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Existence of the Solutions of Nonlinear Fractional Differential Equations Using the Fixed Point Technique in Extended *b*-Metric Spaces

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Abstract: The purpose of this paper is to prove fixed point theorems for cyclic-type operators in extended *b*-metric spaces. The well-posedness of the fixed point problem and limit shadowing property are also discussed. Some examples are given in order to support our results, and the last part of the paper considers some applications of the main results. The first part of this section is devoted to the study of the existence of a solution to the boundary value problem. In the second part of this section, we study the existence of solutions to fractional boundary value problems with integral-type boundary conditions in the frame of some Caputo-type fractional operators.

Keywords: fixed point; extended *b*-metric space; well-posedness; limit shadowing property; fractional differential equation; boundary value problem

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1. Introduction and Preliminaries

In 1922, Banach proved an interesting fixed point theorem for metric spaces (see [1]), known as the famous "Banach contraction principle". Since then, different generalisations of this theorem have been established.

Generalising the Banach contraction principle has been considered in a variety of ways. One of these is the consideration of different types of operators that satisfy some contraction conditions. Recently, different authors proved fixed point theorems for operators that satisfy a cyclic-type contraction condition. One important paper that deals with fixed point theory for cyclic contractions is [2], where some fixed point results for cyclic mappings are proved. The results are then extended in the paper [3], where the authors considered generalisation of the contraction condition. R. George et al. in [4] considered various types of cyclic contractions, such as Kannan, Chatterjee, and Ćirić, and proved the existence and uniqueness theorems for these classes of operators. Other results that involve the notion of cyclic contraction, including applications to integral equations, can be found in [5–8]. We also note that the cyclic operator idea has been applied in deriving synchronisation conditions of complex dynamical systems—see [9].

Concerning the other direction in generalising the "Banach contraction principle"—changing the working space, a popular concept is that of *b*-metric space (also known as quasimetric space). This was introduced by Bakhtin in 1989 in [10] and formally defined by Czerwik in 1993 in [11]. Since then, many authors have proved different fixed point theorems in the context of a *b*-metric space (see [12–14]). One of the major difference between the concepts of metric and *b*-metric is that fact that the latter is not necessary continuous (see [15,16]).

The purpose of this paper is to extend the previous results to the class of extended *b*-metric spaces and also to discuss the well-posedness and the limit shadowing property of the fixed point problem. Some examples are provided in order to support the results.

The standard notations and terminologies in nonlinear analysis are used throughout this paper. We recall some essential definitions and fundamental results. We begin with the definition of the b-metric space.

Definition 1 (Bakhtin [10], Czerwik [17])**.** *Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

We notice that the notion reduces to that of a metric space if $s = 1$. Hence, this notion is a generalisation of that of the metric space.

A classical example of a b-metric is the following:

Example 1 (Berinde see [18])**.** *The space $L_p (0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ such that*

$$\int_0^1 |x(t)|^p dt < \infty,$$

is a b-metric space if we take

$$d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L_p.$$

The constant s is $2^{1/p}$.

For other examples regarding the notions of b-metric and extended b-metric, see [10,12,17–19].

With the paper [4] as a starting point, we consider the case of cyclic operators in extended b-metric spaces. In [2], we found a generalisation of the well-known Banach contraction principle, where the notion of cyclic contraction is inductively introduced for the first time. Let us recall the definition of the cyclic operator in the context of a complete metric space as follows:

Definition 2 ([2])**.** *Let (X, d) be a b-metric space. Let p be a positive integer; $p \geq 2$, $A_1, A_2, ..., A_p$ be nonempty and closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Then, T is called a cyclic operator if*

1. $A_i, i \in \{1, 2, ...p\}$ *are nonempty subsets;*
2. $T(A_1) \subseteq A_2, ..., T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

In [20], T. Kamran et al. introduced the notion of extended b-metric space as follows:

Definition 3. *Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. The function $d_\theta : X \times X \to [0, \infty)$ is said to be an extended b-metric if the following conditions are satisfied:*

1. $d_\theta(x, y) = 0$ *if and only if* $x = y$;
2. $d_\theta(x, y) = d(y, x)$;
3. $d_\theta(x, z) \leq \theta(x, z)[d(x, y) + d(y, z)]$.

for all $x, y, z \in X$. A pair (X, d_θ) is called an extended b-metric space.

It is obvious that if $\theta(x, z) = s$ with $s \geq 1$, the notion reduces to that of b-metric space. As a remark, we must emphasise the symmetry of the extended b-metric, which appears in the second axiom in the previous definition.

In this paper, for the function $T : X \to X$ and $x_0 \in X$, $\mathcal{O}(x_0) = \{x_0, Tx_0, T^2x_0, T^3x_0, \dots\}$ represents the orbit of x_0.

The operator T is a contraction if $k \in [0,1)$ exists such that $d(f(x), f(y)) \le kd(x,y)$, for all $x, y \in X$.

In the following, the concepts of convergence, Cauchy sequence, and completeness are introduced in the framework of an extended b-metric space.

Definition 4. *Let (X, d_θ) be an extended b-metric space. Then, a sequence $\{x_n\}$ in X is said to be*

(i) *Convergent if and only if $x \in X$ exists such that $d_\theta(x_n, x) \to 0$ as $n \to \infty$ we write $\lim\limits_{n\to\infty} x_n = x$.*
(ii) *Cauchy if and only if $d_\theta(x_n, x_m) \to 0$ as $n, m \to \infty$.*

The extended b-metric space (X, d_θ) is complete if every Cauchy sequence converges in X. We note that the extended b-metric d_θ is not in general a continuous function.

Lemma 1. *Let (X, d_θ) be an extended b-metric space. Then, every convergent sequence has a unique limit.*

2. Fixed Point Results

We begin this section with the following main results:

Theorem 1. *Let (X, d_θ) be a complete extended b-metric space with d_θ, a continuous functional. Let $\{A_i\}_{i=1}^{p}$, where p is a positive integer, be nonempty closed subsets of X, and suppose $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$, is a cyclic operator that satisfies the following conditions:*

(i) $T(A_i) \subseteq A_{i+1}$, *for all* $i \in \{1, 2, \dots, p\}$;
(ii) $d(Tx, Ty) \le \lambda d(x, y)$ *for all* $x \in A_i, y \in A_{i+1}$ *where* $\lambda \in [0,1)$ *be such that for each* $x \in X$, $\lim\limits_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{\lambda}$ *where* $x_n = T^n(x)$, $n = 1, 2, \dots$.

Thus, T has a fixed point x^*. Moreover, for each $y \in X$, $T^n y \to x^*$.

Proof. Let $x_0 \in \bigcup\limits_{i=1}^{p} A_i$ if $i \in \{1, 2, \dots, p\}$ exists such that $x_0 \in A_i$.

From hypothesis, (i) we have $x_1 = T(x_0) \in A_{i+1}$.

Thus, we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \ge 0$. We can show that $\{x_n\}$ is a Cauchy sequence.

If $x_n = x_{n+1}$, then x_n is a fixed point of T. We suppose that $x_n \ne x_{n+1}$ for all $n \ge 0$.

From (ii), it follows that

$$d(x_n, x_{n+1}) = d(Tx_{n+1}, Tx_n) \le \lambda d(x_{n-1}, x_n).$$

If we repeat the process we obtain

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1). \tag{1}$$

Additionally, we assume that x_0 is a nonperiodic point of T. If $x_0 = x_n$ using (1), for any $n \ge 2$, we obtain

$$d(x_0, T(x_0)) = d(x_n, Tx_n).$$

Thus, $d(x_0, x_1) = d(x_n, Tx_{n+1})$ and $d(x_0, x_1) \le \lambda^n d(x_0, x_1)$, a contradiction.

Therefore, $d(x_0, x_1) = 0$, i.e., $x_0 = x_1$, and x_0 is a fixed point of T. Thus, we assume that $x_n \ne x_m$ for all $n, m \in \mathbb{N}$ with $m \ne n$.

For any m, n with $m > n$ we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \theta(x_n, x_m)\lambda^n d_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\lambda^{n+1} d_\theta(x_0, x_1) + \\ &+ \theta(x_n, x_m)\theta(x_{n+1}, x_m) \cdots \theta(x_{m-1}, x_m)\lambda^{m-1} d_\theta(x_0, x_1) \\ &\leq d_\theta(x_0, x_1)[\theta(x_1, x_m)\theta(x_2, x_m \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\lambda^n + \\ &+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m)\lambda^{n+1} + \cdots \\ &+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\theta(x_{n+1}, x_m) \cdots \theta(x_{m-1}, x_m)\lambda^{m-1}]. \end{aligned}$$

Since $\lim\limits_{n,m\to\infty} \theta(x_{n+1}, x_m)\lambda < 1$, the series $\sum\limits_{n=1}^{\infty} \lambda^n \prod\limits_{r=1}^{n} \theta(x_r, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let $S = \sum\limits_{n=1}^{\infty} \lambda^n \prod\limits_{r=1}^{n} \theta(x_r, x_m)$, $S_n = \sum\limits_{j=1}^{n} \lambda^j \prod\limits_{r=1}^{j} \theta(x_r, x_m)$.

Thus, for $m > n$ we have $d_\theta(x_n, x_m) \leq d_\theta(x_0, x_1)[S_{m-1}, S_n]$.

Letting $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence in $\bigcup\limits_{i=1}^{p} A_i$, a subspace of the complete extended b-metric space X. Therefore, there exists $x^* \in \bigcup\limits_{i=1}^{p} A_i$ such that $d_\theta(x_n, x^*) \to 0$, as $n \to \infty$. Then, $\lim\limits_{n\to\infty} x_n = x^*$.

The sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, \dots, p\}$. Therefore, $x^* \in \bigcap\limits_{i=1}^{p} A_i$.

We shall now show that x^* is a fixed point of T. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_\theta(Tx^*, x^*) &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, x_n) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[\lambda d_\theta(x^*, x_{n-1}) + d_\theta(x_n, x^*)]. \end{aligned}$$

We note that $d_\theta(Tx^*, x^*) \leq 0$ as $n \to \infty$. Hence, $d_\theta(Tx^*, x^*) = 0$, which is equivalent to $x^* = Tx^*$. Thus, we proved that x^* is the fixed point of T.

For the uniqueness, let ν be another fixed point of T.

By hypothesis (ii), we obtain $d_\theta(x^*, \nu) = d_\theta(Tx^*, T\nu) \leq \lambda d_\theta(x^*, \nu) < d_\theta(x^*, \nu)$, which is a contradiction.

Then, $d_\theta(x^*, \nu) = 0$ and $x^* = \nu$. The fixed point is unique. □

In the following theorem, we present a result which assures the well-posedness of the fixed point problem.

Theorem 2. *Let $T : \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ be a cyclic operator defined as in Theorem 1. Then, the fixed point problem for T is well-posed, i.e., a sequence $\{x_n\} \in \bigcup\limits_{i=1}^{p} A_i$ with $d_\theta(x_n, Tx_n) \to 0$ as $n \to \infty$ exists; then, $x_n \to x^*$ as $n \to \infty$.*

Proof. Applying Theorem 1, for any initial value $x_0 \in \bigcup\limits_{i=1}^{p} A_i$, $x^* \in \bigcap\limits_{i=1}^{p} A_i$ exists, which is the unique fixed point of T. Thus $d_\theta(x_n, x^*)$ is well defined.

We consider the following inequality:

$$\begin{aligned} d_\theta(x_n, x^*) &\leq \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + d_\theta(Tx_n, x^*)] \\ &= \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + d_\theta(Tx_n, Tx^*)] \\ &\leq \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + \lambda d_\theta(x_n, x^*)]. \end{aligned}$$

Then, we have $[1 - \theta(x_n, x^*)\lambda]d_\theta(x_n, x^*) \leq \theta(x_n, x^*)d_\theta(x_n, Tx_n)$, and we obtain $d_\theta(x_n, x^*) \leq \frac{\theta(x_n, x^*)}{1-\theta(x_n, x^*)\lambda} d_\theta(x_n, Tx_n)$.

Letting $n \to \infty$, the hypothesis that $d_\theta(x_n, Tx_n) \to 0$ is formed. Hence, $d_\theta(x_n, x^*) = 0$. Thus, our conclusion is supported. $\square$

The next theorem assures the limit shadowing property of the cyclic operator.

Theorem 3. *Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$, be a cyclic operator defined as in Theorem 1. Then, T has the limit shadowing property, i.e., if a convergent sequence $\{y_n\} \in \bigcup_{i=1}^{p} A_i$ with $d(y_{n+1}, Ty_n) \to 0$, as $n \to \infty$ exists, then $x \in \bigcup_{i=1}^{p} A_i$ exists such that $d(y_n, T^n x) \to 0$, as $n \to \infty$.*

Proof. As in the proof of Theorem 1, for any initial value $x \in \bigcup_{i=1}^{p} A_i$, $x^* \in \bigcap_{i=1}^{p} A_i$ is the unique fixed point of T. Thus, $d(y_n, x^*)$ and $d(y_{n+1}, x^*)$ are well defined.

Let $y \in X$ exist as the limit of the convergent sequence $\{y_n\} \in \bigcup_{i=1}^{p} A_i$.

We consider the following estimation:

$$\begin{aligned} d_\theta(y_{n+1}, x^*) &\le \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + d_\theta(Ty_n, x^*)] \\ &= \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + d_\theta(Ty_n, Tx^*)] \\ &\le \theta(y_{n+1}, x^*)[d_\theta(y_{n+1}, Ty_n) + \lambda(d_\theta(y_n, x^*)]. \end{aligned}$$

Letting $n \to \infty$, from the hypothesis, we have $d_\theta(y_{n+1}, Ty_n) \to 0$.
Thus, $d_\theta(y, x^*) \le \lim_{n\to\infty} \theta(y_{n+1}, x^*)\lambda d_\theta(y, x^*)$.

Since $\lim_{n\to\infty} \theta(y_{n+1}, x^*)\lambda < 1$, this inequality is true only for the case of $d_\theta(y, x^*) = 0$. Thus, $y = x^*$ and we have $d_\theta(y_n, T^n x) \to d(y, x^*) = 0$ as $n \to \infty$. $\square$

In order to support our results, let us present the following example:

Example 2. *Let $X = \mathbb{R}^+$ endowed with $d_\theta : X \times X \to \mathbb{R}^+$ defined by $d_\theta = |x - y|^3$, and let $\theta : X \times X \to [1, \infty)$ defined by $\theta(x, y) = x + y + 2$. It is easy to check that (X, d_θ) is a complete extended b-metric space.*

Let $A_1 = [0, 1]$, $A_2 = [0, \frac{1}{2}]$, $A_3 = [0, \frac{1}{3}]$ be three subsets of $X = \mathbb{R}^+$.

Define $T : \bigcup_{i=1}^{3} A_i \to \bigcup_{i=1}^{3} A_i$ by $Tx = \frac{x}{2}$. Obviously, $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_1$. Thus, $\bigcup_{i=1}^{3} A_i$ is a cyclic representation with respect to T.

The contraction condition is also verified.

$$d_\theta(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}|^3 = |\frac{1}{2}(x - y)|^3 \le \frac{1}{8}|x - y|^3 = \frac{1}{8} d_\theta(x, y).$$

Taking into account for each $x \in \bigcup_{i=1}^{3} A_i$, $T^n x = \frac{x}{2^n}$, we obtain

$$\begin{aligned} \lim_{n,m\to\infty} \theta(x_n, x_m) &= \lim_{n,m\to\infty} \theta(\frac{x}{2^n}, \frac{x}{2^m}) \\ &= \lim_{n,m\to\infty} (\frac{x}{2^n} + \frac{x}{2^m} + 2) = 2 < 8. \end{aligned}$$

Therefore, all conditions of Theorem 1 are satisfied, meaning that $0 \in \bigcap_{i=1}^{3} A_i$ is the unique fixed point of T.

3. Applications to Nonlinear Fractional Differential Equations

Our first application of this section is devoted to the existence of a solution of a boundary value problem. Thus, we recall the following problem given by Nieto and Lopez in [21].

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u(a), \end{cases} \tag{2}$$

where $a > 0$ and $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. A solution to (2) is the function $u \in C^1([0, a], \mathbb{R})$ satisfying (2), where $C^1([0, a], \mathbb{R})$ is the set of all continuous differentiable functions on $[0, a]$. We suggest that (2) has a lower solution if function $u \in C^1([0, a], \mathbb{R})$ exists, satisfying

$$\begin{cases} u'(t) \leq f(t, u(t)) \\ u(0) \leq u(a). \end{cases}$$

It is well known [22] that the existence of a lower solution a and an upper solution b with $a \leq b$ implies the existence of a solution of the boundary value problem between a and b.

In [21], we find the following results:

Theorem 4. *Let $a > 0$. Let $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping. Assume that $\alpha > 0$, $\beta > 0$ with $\beta < \alpha$ exist such that for any $x, y \in \mathbb{R}$,*

$$0 \leq f(t, x) + \alpha x - (f(t, y) + \alpha y) \leq \beta(x - y).$$

Thus, the existence of a lower solution of (2) provides the existence of a unique solution of (2).

Furthermore, let us provide a generalisation of Theorem 4 using cyclic operators for the case of extended b-metric spaces.

Theorem 5. *Let $a > 0$. Let $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ be a continuous mapping. Assume that $\alpha > 0$, $\beta > 0$ with $\beta < \alpha$ exsit such that for any $x, y \in \mathbb{R}$,*

$$0 \leq f(t, x) + \alpha x - (f(t, y) + \alpha y) \leq \beta(x - y).$$

Thus, problem (2) has a unique solution.

Proof. We can rewrite problem (2) as follows:

$$\begin{cases} u'(t) + \alpha u(t) = f(t, u(t)) + \alpha u(t) \\ u(0) = u(a). \end{cases}$$

This problem is equivalent to the following integral equation:

$$u(t) = \int_0^a Q(t, s)(f(s, u(s)) + \alpha u(s))ds,$$

where

$$Q(t, s) = \begin{cases} \frac{e^{\alpha(a+s-t)}}{e^{\alpha a} - 1}, \ 0 \leq s \leq t \leq a \\ \frac{e^{\alpha(s-t)}}{e^{\alpha a} - 1}, \ 0 \leq t \leq s \leq a. \end{cases}$$

and $u \in C^1([0, 1], \mathbb{R})$.

Let $X = C([0, a], \mathbb{R})$. Then, X is a complete extended b-metric space considering $d_\theta(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2$, with $\theta(x, y) = 2|x(t)| + |y(t)| + 1$, where $\theta : X \times X \to [1, \infty)$.

Let $A_1 = A_2 = A_3 = X = C([0, a], \mathbb{R})$ three closed subsets of the space (X, d_θ).

Let us define the operator $T : \bigcup_{i=1}^{3} A_i \to \bigcup\limits_{i=1}^{3} A_i$ as follows:

$$Tu(t) = \int\limits_0^a Q(t,s)(f(s,u(s)) + \alpha u(s))ds$$

for $u \in C([0,a],\mathbb{R})$ and $t \in [0,a)$.

For $x, y \in C([0,a],\mathbb{R})$ and $t \in [0,a]$ we have

$$|f(t,x(t)) + \alpha x(t) - f(t,y(t)) - \alpha y(t)| \leq \sqrt{\beta}|x(t) - y(t)|. \tag{3}$$

Clearly, $T(A_1) \subseteq A_2$, $T(A_2) \subseteq A_3$, $T(A_3) \subseteq A_1$. Thus, T is a cyclic operator on $\bigcup\limits_{i=1}^{3} A_i$. For any $x, y \in \bigcup\limits_{i=1}^{3} A_i$ we have the following estimation:

$$\begin{aligned}
|Tx(t) - Ty(t)|^2 &\leq \int\limits_0^a Q(t,s)|f(s,x(s)) + \alpha x(s) - f(t,y(s)) - \alpha y(s)|^2 ds \\
&\leq \int\limits_0^a Q(t,s)\beta|x(s) - y(s)|^2 ds \\
&\leq \beta d_\theta(x,y) \sup_{0 \leq t \leq a} \int\limits_0^a Q(t,s)ds \\
&= \frac{\beta}{\alpha} d_\theta(x,y).
\end{aligned}$$

Thus, for $x, y \in C([0,a],\mathbb{R})$, we have $d_\theta(Tx,Ty) \leq \frac{\beta}{\alpha} d_\theta(x,y)$.

Since $\lim\limits_{n,m\to\infty} \theta(x_n(t), x_m(t)) = 1 < \frac{\alpha}{\beta}$, we fulfilled all of the conditions of Theorem 3. Hence, using Theorem 3, we obtained the existence and uniqueness of fixed points of T. $\square$

Remark 1. *Theorem 5 still holds (with the reverse inequality) if we replace the existence of a lower solution of the boundary value problem by the existence of an upper solution of the same problem.*

In the last part of this section, we present an application of our main theorem for nonlinear fractional differential equations. Some results concerning the fixed point technique for determining the solutions of fractional differential equations can also be found in [19,23].

In [24,25], the definition of the Caputo derivative of functional $g : [0,\infty) \to \mathbb{R}$ of order $\beta > 0$ is given, where g is a continuous function as follows:

$$^{C}D^{\beta}(g(t)) := \frac{1}{\Gamma(n-\beta)} \int\limits_0^t (t-s)^{n-\beta-1} g^{(n)}(s)ds \quad (n-1 < \beta < n,\ n = [\beta]+1), \tag{4}$$

where $[\beta]$ represents the integer part of the positive real number β, and Γ is a gamma function. Let us recall the Caputo type nonlinear fractional differential equation

$$^{C}D^{\beta}(x(t)) = f(t,x(t)) \tag{5}$$

with the integral boundary conditions:

$$x(0) = 0, \quad x(1) = \int\limits_0^\eta x(s)ds,$$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $x \in C[0,1]$ and $T : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous given function (see [26]). Since f is continuous, Equation (5) is inverted as the following integral equation:

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} f(p,x(p))dp \right) ds. \end{aligned} \tag{6}$$

In addition, we provide an existence theorem.

Theorem 6. *Taking into account the nonlinear fractional differential Equation (5), for every $x, y \in C[0,1]$ and $M : [0,1] \times \mathbb{R} \to \mathbb{R}$ a given continuous mapping, we obtain*

$$|M(s,x(s)) - M(s,y(s))| \leq \frac{\Gamma(\beta+1)}{\sqrt{50}} |x(s) - y(s)|, \qquad for\ all\ s \in [0,1].$$

Thus, the Caputo type nonlinear fractional differential Equation (5) has a unique solution. Moreover, for each $x_0 \in C[0,1]$, the sequence of the successive approximation $\{x_n\}$ defined by

$$\begin{aligned} x_n(t) = &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} M(s,x_{n-1}(s))ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} M(s,x_{n-1}(s))ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} M(p,x_{n-1}(p))dp \right) ds. \end{aligned}$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear fractional differential equation of Caputo type (5).

Proof. Let $X = C[0,1]$. The operator is defined as follows: $T : \bigcup_{i=1}^{3} A_i \to \bigcup_{i=1}^{3} A_i$ as follows

$$\begin{aligned} Tx(t) = &\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} M(s,x(s))ds \\ &- \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^1 (1-s)^{\beta-1} M(s,x(s))ds \\ &+ \frac{2t}{(2-\eta^2)(\Gamma(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} M(p,x(p))dp \right) ds. \end{aligned}$$

Thus, (X, d_θ) is a complete extended b-metric space with respect to $d_\theta(x,y) = \|x - y\|_\infty = \sup_{t\in[a,b]} |x(t) - y(t)|^2$, where $\theta : X \times X \to [1,\infty)$ is defined by $\theta(x,y) = |x(t)| + |y(t)| + 1$.

Let $A_1 = A_2 = A_3 = X = C[0,1]$ three nonempty subsets of X. Obviously, A_1, A_2, A_3 are closed subsets of (X, d_θ). Clearly, $T(A_1) \subset A_2, T(A_2) \subset A_3$ and $T(A_3) \subset A_1$. Thus, T is a cyclic operator on $\bigcup\limits_{i=1}^{3} A_i$.

Assuming $x, y \in \bigcup\limits_{i=1}^{3} A_i$ and $t \in [0,1]$, we obtain

$$
\begin{aligned}
& d_\theta(Tx, Ty) = |Tx(t) - Ty(t)|^2 \\
& = |\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} M(s, x(s))ds \\
& - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} M(s, x(s))ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} M(p, x(p))dp \right) ds \\
& - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} M(s, y(s))ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} M(s, y(s))ds \\
& - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} M(p, y(p))dp \right) ds|^2.
\end{aligned}
$$

Using the properties of the module, we obtain

$$
\begin{aligned}
& d_\theta(Tx, Ty) \leq |\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [M(s, x(s)) - M(s, y(s))]ds \\
& - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} [M(s, x(s)) - M(s, y(s))]ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-p)^{\beta-1} [M(p, x(p)) - M(p, y(p)]dp \right) ds|^2 \\
& \leq \frac{1}{\Gamma^2(\beta)} \int_0^t |t-s|^{2(\beta-1)} |M(s, x(s)) - M(s, y(s))|^2 ds \\
& + \frac{4t^2}{(2-\eta^2)^2\Gamma^2(\beta)} \int_0^1 (1-s)^{2(\beta-1)} |M(s, x(s)) - M(s, y(s))|^2 ds \\
& + \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{2(\beta-1)} |M(p, x(p)) - M(p, y(p)|^2 dp \right) ds.
\end{aligned}
$$

Taking the supremum over $s \in [0,1]$, we obtain

$$
\begin{aligned}
d_\theta(Tx,Ty) \le & \frac{1}{\Gamma^2(\beta)} \int_0^t |t-s|^{2(\beta-1)} \frac{\Gamma^2(\beta+1)}{50} \sup_{s\in[0,1]} |x(s)-y(s)|^2 ds \\
& + \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))} \int_0^1 (1-s)^{2(\beta-1)} \frac{\Gamma^2(\beta+1)}{50} \sup_{s\in[0,1]} |x(s)-y(s)|^2 ds \\
& + \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))} \int_0^\eta \left(\int_0^s (s-p)^{2(\beta-1)} \frac{\Gamma^2(\beta+1)}{50} \sup_{s\in[0,1]} |x(s)-y(s)|^2 dp \right) ds \\
& \le \frac{\Gamma^2(\beta+1)}{50} d_\theta(x,y) \times \sup_{s\in[0,1]} [\frac{1}{\Gamma^2(\beta)} \int_0^t |t-s|^{2(\beta-1)} ds \\
& + \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))} \int_0^1 (1-s)^{2(\beta-1)} ds + \frac{4t^2}{(2-\eta^2)^2(\Gamma^2(\beta))} \int_0^\eta \int_0^s (s-p)^{2(\beta-1)} dp\, ds] \\
& \le \frac{1}{2} d_\theta(x,y).
\end{aligned}
$$

Since $\lim_{n,m\to\infty} \theta(x_n, x_m) = 1 < 2$, we fulfilled all of the conditions of Theorem 1. Thus, a unique solution of the Caputo-type nonlinear fractional differential Equation exists (5). □

4. Conclusions

Fixed point theory is a powerful tool for proving the existence and uniqueness of different types of equations. Recently, there has been an increase in papers that use the concept of the cyclic operator. This notion has many projections in physics and astrophysics. We also know that one of the most researched areas of mathematics is partial differential calculus. It is used in modeling many real world phenomena. This paper unifies both fields. First, a fixed point result translating the Banach contraction principle for the case of cyclic operators in extended b-metric spaces is given. Then, it is proved that one can obtain fixed point results in extended b-metric spaces for the case of this type of operator. The Application Section 3 is devoted to the study of the existence and uniqueness of a boundary value problem given by Nieto and Lopez in [21]. Then, an application to fractional differential equations is presented.

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