), \beta(\epsilon_i, \epsilon_j) > \Big) \\
&= (< (\mathfrak{a}_{ij})_{n \times n}, (\mathfrak{a}_{ij})_{n \times n} >, < (\alpha_{ij})_{n \times n}, (\beta_{ij})_{n \times n} >),
\end{aligned}$$

where $i, j = 1, 2, ..., n$. Then $\mathcal{LDF}-$relation $\mathcal{R}_{\mathfrak{D}}$ is reflexive, if:

$$\mathfrak{a}_{ii} = \alpha_{ii} = 1, \text{ and } \mathfrak{b}_{ii} = \beta_{ii} = 0.$$

Definition 16. *The LDF-relation $\mathcal{R}_{\mathfrak{D}}$ is called symmetric, if:*

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_1,\epsilon_2) = \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_2,\epsilon_1),$$

$$\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_1,\epsilon_2) = \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(\epsilon_2,\epsilon_1),$$

$$\alpha(\epsilon_1,\epsilon_2) = \alpha(\epsilon_2,\epsilon_1),$$

$$\beta(\epsilon_1,\epsilon_2) = \beta(\epsilon_2,\epsilon_1)$$

for all $\epsilon_1,\epsilon_2 \in \mathcal{Q}$.

Since a relation is symmetric, if and only if its matrix is the same as its transpose. So, $\mathcal{R}_{\mathfrak{D}}$ is symmetric, if and only if,

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} = (\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}})^T, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} = (\delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}})^T \text{ and } \alpha = \alpha^T, \beta = \beta^T.$$

Definition 17. *The LDF-relation $\mathcal{R}_{\mathfrak{D}}$ is called transitive, if $\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{R}_{\mathfrak{D}} \subseteq \mathcal{R}_{\mathfrak{D}}$, that is,*

$$\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} \subseteq \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}, \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \hat{\circ} \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} \supseteq \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}, \text{ and } \alpha \hat{\circ} \alpha \subseteq \alpha, \beta \hat{\circ} \beta \supseteq \beta$$

Definition 18. *The LDF-relation $\mathcal{R}_{\mathfrak{D}}$ is said to be an equivalence LDF-relation, if $\mathcal{R}_{\mathfrak{D}}$ is reflexive, symmetric and transitive.*

For illustration, we construct the following Example.

Example 2. *Let $\mathcal{Q} = \{\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3\}$. Consider an $\mathcal{LDF}-$relation $\mathcal{R}_{\mathfrak{D}}$ on $\mathcal{Q}$ as follows:*

$$\vartheta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}} = \begin{pmatrix} 1 & 0.98 & 0.46 \\ 0.98 & 1 & 0.67 \\ 0.46 & 0.67 & 1 \end{pmatrix}, \vartheta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}} = \begin{pmatrix} 0 & 0.47 & 0.32 \\ 0.47 & 0 & 0.71 \\ 0.32 & 0.71 & 0 \end{pmatrix},$$

In addition,

$$\alpha = \begin{pmatrix} 1 & 0.33 & 0.26 \\ 0.33 & 1 & 0.47 \\ 0.26 & 0.47 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 0.46 & 0.63 \\ 0.46 & 0 & 0.34 \\ 0.63 & 0.34 & 0 \end{pmatrix}.$$

Then, it can be easily seen that $\mathcal{R}_{\mathfrak{D}}$ is an equivalence LDF-relation.

4. Application of LDF-Relations in Decision Making (DM)

Since LDF-relations are LDFSs, so its applications can be found in the field of AI, engineering, medical, DM and MADM [26]. DM as an abstract technique results best alternative among various choices. In this section, an algorithm is produced to solve some DM problems by utilizing the concept of LDF-relation, in the motivation of Naeem et al. [19], which is supported by a numerical example.

First, we define the score function on LDF-relations, in the motivation of Riaz et al. [26].

Definition 19. *Let $\mathcal{R}_{\mathfrak{D}} = (< \delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2), \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2) >, < \alpha(v_1,v_2), \beta(v_1,v_2) >)$ be a LDF-relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$. Define the score function on $\mathcal{R}_{\mathfrak{D}}$ by a map*

$$\mathfrak{S} : \mathcal{LDFR}(\mathcal{Q}_1 \times \mathcal{Q}_2) \to [-1,1]$$

given as follows:

$$\mathfrak{S}(\mathcal{R}_{\mathfrak{D}}) = \frac{1}{2}[(\delta^{\mathcal{M}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2) - \delta^{\mathcal{N}}_{\mathcal{R}_{\mathfrak{D}}}(v_1,v_2)) + (\alpha(v_1,v_2) - \beta(v_1,v_2))]$$

Now, we propose an Algorithm 1 to DM approach in view of LDF-relations as follows:

Algorithm 1

(1) Input the data sets $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$.
(2) Compute the LDF-relations $\mathcal{R}_{\mathfrak{D}}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$, and $\mathcal{P}_{\mathfrak{D}}$ from $\mathcal{Q}_2$ to $\mathcal{Q}_3$.
(3) Perform the composition operation $\hat{\circ}$ among $\mathcal{R}_{\mathfrak{D}}$ and $\mathcal{P}_{\mathfrak{D}}$, that is, $\mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{\mathfrak{D}}$.
(4) Compute the error or hesitation values of according to Definition 10, that is, $\varepsilon_{ik} = \gamma_{ik}\pi_{ik} = 1-(\theta_{ik}\eta_{ik}^{\mathcal{M}} + \theta'_{ik}\eta_{ik}^{\mathcal{N}})$, where $\eta_{ik}^{\mathcal{M}} = \delta_{ij}^{\mathcal{M}}\hat{\circ}\delta_{jk}^{\mathcal{M}}$, $\eta_{ik}^{\mathcal{N}} = \delta_{ij}^{\mathcal{N}}\hat{\circ}\delta_{jk}^{\mathcal{N}}$, and $\theta_{ik} = \alpha_{ij}\hat{\circ}\alpha'_{jk}$, $\theta'_{ik} = \beta_{ij}\hat{\circ}\beta'_{jk}$, and $1 \le i \le |\mathcal{Q}_1|$, $1 \le j \le |\mathcal{Q}_2|$, and $1 \le k \le |\mathcal{Q}_3|$, where $|\mathcal{Q}_l|$, $l = 1,2,3$, represents the number of elements of $|\mathcal{Q}_l|$.
(5) Compute the association grades among the elements of the sets $\mathcal{Q}_1$ and $\mathcal{Q}_3$ by using $\ddot{\mathcal{A}} = \eta_{ik}^{\mathcal{M}} - \eta_{ik}^{\mathcal{N}}\varepsilon_{ik}$.
(6) Find out the pair $(\mathfrak{q}_i, \mathfrak{q}_k)$, where $\mathfrak{q}_i \in \mathcal{Q}_1$, $\mathfrak{q}_k \in \mathcal{Q}_3$ having the maximum association grade value $\ddot{\mathcal{A}}_{ik}$.
(7) Decision: The pair $(\mathfrak{q}_i, \mathfrak{q}_k)$ is the optimal choice.

To explain the above algorithm, the following example is elaborated.

Example 3. *Suppose that a person Mr. X wants to purchase a new brand one canal double story bungalow and the property dealer visited four bungalows $\mathcal{Q}_1 = \{u_1, u_2, u_3, u_4\}$ as per his requirement $\mathcal{Q}_2 = \{l_1 =$ near to play ground$, l_2 =$ near to park $, l_3 =$ near to main service road $\}$ in reasonable price, where the set of prices is $\mathcal{Q}_3 = \{p_1 =$ low $, p_2 =$ medium $, p_3 =$ high $\}$.*

Now, we consider an LDF-relation $\mathcal{R}_{\mathfrak{D}}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$ which describes the location of bungalows in a certain membership and non-membership degree functions $\delta_{\mathcal{R}_{\mathfrak{D}}}^{\mathcal{M}}$, and $\delta_{\mathcal{R}_{\mathfrak{D}}}^{\mathcal{M}}$ together with the parametric values $\alpha =$ good location and $\beta =$not good location, to the locations, respectively, in the Table 7.

Table 7. LDF-relation $\mathcal{R}_{\mathfrak{D}}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_2$.

$\mathcal{R}_{\mathfrak{D}}$	l_1	l_2	l_3
u_1	$((0.86, 0.34), (0.75, 0.24))$	$((0.56, 0.49), (0.50, 0.37))$	$((0.78, 0.35), (0.65, 0.25))$
u_2	$((0.75, 0.34), (0.60, 0.24))$	$((0.46, 0.74), (0.28, 0.60))$	$((0.45, 0.41), (0.32, 0.27))$
u_3	$((0.56, 0.44), (0.48, 0.26))$	$((0.34, 0.66), (0.25, 0.53))$	$((0.78, 0.59), (0.61, 0.49))$
u_4	$((0.95, 0.11), (0.80, 0.10))$	$((0.99, 0.21), (0.88, 0.08))$	$((0.86, 0.35), (0.75, 0.24))$

In addition, we consider the LDF-relation $\mathcal{P}_{\mathfrak{D}}$ from $\mathcal{Q}_2$ to $\mathcal{Q}_3$ which describes the relationship among the locations of bungalows and their prices by the membership and non-membership F-relations $\delta_{\mathcal{P}_{\mathfrak{D}}}^{\mathcal{M}}, \delta_{\mathcal{P}_{\mathfrak{D}}}^{\mathcal{N}}$ together with parametric values $\alpha' =$ reasonable price, $\beta' =$ not reasonable price in Table 8.

Table 8. LDF-relation $\mathcal{P}_{\mathfrak{D}}$ from $\mathcal{Q}_2$ to $\mathcal{Q}_3$.

$\mathcal{P}_{\mathfrak{D}}$	p_1	p_2	p_3
l_1	$((0.86, 0.50), (0.70, 0.25))$	$((0.89, 0.42), (0.75, 0.20))$	$((0.75, 0.31), (0.65, 0.20))$
l_2	$((0.65, 0.42), (0.60, 0.18))$	$((0.78, 0.32), (0.62, 0.17))$	$((0.75, 0.27), (0.65, 0.15))$
l_3	$((0.70, 0.40), (0.41, 0.28))$	$((0.86, 0.21), (0.48, 0.21))$	$((0.89, 0.10), (0.56, 0.10))$

By simple calculations of the composition 14, LDF-relation $\mathcal{R}_{\mathfrak{D}}\hat{\circ}\mathcal{P}_{\mathfrak{D}}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_3$ given in Table 9 describes the relationship among the bungalows and their prices according to the locations.

Table 9. Composition LDF-relation $\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{P}_{\mathfrak{D}}$ from $\mathcal{Q}_1$ to $\mathcal{Q}_3$.

$\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{P}_{\mathfrak{D}}$	p_1	p_2	p_3
u_1	$((0.86, 0.40), (0.70, 0.25))$	$((0.86, 0.35), (0.75, 0.24))$	$((0.78, 0.34), (0.65, 0.24))$
u_2	$((0.75, 0.41), (0.60, 0.25))$	$((0.75, 0.41), (0.60, 0.24))$	$((0.75, 0.34), (0.60, 0.24))$
u_3	$((0.70, 0.50), (0.48, 0.26))$	$((0.78, 0.44), (0.48, 0.26))$	$((0.78, 0.44), (0.56, 0.26))$
u_4	$((0.86, 0.40), (0.70, 0.18))$	$((0.89, 0.32), (0.75, 0.17))$	$((0.86, 0.27), (0.65, 0.15))$

Now, by using the Definition 19, hesitation degrees $\eta_{ik} = 1 - (\gamma_{ik}\theta_{ik}^{\mathcal{M}} + \gamma'_{ik}\theta_{ik}^{\mathcal{N}})$ of $\mathcal{R}_{\mathfrak{D}} \hat{\circ} \mathcal{P}_{\mathfrak{D}}$ are given in Table 10.

Table 10. Hesitation degrees $\eta_{ik} = 1 - (\gamma_{ik}\theta_{ik}^{\mathcal{M}} + \gamma'_{ik}\theta_{ik}^{\mathcal{N}})$.

η_{ik}	p_1	p_2	p_3
u_1	0.298	0.271	0.4114
u_2	0.4475	0.4516	0.4684
u_3	0.534	0.5112	0.4488
u_4	0.326	0.2781	0.3951

Next, the association grades among objects of $\mathcal{Q}_1$ and $\mathcal{Q}_3$ by using the formulae $\ddot{A}_{ik} = \theta_{ik}^{M} - \theta_{ik}^{N}\eta_{ik}$ are given in Table 11.

Table 11. Association grades with $\ddot{A}_{ik} = \theta_{ik}^{M} - \theta_{ik}^{N}\eta_{ik}$.

$\ddot{A}_{ik}$	p_1	p_2	p_3
u_1	0.7408	0.76515	0.640124
u_2	0.566525	0.564844	0.573904
u_3	0.433	0.555072	0.582528
u_4	0.7296	0.80104	0.75335

Clearly, the pair (u_4, p_2) have the highest association grade. Thus, u_4 is the optimal choice for Mr. X to purchase property in good location and reasonable price. For confirmation of our result, we calculate the score values among the objects of $\mathcal{Q}_1$ and $\mathcal{Q}_3$ by using the Definition 19 are computed in Table 12.

Table 12. Score values.

$\mathfrak{S}_{ik}$	p_1	p_2	p_3
u_1	0.455	0.51	0.425
u_2	0.345	0.35	0.385
u_3	0.21	0.28	0.32
u_4	0.49	0.575	0.545

It can be easily seen in the last row the pair (u_4, p_2) has the highest score value. Thus, our decision is true. Hence, our results are valid, and thus our proposed algorithm is a reliable method.

5. Conclusions

Binary relations play an important role in various fields of pure and applied sciences. This manuscript is devoted to studying the concept of LDF-relation in the motivation of

Riaz and Hashmi's work. This new concept of LDF-relation removes the limitations of IF-relation and enhances the space of membership and non-membership grades by adding the reference or control parameters. Some primary operations are defined and certain important results are established. With the help of these operations, it is investigated that the set of all LDF-relations give rise to some algebraic structures namely, semigroup, semiring and hemiring. Moreover, the concept of score function on an LDF-relation is introduced. Moreover, the notion of score function of LDF-relations is introduced to analyze the symmetry of the optimal decision and ranking of feasible alternatives. As an application of proposed LFD-relations in DM, an algorithm is rendered together with a numerical example. In future studies, this new work may be applied to various directions of MCDM and rough set theory using different hybrid techniques, for further research work. LDF-relation comes up with a rigorous mathematical model for modeling uncertainties in decision-making problems, including AI, robotics, machine learning, medical analysis, medicine, economics, and many other real life problems. We hope that the proposed model of LDF-relations and all the ideas in this paper shall exist as an establishment for LDFS theory and will lead to new fruitful results.

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