

## Article

# An Application of the Madelung Formalism for Dissipating and Decaying Systems

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**Abstract:** This paper is concerned with the modeling and analysis of quantum dissipation and diffusion phenomena in the Schrödinger picture. We derive and investigate in detail the Schrödinger-type equations accounting for dissipation and diffusion effects. From a mathematical viewpoint, this equation allows one to achieve and analyze all aspects of the quantum dissipative systems, regarding the wave equation, Hamilton–Jacobi and continuity equations. This simplification requires the performance of “the Madelung decomposition” of “the wave function”, which is rigorously attained under the general Lagrangian justification for this modification of quantum mechanics. It is proved that most of the important equations of dissipative quantum physics, such as convection–diffusion, Fokker–Planck and quantum Boltzmann, have a common origin and can be unified in one equation.

**Keywords:** decaying systems; Schrödinger equation; Madelung formulation



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## 1. Introduction

Since the early 20th century, the challenging problem of dissipation and diffusion modeling has been widely studied in quantum theory because a comprehensive understanding of dissipation in quantum mechanics is fundamental to the foundations of this theory [1,2]. The quantum–mechanical treatment of dissipative processes and other nonequilibrium phenomena has been the subject of much attention due to its applicability in various fields such as solid-state and statistical physics, incoherent solitons, photochemistry, Brownian dynamics, heavy-ion scattering, quantum gravity theories, dynamical modes of plasma physics, propagation of optical pulses and damping effects in nonlinear media [3–5].

In 1926 (the same year Schrödinger published his celebrated articles), Madelung reformulated the Schrödinger equation into a set of real, non-linear partial differential equations comparable with the Euler equations which were used in hydrodynamics. Madelung showed that the two equations were mathematically equivalent [6,7], and if one writes the wave function in the form of  $e^{R+iS}$ , the Schrödinger equation implies that, first,  $S$  is governed by a classical Hamilton–Jacobi-like equation, or alternatively that  $\vec{v} = \nabla S$  is formulated by a Newton-like equation; second,  $\rho$  (which is defined as  $\rho(x, t) = |\psi|^2 = R(x, t)^2$ ) is governed by a classical continuity equation [8]. The only formal difference between these equations and their purely classical counterparts is the existence of an additional “quantum” potential. Since that time these equations have provided the basis for numerous classical interpretations of quantum mechanics, including the hydrodynamic interpretation first proposed by Madelung [6], the theory of stochastic mechanics due to Nelson and others [3,8–12], the hidden-variable and double-solution theories of Bohm and de Broglie [13–15] and quite possibly other interpretations as well [8,16,17]. In some of these theories, such as the hydrodynamic interpretation and stochastic mechanics, the Madelung equations are taken as fundamental, and the Schrödinger equation is viewed as a mathematical consequence [8].

In the Schrödinger picture, quantum diffusion and dissipation effects have been effectively modeled by nonlinear terms of the type  $\lambda \Delta_\psi \psi$ , which was first formulated

by Kostin [18,19] to describe nonlinear Schrödinger–Langevin dynamics, where  $\lambda$  is a friction constant and  $\Delta_\psi = -i \log \frac{\psi}{|\psi|}$  is the argument of the complex wave function  $\psi(x, t)$ , and also later by logarithmic nonlinearities with the form  $\log(|\psi|^2)\psi$ , which was studied by Bialynicki-Birula and Mycielski [20,21]. They suggested that this logarithmic form maintains the lack of correlation between noninteracting particles. In a system under observation, there are many degrees of freedom such that information would be lost in the coupling process, which leads to dissipation [22]. Caldeira and Leggett showed by using the influence-functional method [23] that dissipation tends to destroy quantum interference in a time scale shorter than the relaxation time of the system [22]. This result has given justification for the use of logarithmic nonlinear wave equations [12,18,22,24] to describe quantum dissipation. These equations are acceptable as a proper, practical bath functional in time-dependent density functional theory for open quantum systems [22,24].

In this paper the main purpose is to obtain a deeper understanding of some equations of quantum dissipation and their interrelations in a more satisfactory way based on the influence-functional method. We start from the fact that the Lagrangian density equation can be generalized to accommodate arbitrary wave functions. This is done using a substitution prescription for the principal function by applying a general complex function of  $e^{\beta(x,t)}$  to introduce a wide-ranging Schrödinger-type wave equation and their corresponding Hamilton–Jacobi and continuity equations to access desirable results. It is shown that this approach provides a unified framework to aggregate and reproduce the wide class of Schrödinger-type equations compatible with the convection–diffusion, quantum Boltzmann and Fokker–Planck continuity equations; this leads to a deeper understanding of the nature of dissipative systems. We use Madelung’s fluid dynamical formulation of the Schrödinger equation, in order to quantize the generalized Hamilton–Jacobi equation [25], which is intimately related to Hamilton–Jacobi theory. In this work, it is shown that the spatial dependence of the real part of  $e^{\beta(x,t)}$  leads to the dissipative form of quantum potential, which is first introduced here.

This paper includes two main parts. First, the Schrödinger-type equation for dissipative systems via Lagrangian density and Euler–Lagrange equations is derived, and the related Madelung decomposition is discussed. In the next section it is shown that by assigning appropriate functions for  $\beta(x, t)$ , a wide range of important equations in nonequilibrium quantum physics, such as convection–diffusion, Fokker–Planck and quantum Boltzmann, are obtained.

## 2. Schrödinger-Type Equations of Dissipative Systems

### 2.1. Derivation

The Lagrangian approach based on the Principle of Least Action has been a unifying principle in almost all areas of physics to obtain dynamical equations. The Lagrangian  $L$  is a functional of field amplitude  $\psi(x, t)$ . It can usually be expressed as the integral over all space of a Lagrangian density  $\mathcal{L}(\psi)$ . If the field Lagrangian density  $\mathcal{L}(\psi)$  is given, we can obtain the dynamical field equation from the Euler–Lagrange equations [26–29],

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( \frac{\partial \mathcal{L}}{\partial (\partial \psi / \partial x_i)} \right) = 0, \quad (1)$$

where  $\frac{\partial}{\partial \psi}$  is a partial derivative.

Based on the Lagrangian approach, one can obtain the Schrödinger equation by its corresponding Lagrangian density [26,27], which is

$$\mathcal{L}_{sh}(\psi) = i\hbar\psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \sum_{i=1}^3 \left( \frac{\partial \psi^*}{\partial x_i} \right) \left( \frac{\partial \psi}{\partial x_i} \right),$$

If we define  $\dot{\psi} = \frac{\partial \psi}{\partial t}$  and the gradient as  $\vec{\nabla} \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ , the above equation can be written as

$$\mathcal{L}_{sh}(\psi) = i\hbar \psi^* \dot{\psi} - \frac{\hbar^2}{2m} \vec{\nabla} \psi^* \cdot \vec{\nabla} \psi, \quad (2)$$

Here  $\psi(x, t)$  is a complex function, and we can treat  $\psi$  and  $\psi^*$  as independent fields. Now we can put the Lagrangian density (2) in the Euler–Lagrange Equation (1) for the field  $\psi^*$  to obtain the Schrödinger equation (or  $\psi$  for its complex conjugate), and end up with the Schrödinger equation for free particles [26,27,30]:

$$-i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi \quad (3)$$

where  $\nabla^2 \equiv \left( \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right)$  is the Laplacian.

**Theorem 1.** If  $\psi(x, t)$  satisfied Equation (3), and it can be written in the form  $\psi(x, t) = e^{-\beta(x, t)} \psi_D(x, t)$  where  $\beta(x, t) \in \mathbb{C}$  and  $x \in \mathbb{R}^3$ ,  $t \in \mathbb{R}$ , then  $\psi(x, t)$  is also the solution of the following differential equation,

$$-i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \left( \vec{\nabla} + \vec{\nabla} \beta \right)^2 \psi. \quad (4)$$

**Proof.** By putting  $\psi_D = e^{\beta(x, t)} \psi(x, t)$  in  $\mathcal{L}_{sh}$  we have

$$\begin{aligned} \mathcal{L}_{sh}(\psi_D) &= i\hbar (e^\beta \psi)^* \frac{\partial (e^\beta \psi)}{\partial t} - \frac{\hbar^2}{2m} \vec{\nabla} (e^\beta \psi)^* \cdot \vec{\nabla} (e^\beta \psi) \\ &= i\hbar e^{\beta^*} e^\beta \psi^* \left( \frac{\partial \beta}{\partial t} \psi + \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} e^{\beta^*} e^\beta \left( \psi^* \vec{\nabla} \beta^* + \vec{\nabla} \psi^* \right) \cdot \left( \psi \vec{\nabla} \beta + \vec{\nabla} \psi \right) \end{aligned}$$

Now  $e^{\beta^*} e^\beta = e^{(\beta_r - i\beta_i) + (\beta_r + i\beta_i)} = e^{2\beta_r}$ , where  $\beta_r(x, t)$  and  $\beta_i(x, t)$  are the real and imaginary parts of  $\beta(x, t)$ . By defining  $\mathcal{L}_D(\psi) = \mathcal{L}_{sh}(\psi_D)$ , the above equation gives

$$\mathcal{L}_D(\psi) = i\hbar e^{2\beta_r} \psi^* \left( \frac{\partial \beta}{\partial t} \psi + \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} e^{2\beta_r} \left( \vec{\nabla} \beta^* \cdot \vec{\nabla} \beta \psi^* \psi + \vec{\nabla} \beta \cdot \vec{\nabla} \psi^* \psi + \vec{\nabla} \beta^* \cdot \vec{\nabla} \psi \psi^* + \vec{\nabla} \psi^* \cdot \vec{\nabla} \psi \right). \quad (5)$$

This is the Lagrangian density of dissipation wave function. Now to obtain the dynamical equation of dissipation wave function, we substitute  $\mathcal{L}_D$  in the Euler–Lagrange Equation (1) to obtain

$$\begin{aligned} i\hbar \frac{\partial \beta}{\partial t} \psi + i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} (\vec{\nabla} \beta^* \cdot \vec{\nabla} \beta) \psi - \frac{\hbar^2}{2m} \vec{\nabla} \beta^* \cdot \vec{\nabla} \psi + \frac{\hbar^2}{2m} 2(\vec{\nabla} \beta_r \cdot \vec{\nabla} \beta) \psi \\ + \frac{\hbar^2}{2m} (\nabla^2 \beta) \psi + \frac{\hbar^2}{2m} \vec{\nabla} \beta \cdot \vec{\nabla} \psi + \frac{\hbar^2}{2m} 2\vec{\nabla} \beta_r \cdot \vec{\nabla} \psi + \frac{\hbar^2}{2m} \nabla^2 \psi = 0. \end{aligned} \quad (6)$$

To simplify the above equation we replace  $\beta = (\beta_r + i\beta_i)$  and  $\beta^* = (\beta_r - i\beta_i)$  in some terms in Equation (6), then we have

$$\begin{aligned} \left\{ i\hbar \frac{\partial \beta}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \beta \right\} \psi + \left\{ i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi \right\} - \frac{\hbar^2}{2m} (\vec{\nabla} \beta_i)^2 \psi + \frac{\hbar^2}{2m} (\vec{\nabla} \beta_r)^2 \psi + i\frac{\hbar^2}{2m} 2\vec{\nabla} \beta_r \cdot \vec{\nabla} \beta_i \psi + \frac{\hbar^2}{2m} \vec{\nabla} \beta_r \cdot \vec{\nabla} \psi \\ + i\frac{\hbar^2}{2m} \vec{\nabla} \beta_i \cdot \vec{\nabla} \psi = 0. \end{aligned} \quad (7)$$

By using the two following equations

$$(\vec{\nabla} \beta)^2 \psi = (\vec{\nabla} \beta_r + i\vec{\nabla} \beta_i)^2 \psi = \left( (\vec{\nabla} \beta_r)^2 - (\vec{\nabla} \beta_i)^2 + i2\vec{\nabla} \beta_r \cdot \vec{\nabla} \beta_i \right) \psi \quad (8)$$

and

$$\vec{\nabla} \beta \cdot \vec{\nabla} \psi = (\vec{\nabla} \beta_r + i\vec{\nabla} \beta_i) \cdot \vec{\nabla} \psi \quad (9)$$

Equation (7) is rephrased in terms of  $\beta$  as

$$\left\{ i\hbar \frac{\partial \beta}{\partial t} \psi + i\hbar \frac{\partial \psi}{\partial t} \right\} + \frac{\hbar^2}{2m} \nabla^2 \psi + \frac{\hbar^2}{2m} \nabla^2 \beta \psi + \frac{\hbar^2}{2m} (\vec{\nabla} \beta)^2 \psi + \frac{\hbar^2}{2m} 2\vec{\nabla} \beta \cdot \vec{\nabla} \psi = 0.$$

Finally, to shorten Equation (9) and have an optimal form we use

$$(\vec{\nabla} + \vec{\nabla} \beta)^2 \psi = \nabla^2 \psi + \psi \nabla^2 \beta + (\vec{\nabla} \beta)^2 \psi + 2\vec{\nabla} \beta \cdot \vec{\nabla} \psi$$

Therefore, we conclude

$$-i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} (\vec{\nabla} + \vec{\nabla} \beta)^2 \psi + i\hbar \frac{\partial \beta}{\partial t} \psi \quad (10)$$

Equation (10) is a dissipative Schrödinger-like equation (DSE).  $\square$

Theorem 1, to some extent, is comparable to the Stoker [25] method, which has a variable function from which different equations can be obtained. Stoker's method was rudimentary and was not covered thoroughly. In the quantum hydrodynamical framework, Nassar [22,31] proposed a generalized nonlinear equation covering some of the famous equations due to Kostin [18], Süssmann and Hasse [32], Bialynicki-Birula–Mycielski [20], Stoker–Albrecht [25] and Schuch–Chung–Hartmann [33]. His equation had a variable parameter to produce different equations. Zander, Plastino and Díaz-Alonso [34] have investigated the nonlinear equation proposed by Nassar [22], and in its corresponding Hamilton–Jacobi and continuity parts some terms are left undefined. Recently, Gonçalves and Olavo [35] have derived, from first principles, a generalized Schrödinger equation that encompasses dissipative phenomena. Their results are not applicable to different equations of quantum dissipation systems.

Our approach is based on a variable complex valued function of  $e^{\beta(x,t)}$ , which can produce Fokker–Plank, convection–diffusion and quantum Boltzmann equations. These equations are only some examples of the applicability of this approach.

In this work, to the best of our knowledge, for the first time we present:

(A) a formalism which aggregates most of the important equations of dissipative quantum systems in a comprehensive manner; this leads to a deeper understanding of the nature of dissipative systems, see Section 3;

(B) a general form for quantum potential that would appear when the dissipation parameter depends on space, such as plasma currents where the quantum Boltzmann equation is used [36], see Sections 2.2 and 3.4.

**Corollary 1.** *if  $\beta = -\gamma t$  ( $\gamma > 0$ ), we have  $e^{-\gamma t} \psi$  which describes a decaying wave function, then its relevant wave equation is*

$$-i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi - i\hbar \gamma |\psi|. \quad (11)$$

**Corollary 2.** *if  $\beta = i \frac{q\varphi}{\hbar c}$  (where  $\frac{q\varphi}{\hbar c}$  is the Aharonov–Bohm parameter [27]), we have  $e^{i \frac{q\varphi}{\hbar c}} \psi$  which describes an Aharonov–Bohm effect, and its relevant wave equation is the same as the standard Schrödinger equation.*

## 2.2. Madelung Decomposition of DSE

The equations of quantum hydrodynamics (Madelung equations) are Madelung's corresponding alternative formulation of the Schrödinger equation [6], which is written in terms of hydrodynamical variables. The derivation of Madelung equations is similar to the de Broglie–Bohm formulation, which represents the Schrödinger equation as a quantum Hamilton–Jacobi equation [37]. The Madelung equations, by their virtue of being

formulated in the language of Newtonian mechanics, make it possible to construct a wide class of quantum theories by making the same coordinate-independent modifications found in Newtonian mechanics, without any need to construct a quantization algorithm [38].

Consider the one-dimensional time-dependent Schrödinger Equation (3) and write the polar form of wave function in terms of amplitude  $R(x, t)$  and phase function (or action function)  $S(x, t)$ , i.e.,

$$\psi(x, t) = R(x, t)e^{\frac{iS(x, t)}{\hbar}}, \quad (12)$$

in which both are real valued functions. The probability density associated with this wave function is  $\rho(x, t) = R(x, t)^2$ .

By substituting this wave function into the Schrödinger Equation (3), one obtains a system of two coupled partial differential equations, which are the Madelung hydrodynamical formulation of wave mechanics [7,15,25,39]. Now by equating the real and imaginary parts, first we find the continuity equation for the imaginary part,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}} = 0, \quad (13)$$

in which the probability flux is  $\vec{\mathcal{J}}$ , defined as  $\vec{\mathcal{J}} = \rho \frac{\vec{\nabla} S}{m}$ , and  $\vec{v} = \frac{\vec{\nabla} S}{m}$  is the flow velocity (or drift velocity) of the probability current.

The second equation or the real part is the quantum Hamilton–Jacobi equation (for  $V = 0$ ), given by

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}(\vec{\nabla} S)^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}; \quad (14)$$

in which the total energy is equal to the kinetic energy plus a quantum potential  $Q$  ( $Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$ ). Because of the explicit dependence of the quantum potential on  $\hbar$ , it brings all quantum effects into hydrodynamic formulation [39].

Now by applying Madelung decomposition on DSE (10), and after some manipulations, with the use of the identity

$$(\vec{\nabla} + \vec{\nabla} \beta_r)^2 R = \nabla^2 R + R \nabla^2 \beta_r + 2 \vec{\nabla} \beta_r \cdot \vec{\nabla} R + (\vec{\nabla} \beta_r)^2 R \quad (15)$$

the general quantum Hamilton–Jacobi and the general continuity equations, respectively, become

$$-\frac{\partial}{\partial t}(S + \hbar \beta_i) = \frac{1}{2m}(\vec{\nabla} S + \hbar \vec{\nabla} \beta_i)^2 - \frac{\hbar^2}{2m} \frac{(\vec{\nabla} + \vec{\nabla} \beta_r)^2 R}{R} \quad (16)$$

and

$$\left(\frac{\partial}{\partial t} + 2 \frac{\partial \beta_r}{\partial t}\right) \rho + (\vec{\nabla} + 2 \vec{\nabla} \beta_r) \cdot \left(\rho \frac{\vec{\nabla} S + \hbar \vec{\nabla} \beta_i}{m}\right) = 0 \quad (17)$$

Equation (16) is the Hamilton–Jacobi equation, in which on the left-hand side is the total energy and the right-hand side includes a dissipative form of the kinetic energy plus the general form of quantum potential. It is noteworthy to emphasize, when  $\vec{\nabla} \beta_r \neq 0$ , the quantum potential equation appears as

$$Q_D = -\frac{\hbar^2}{2m} \frac{(\vec{\nabla} + \vec{\nabla} \beta_r)^2 R}{R}. \quad (18)$$

This means that the spatial part of  $\beta_r$  changes the quantum properties and modifies the effect of the quantum potential. Since  $\beta_r$  is responsible for dissipative phenomena, one could refer (18) as a dissipative quantum potential (DQP).

It can be shown that the mean value of the DQP is proportional to the Fisher information probability density about the observable  $\hat{x}$ . By definition of Fisher information

$$\mathcal{F}(\theta) := E \left[ -\frac{\partial^2}{\partial \theta^2} \log f(X|\theta) \right] = - \int f \nabla^2 (\ln f) d^3x \quad (19)$$

and since

$$\nabla^2 (\ln f) = \frac{\nabla^2 f}{f} - \left( \frac{\nabla f}{f} \right)^2 \quad (20)$$

it can be proved that, similar to quantum potential [40–42], the mean value of the DQP is proportional to Fisher information

$$\langle Q_D \rangle = \int \psi_D^* Q_D \psi_D dr = \frac{\hbar^2}{8m} \mathcal{F} \quad (21)$$

and this is important because quantum fluctuations and quantum geometry are related to the quantum potential via Fisher information [43,44].

Temporal and spatial dependence of the imaginary part  $\beta$  (which adds to the phase) changes the total energy and the kinetic energy, respectively. Additionally, the spatial dependence of the real part  $\beta$  changes the well-known quantum potential, but its temporal part has no effect on the quantum Hamilton–Jacobi Equation (16).

Equation (17) is the origin of several important equations in open quantum systems. In the following sections, Equation (17) will be further discussed.

### 3. Applications

In this section, we will provide further discussion of Equations (16) and (17). We show that by selecting an appropriate function for  $\beta(x, t)$ , a wide range of important equations in quantum physics are obtained. Moreover, with enough knowledge about the wave function, it would be possible to predict the nature of the Hamilton–Jacobi, continuity and more importantly wave equations and vice versa.

For example, here is a brief description of the Berry’s phase. Berry’s phase [43] is a quantum phase effect arising in systems that undergo a slow, cyclic evolution. In an adiabatic evolution of the Hamiltonian, a quantum system in an  $n^{\text{th}}$  eigenstate, remains in this  $n^{\text{th}}$  eigenstate of the Hamiltonian, while picking up a phase factor. Under adiabatic approximation, the coefficient of the  $n^{\text{th}}$  eigenstate is given by

$$C_n(t) = C_n(0) \exp \left[ - \int_0^t \langle \psi_n(t') | \dot{\psi}_n(t') \rangle dt' \right] = C_n(0) e^{i\gamma_n(t)} \quad (22)$$

where  $\gamma_n(t)$  is the Berry’s phase with respect of parameter  $t$ . According to (22) and comparison with (16) and (17) one concludes that the adiabatic evolution has no effect on the continuity part, but the Hamilton–Jacobi equation takes the form of

$$-\frac{\partial S}{\partial t} = \frac{1}{2m} (\vec{\nabla} S)^2 - \frac{\hbar^2}{2m} \frac{\vec{\nabla}^2 R}{R} + \hbar \frac{\partial \gamma(t)}{\partial t} \quad (23)$$

and the corresponding wave equation is obtained as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi - \hbar \frac{\partial \gamma(t)}{\partial t} \psi \quad (24)$$

In the following, for better clarification, we will apply this approach to obtain the continuity equation, the Fokker–Planck equation, the convection–diffusion equation and the quantum Boltzmann equation. The important point is that they are all derived from a single equation produced by the “Lagrangian density of the Srodinger equation”. Therefore,

Equation (10) can be considered as a consistent and constructive generalization for the Schrödinger equation.

### 3.1. Continuity Equation with Source or Sink

In the simplest example we take

$$\beta_i = 0 \text{ and } \beta_r = \gamma t, \gamma \in \mathbb{R} \quad (25)$$

so, the wave function is  $\psi_D = e^{\gamma t} \psi$ , in which  $\gamma$  is a real number. Therefore, the corresponding wave equation takes the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + i\gamma \hbar \psi. \quad (26)$$

In this case, the Hamilton–Jacobi equation remains without change

$$-\frac{\partial S}{\partial t} = \frac{1}{2m} (\vec{\nabla} S)^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}, \quad (27)$$

and, as expected, the probability current is no longer conserved

$$\frac{\partial R^2}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}} = -2\gamma \rho \quad (28)$$

where  $2\gamma\rho$  represents source or sink for the current [17].

### 3.2. The Fokker–Planck Equation

As an important example, we take

$$\beta_r = 0 \text{ and } \beta_i = \ln R(x, t) \quad (29)$$

where  $R(x, t)$  is the real valued amplitude of the wave function, and upon substituting in DSE (10), its corresponding wave equation takes the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi - i\frac{\hbar^2}{2m} \nabla^2 (\ln|\psi|) \psi - i\frac{\hbar^2}{m} \vec{\nabla} \ln|\psi| \cdot \vec{\nabla} \psi + \frac{\hbar^2}{2m} (\vec{\nabla} \ln|\psi|)^2 \psi - \hbar \frac{\partial \ln|\psi|}{\partial t} \psi. \quad (30)$$

Next, by setting (29) in Equations (16) and (17), we obtain its relevant continuity and the Hamilton–Jacobi equations, respectively

$$-\frac{\partial}{\partial t} (S + \hbar \ln R) = \frac{1}{2m} (\vec{\nabla} S + \hbar \vec{\nabla} \ln R)^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \quad (31)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}} = -\frac{\hbar}{2m} \nabla^2 \rho. \quad (32)$$

Equation (31) is the Hamilton–Jacobi equation of a diffusive system, whose total energy is on the left-hand side and total kinetic energy plus quantum potential is on the right-hand side. Total kinetic energy includes kinetic energy corresponding to drift velocity in which there is an added non-classical, stochastic diffusion velocity (either of Markovian or of non-Markovian type) [1].

Equation (32) is the well-known Fokker–Planck equation. Some fundamental considerations of quantum theory suggest a general, complex nonlinear Schrödinger equation (outside the classes most often studied), which follows from admitting quantum diffusion currents, so that its probability density satisfies the Fokker–Planck equation [17,45–47].

It is worth mentioning that the diffusion current, and thus the Fokker–Planck equation, are due to the presence of  $\ln R(x, y)$  in the phase of the wave function

$$\begin{aligned}\psi &\rightarrow \psi_D = e^{(\beta_r + i\beta_i)}\psi \\ \psi_D &= Re^{i\kappa \ln R} e^{i\frac{S}{\hbar}} = Re^{\frac{i(S + \hbar\kappa \ln R)}{\hbar}}\end{aligned}$$

where  $\kappa$  is a real number.

Doebner and Goldin in [45] propose a group-theoretical justification for a nonlinear modification of quantum mechanics, as the most general class of Schrödinger-type equations compatible with the Fokker–Planck continuity equation; they start with

$$i\hbar \frac{\partial \psi}{\partial t} = H_0 \psi + \mathcal{R}[\psi]\psi + i\mathcal{I}[\psi]\psi \text{ with } \mathcal{I}[\psi] = \frac{\hbar D}{2} \left( \frac{\nabla^2(\bar{\psi}\psi)}{\bar{\psi}\psi} \right) \quad (33)$$

where  $\mathcal{R}[\psi]$  and  $\mathcal{I}[\psi]$  are the real and imaginary parts of the nonlinear functional multiplying  $\psi$ ; and  $\mathcal{I}[\psi]$  is the key phrase to create the Fokker–Planck equation. Then, they declare that the theory at this point gives no further information about  $\mathcal{R}[\psi]$ , but it is reasonable to assume that  $\mathcal{R}[\psi]$  is of a form similar to  $\mathcal{I}[\psi]$ .

If Equation (30) is rewritten as (34), its compliance with the Doebner–Goldin Equation (33) is determined [35,45,46,48]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + \left\{ \frac{\hbar}{m} \frac{\vec{\nabla}|\psi|}{|\psi|} \vec{\nabla} S + \frac{\hbar^2}{2m} \left( \frac{\vec{\nabla}|\psi|}{|\psi|} \right)^2 + \hbar \frac{\partial \ln|\psi|}{\partial t} \right\} \psi + i \left\{ \frac{-\hbar^2}{2m} \left( \frac{\nabla^2 \psi^2}{2\psi^2} \right) \right\} \psi \quad (34)$$

The general Lagrangian justification for this nonlinear modification of quantum mechanics seems to suggest it as a minimal nonlinear generalization of the Schrödinger equation; it contains the least terms and still satisfies the Fokker–Planck equation. In addition, it satisfies the general requirements of a nonlinear Schrödinger equation: (a) the probability is conserved, (b) the equation is homogeneous, (c) non-interacting particle subsystems remain uncorrelated, (d) plane waves are solutions for the free equation and (e) the free equation is also time and space translation invariant [45,48].

Logarithmic terms which appear in (34) have a great importance in quantum friction and diffusion effects. Dissipation tends to destroy quantum interference in a time scale shorter than the relaxation time of the system [21]. This result has given justification for the use of logarithmic nonlinear wave equations [12,18,22,24] to describe quantum dissipation. Additionally, the logarithmic form guarantees non-interacting particle subsystems remain uncorrelated [20].

### 3.3. Convection–Diffusion Equation

If we combine the diffusion (the Fokker–Planck equation) (32) with source or sink (28), the result would be a convection–diffusion equation. Thus, based on previous sections we pick  $\psi_D = e^{(\beta_r + i\beta_i)}\psi = e^{\frac{\gamma}{2}t + i\ln R}\psi$  ( $\gamma \in \mathbb{R}$ ), and substituting in (10), (16) and (17), respectively, the wave equation, the Hamilton–Jacobi equation and continuity equations are obtained as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} (\vec{\nabla} + i\vec{\nabla} \ln|\psi|)^2 \psi + \hbar \frac{\partial}{\partial t} \ln|\psi| \psi + i\hbar \frac{\gamma}{2} \psi \quad (35)$$

$$-\frac{\partial}{\partial t} (S + \hbar \ln R) = \frac{1}{2m} (\vec{\nabla} S - \hbar \vec{\nabla} \ln R)^2 - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \quad (36)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{J}} = -\frac{\hbar}{2m} \nabla^2 \rho - 2\gamma \rho. \quad (37)$$

Equation (37) is the convection–diffusion equation [49], and  $2\gamma\rho$  is the source or sink of the probability current.

### 3.4. Quantum Boltzmann Equation

As a last example, we take  $\Gamma(x, t)$  as a general real function and again the diffusion factor of  $i \ln R(x, t)$  we have

$$\beta = \frac{-\Gamma(x, t)}{2} + i \ln R(x, t), \quad (38)$$

Therefore, wave Equation (10) takes the form of

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} (\vec{\nabla} - \vec{\nabla} \Gamma + i \vec{\nabla} \ln |\psi|)^2 \psi + \hbar \frac{\partial \ln |\psi|}{\partial t} \psi + \frac{i\hbar}{2} \frac{\partial \Gamma}{\partial t} \psi \quad (39)$$

and corresponding Hamilton–Jacobi and continuity equations become

$$-\frac{\partial}{\partial t} (S + \hbar \ln R) = \frac{1}{2m} (\vec{\nabla} S + \hbar \vec{\nabla} \ln R)^2 - \frac{\hbar^2}{2m} \frac{\left( \vec{\nabla} - \frac{\vec{\nabla} \Gamma}{2} \right)^2 R}{R} \quad (40)$$

$$\left( \frac{\partial}{\partial t} - \frac{\partial \Gamma}{\partial t} \right) \rho + (\vec{\nabla} - \vec{\nabla} \Gamma) \cdot \left( \vec{\mathcal{J}} + \frac{\hbar}{2m} \rho \vec{\nabla} \ln \rho \right) = 0. \quad (41)$$

We see in Equation (40) that DQP is revealed. It means that if  $\vec{\nabla} \Gamma(x, t) \neq 0$ , the well-known quantum potential will take a new form of DQP.

Equation (41) is the continuity equation, and with some manipulations it takes a familiar form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) - \frac{\hbar}{2m} \nabla^2 \rho = \rho \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \frac{\hbar}{2m} \vec{\nabla} \ln \rho \cdot \vec{\nabla} \right) \Gamma. \quad (42)$$

where  $\vec{v} = \frac{\vec{\nabla} S}{m}$  is the drift velocity. The left-hand side of the above equation is the Fokker–Planck equation and the right-hand side resembles the quantum Boltzmann equation

$$\left[ \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \vec{F} \cdot \vec{\nabla}_p \right] f(\vec{x}, \vec{p}, t) = \mathcal{Q}[f](\vec{x}, \vec{p}). \quad (43)$$

The quantum Boltzmann equation gives the non-equilibrium time evolution of a gas of quantum mechanically interacting particles. In Equation (43),  $f(\vec{x}, \vec{p}, t)$  is a general distribution function,  $\vec{F}$  is external applied force and  $\mathcal{Q}$  is quantum collision operator, accounting for the interactions between the gas particles; if it is zero then the particles do not collide. Thus, the Fokker–Planck term at the left-hand side of (42) represents the effects of particle collisions [50].

Go back to the right-hand side of Equation (42). Since we can write

$$\vec{\nabla} \Gamma = \frac{\partial \Gamma}{\partial x} \frac{\vec{\nabla} S}{\partial \nabla S} = \nabla^2 S \frac{\vec{\nabla} \Gamma}{\partial \nabla S}, \quad (44)$$

then by substituting (44) and (43) into the right-hand side of (42), we have

$$\left( \vec{F} \cdot \frac{\partial}{\partial \vec{p}} \right) f = \left( \frac{\hbar}{2m^2} \nabla^2 S \vec{\nabla} \ln \rho \cdot \frac{\partial}{\partial \nabla S} \right) \Gamma. \quad (45)$$

Thus, we can write Equation (42) as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) - \frac{\hbar}{2m} \nabla^2 \rho = \rho \left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} + \frac{\hbar}{2m} (\vec{\nabla} \cdot \vec{v}) \vec{\nabla} \ln \rho \cdot \vec{\nabla}_p \right) \Gamma. \quad (46)$$

So, the external applied force is defined as  $\vec{F} = \frac{\hbar}{2m}(\vec{\nabla} \cdot \vec{v})\vec{\nabla} \ln \rho$ , which is related to both drift and diffusion velocities.

The Fokker–Planck equation, which was firstly derived to treat the Brownian motion of molecules, has been extensively used to evaluate the collision term of the Boltzmann equation for describing small-angle binary collisions of the inverse-square type of force [36]. In stellar dynamics, Chandrasekhar first discussed this theory for stochastic effects of gravity [36,51]. The applications of this equation to classical plasma physics were first treated by Landau, Spitzer, as well as Cohen, Spitzer and Routly, and an elegant mathematical treatment was completed by Rosenbluth, MacDonald and Judd [36,52].

#### 4. Summery and Conclusions

In this approach based on the influence-functional method we gain a better understanding on quantum friction and diffusion effects in a more satisfactory way. We generalize the Lagrangian density equation to accommodate arbitrary wave functions, to provide an approach to general quantum dissipation and diffusion modeling by introducing a dissipative Schrödinger-type equation (DSE). The key point of this approach is  $\beta(x, t)$ , as a complex function, which is responsible for taking the problem from equilibrium phenomena to nonequilibrium ones. The approach provides the achievable analysis, for all aspects of the phenomena, concerning the wave equation, the Hamilton–Jacobi equation and the continuity equation. It is shown that by applying a general complex function of  $e^{\beta(x, t)}$  one can produce a wide-ranging Schrödinger-type wave equation and their corresponding Hamilton–Jacobi and continuity equations to access desirable results. To show the widespread applications of this approach, we provided some examples of the Berry phase, the continuity equation, the Fokker–Planck equation, the convection–diffusion equation and the quantum Boltzmann equation. All of these equations are obtained by step-by-step generalizations of the beta function, see Table 1.

**Table 1.** Dissipation quantum equations and their related  $\beta$  functions.

| Quantum Equations                       | $\beta_r$                                 | $\beta_i$                     |
|-----------------------------------------|-------------------------------------------|-------------------------------|
| Continuity equation with Source or Sink | $\beta_r(x, t) = \gamma t$                | $\beta_i(x, t) = 0$           |
| Fokker–Planck equation                  | $\beta_r(x, t) = 0$                       | $\beta_i(x, t) = \ln R(x, t)$ |
| Convection–diffusion equation           | $\beta_r(x, t) = \frac{\gamma t}{2}$      | $\beta_i(x, t) = \ln R(x, t)$ |
| Quantum Boltzmann equation              | $\beta_r(x, t) = \frac{-\Gamma(x, t)}{2}$ | $\beta_i(x, t) = \ln R(x, t)$ |

As a first step, we chose  $\beta_r(x, t) = \gamma t$  with no imaginary part, then we would have growing/shrinking amplitude; it means the probability current is no longer conserved, and there would be source/sink in the continuity equation.

Some fundamental considerations of quantum theory suggest a general, complex nonlinear Schrödinger equation, which follows from admitting quantum diffusion currents and must be such that its probability density satisfies the Fokker–Planck equation. To describe the diffusion current there is no need for source or sink; so the real part which affects the amplitude is zero, and the imaginary part which controls the phase gets the  $\beta_i = \ln R(x, t)$  value. From this choice the Fokker–Planck equation results in the continuity part, but it also changes the usual appearance of the Hamilton–Jacobi (31). As expected, by quantum friction and diffusion effects, logarithmic nonlinearities appear in the corresponding wave Equation (34).

Now if we have a combination of growing/shrinking amplitude with diffusion current,  $\beta_r(x, t) = \frac{\gamma t}{2}$  and  $\beta_i(x, t) = \ln R(x, t)$ , the result is the convection–diffusion Equation (37).

Finally, choosing for general real part  $\Gamma(x, t)$ , which means  $\beta_r(x, t) = \frac{-\Gamma(x, t)}{2}$  and  $\beta_i(x, t) = \ln R(x, t)$ , this leads to the quantum Boltzmann equation. Because of the spatial dependence of  $\beta_r(x, t)$ , the new appearance for quantum potential (DQP) has been revealed (18). Since the quantum potential depends on the amplitude of a wave function, it is

deformed as DQP for a dissipative system. This form of quantum potential is first presented in this work.

All of above equations are derived from a single equation produced by the Dissipative Schrödinger Equation (10). This unified framework provides a common ground for a better understanding of quantum friction and diffusion effects and allows a deeper understanding of the nature of the dissipative systems. The present approach is useful to the study of non-equilibrium quantum mechanical systems, such as mesoscopic systems.

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