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An Extension of Caputo Fractional Derivative Operator by Use of Wiman's Function

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Abstract: The main aim of this work is to study an extension of the Caputo fractional derivative operator by use of the two-parameter Mittag–Leffler function given by Wiman. We have studied some generating relations, Mellin transforms and other relationships with extended hypergeometric functions in order to derive this extended operator. Due to symmetry in the family of special functions, it is easy to study their various properties with the extended fractional derivative operators.

Keywords: classical Caputo fractional derivative operator; beta function; gamma function; Gauss hypergeometric function; confluent hypergeometric function; Mittag–Leffler function

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1. Introduction

In the field of mathematics, the theory of fractional calculus (FC) has been successfully studied to focus on fractal problems, which are real-life problems in engineering mathematics. FC has become an interesting topic of research since it offers many application opportunities to various areas of science and engineering, such as fluid flow, electrical networks and probability theory. The Caputo derivative operator plays a vital role in fractional calculus as well, because of its applications to different branches of science. Fractional operators and special functions have been receiving renewed attention in recent years, and a remarkable variety of refinements and generalizations are currently available [1–3].

The classical Caputo fractional derivative operator [4,5] is defined as:

$$D_z^u[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} \frac{d^m}{dt^m} f(t)dt, \tag{1}$$

where, $m-1 < \Re(u) < m, m \in N$.

The theoretical extension and modification of this classical operator has been taking place since 2000. In 2016, Kiyamaz et al. [6] have extended it as follows:

$$D_z^{u,r}[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} e^{\left(\frac{-rz^2}{t(z-t)}\right)} \frac{d^m}{dt^m} f(t)dt, \tag{2}$$

where $\Re(r) > 0$ and $m-1 < \Re(u) < m, m \in N$.

Remark 1. *If we set $r = 0$ in (2), we get the classical Caputo fractional derivative operator (1)*

$$D_z^{u,0}[f(z)] = D_z^u[f(z)]. \tag{3}$$

In the above extension, Kiyamaz et al. [6] use the exponential function as a regularizer to extend the classical Caputo fractional derivative operator, and they also discuss various generating relations, Mellin transforms and additional relationships with other special functions. The exponential function is introduced as the kernel in the integral part of the classical Caputo fractional derivative operator.

The 2-parameter Mittag–Leffler function (known as the Wiman's function) [7,8] is defined as follows:

$$E_{r_1,r_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(nr_1 + r_2)}, \qquad \Re(r_1) \geq 0, \Re(r_2) \geq 0, z \in C. \tag{4}$$

The classical Euler beta function [9] is defined as follows:

$$B(x_1, x_2) = \int_0^1 t^{x_1-1}(1-t)^{x_2-1}dt, \qquad \Re(x_1), \Re(x_2) > 0. \tag{5}$$

Very recently, Goyal et al. [10] have extended the classical beta function using the two-parameter Mittag–Leffler function $E_{r_1,r_2}[z]$ given by Wiman [7], studying various properties of this extended beta function. They have introduced the Wiman's function as the kernel in the integral part of the classical Euler beta function:

$$B^{(u)}_{(u_1,u_2)}(y_1, y_2) = \int_0^1 t^{y_1-1}(1-t)^{y_2-1}E_{u_1,u_2}\left(\frac{-u}{t(1-t)}\right)dt. \tag{6}$$

Here, $\min\{\Re(y_1), \Re(y_2)\} > 0$, $\Re(u_1) > 0, \Re(u_2) > 0$, $u \geq 0$; and $E_{u_1,u_2}(z)$ is the *two*-parameter Mittag–Leffler function.

The series and integral representations of the Gauss hypergeometric function ${}_2F_1$ [11] are defined as:

$$F(r_0, r_1, r_2; z) = \sum_{n=0}^{\infty} \frac{B(r_1 + n, r_2 - r_1)}{B(r_1, r_2 - r_1)}(r_0)_n \frac{z^n}{n!}, \tag{7}$$

where $\Re(r_2) > \Re(r_1) > 0$ and $| z |< 1$, and

$$F(r_0, r_1, r_2; z) = \frac{1}{B(r_1, r_2 - r_1)} \int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1}(1-zt)^{-r_0}dt. \tag{8}$$

respectively.

Jain et al. [12] have extended the Gauss hypergeometric function using the extended beta function (6) given by Goyal et al. [10], specifically studying many basic properties such as integral representations and Mellin transforms of this extended hypergeometric function.

The extended Gauss hypergeometric function [12] is defined as:

$$F^{(s)}_{(s_1,s_2)}(q_0, q_1, q_2; z) = \sum_{n=0}^{\infty} \frac{B^{(s)}_{(s_1,s_2)}(q_1 + n, q_2 - p_1)}{B(q_1, q_2 - q_1)}(q_0)_n \frac{z^n}{n!}. \tag{9}$$

Here, $\Re(q_2) > \Re(q_1) > 0$, $\min\{\Re(s_1), \Re(s_2)\} > 0$, $s \geq 0$, $|z| < 1$, and $B^{(s)}_{(s_1,s_2)}(w_1, w_2)$ is the extended beta function.

The integral representation of the extended Gauss hypergeometric function [12] is defined as:

$$F^{(s)}_{(s_1,s_2)}(q_0, q_1, q_2; z) = \frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1}(1-t)^{q_2-q_1-1}(1-zt)^{-q_0}E_{s_1,s_2}\left(\frac{-s}{t(1-t)}\right)dt. \tag{10}$$

Here, $\Re(q_2) > \Re(q_1) > 0$, $\min\{\Re(s_1), \Re(s_2)\} > 0$, $s \geq 0$, and $|z| < 1$.

2. Extension of the Hypergeometric Function

We know a variety of problems in classical mechanics and mathematical physics lead to Picard-Fuchs equations and these equations are solvable in terms of generalised hypergeometric functions, and the monodromy of generalized hypergeometric functions plays important role in describing properties of the solutions. Many combinatorial identities, especially ones involving binomial and related coefficients, are special cases of hypergeometric identities. Also, generalized hypergeometric functions appear in the evaluation of the Watson-integrals which characterized the simplest possible lattice walks and they are potentially useful for the solution of more complicated restricted lattice walk problems. Motivated by the above work in this section, we define a further new extension of the Gauss hypergeometric function by use of the extended beta function (6) given by Goyal et al. [10].

Definition 1.

$$ {}_2F^{(s)}_{1,(s_1,s_2)}(q_0,q_1,q_2;z) = \sum_{n=0}^{\infty} \frac{(q_0)_n (q_1)_n}{(q_1-m)_n} \frac{B^{(s)}_{(s_1,s_2)}(q_1-m+n, q_2-q_1+m)}{B(q_1-m, q_2-q_1+m)} \frac{z^n}{n!}. \tag{11} $$

Here, $\Re(q_2) > \Re(q_1) > m$, $s \geq 0$, $|z| < 1$; *and* $B^{(s)}_{(s_1,s_2)}(w_1,w_2)$ *is the extended beta function.*

3. Extension of the Caputo Fractional Derivative Operator

In literature point of view, many fractional derivative operators proved their importance. Many researchers still are working on introducing new fractional derivative operators and applying these new operators in certain real world problems like fractal space time, fractional derivatives for heat conduction in a fractal medium arising in silkworm cocoon hierarchy, asymptotic perturbation for a linear oscillator of free damped vibrations in fractal medium describe by local fractional derivatives and modelling growths of populations. By the inspiring above work in this section, we define a new extension of the classical Caputo fractional derivative operator using the two-parameter Mittag–Leffler function. We have introduced the Wiman's function as the kernel in the integral part of the classical Caputo fractional derivative operator; as a result, the integral part reduces to the extended beta function defined in [10] after some calculations. We have also established some interesting results for this extended operator.

Definition 2.

$$ D^{u,(r)}_{z,(r_1,r_2)}[f(z)] = \frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right) \frac{d^m}{dt^m} f(t)dt. \tag{12} $$

Here, $\min\{\Re(r_1), \Re(r_2)\} > 0$, $\Re(r) > 0$,$m-1 < \Re(u) < m$, $m \in N$, *and* $E_{r_1,r_2}(z)$ *is the* 2-*parameter Mittag–Leffler function.*

Remark 2.

(i) *If we set* $r_1 = r_2 = 1$ *in* (12), *we get the extended Caputo fractional derivative operator* (2)

$$ D^{u,(r)}_{z,(1,1)}[f(z)] = D^{u,r}_z[f(z)]. \tag{13} $$

(ii) *If we take* $r_1 = r_2 = 1$ *and* $r = 0$ *in* (12), *we get the classical Caputo fractional derivative operator* (1)

$$ D^{u,(0)}_{z,(1,1)}[f(z)] = D^u_z[f(z)]. \tag{14} $$

Theorem 1. *Consider* $m-1 < \Re(u) < m$, $\Re(u) < \Re(k)$. *Then*

$$ D^{u,(r)}_{z,(r_1,r_2)}[z^k] = \frac{\Gamma(k+1)}{\Gamma(k-u+1)} \frac{B^{(r)}_{(r_1,r_2)}(k-m+1, m-u)}{B(k-m+1, m-u)} z^{k-u}. \tag{15} $$

Proof of Theorem 1. From the definition of the extended Caputo fractional derivative operator (12), we have:

$$
\begin{aligned}
D^{u,(r)}_{z,(r_1,r_2)}[z^k] &= \frac{1}{\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)\frac{d^m}{dt^m}t^k dt \\
&= \frac{1}{\Gamma(m-u)}[k(k-1)(k-2)\ldots\ldots(k-m+1)]\int_0^z (z-t)^{m-u-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-m}dt \\
&= \frac{\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)t^{k-m}dt.
\end{aligned}
\tag{16}
$$

On putting $t = xz$ in (16), we get:

$$
D^{u,(r)}_{z,(r_1,r_2)}[z^k] = \frac{\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(m-u)}z^{k-u}\int_0^1 x^{k-m}(1-x)^{m-u-1}E_{r_1,r_2}\left(\frac{-r}{x(1-x)}\right)dx.
\tag{17}
$$

Now, entering the definition of the extended beta function (6) in the above Equation (17), we have:

$$
D^{u,(r)}_{z,(r_1,r_2)}[z^k] = \frac{\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(m-u)}z^{k-u}B^{(r)}_{(r_1,r_2)}(k-m+1, m-u).
\tag{18}
$$

Further exploiting the relationship between the Gamma and beta functions, $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, in Equation (18), we get our desired result of Theorem 1.

$$
D^{u,(r)}_{z,(r_1,r_2)}[z^k] = \frac{\Gamma(k+1)}{\Gamma(k-u+1)}\frac{B^{(r)}_{(r_1,r_2)}(k-m+1, m-u)}{B(k-m+1, m-u)}z^{k-u}.
\tag{19}
$$

□

Remark 3. *If* $k = 0, 1, 2\ldots\ldots, m-1$, *then* $D^{u,(r)}_{z,(r_1,r_2)}[z^k] = 0$.

Theorem 2. *Assume that f(z) is a holomorphic function in the disc* $\mid z \mid < \delta$, *with the Taylor series expansion* $f(z) = \sum_{n=0}^{\infty} b_n z^n$. *Then*

$$
D^{u,(r)}_{z,(r_1,r_2)}[f(z)] = \sum_{n=0}^{\infty} b_n D^{u,(r)}_{z,(r_1,r_2)}[z^n],
\tag{20}
$$

where $m-1 < \Re(u) < m$.

Proof of Theorem 2. From the definition of the extended Caputo fractional derivative operator (12), we have:

$$
D^{u,(r)}_{z,(r_1,r_2)}[f(t)] = \frac{1}{\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)\frac{d^m}{dt^m}f(t)dt.
\tag{21}
$$

Applying the Taylor series expansion of f, we derive:

$$
D^{u,(r)}_{z,(r_1,r_2)}[f(t)] = \frac{1}{\Gamma(m-u)}\int_0^z (z-t)^{m-u-1}E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)}\right)\frac{d^m}{dt^m}\left(\sum_{n=0}^{\infty} b_n t^n\right)dt.
\tag{22}
$$

Since the power series converges uniformly and the integral converges absolutely, when interchanging the order of integration and summation, we are left with:

$$D^{u,(r)}_{z,(r_1,r_2)}[f(t)] = \sum_{n=0}^{\infty} b_n \left[\frac{1}{\Gamma(m-u)} \int_0^z (z-t)^{m-u-1} E_{r_1,r_2}\left(\frac{-rz^2}{t(z-t)} \right) \frac{d^m}{dt^m} t^n dt \right]. \tag{23}$$

Then, from the definition of the extended Caputo fractional derivative operator, the desired result of Theorem 2 is recovered.

$$D^{u,(r)}_{z,(r_1,r_2)}[f(z)] = \sum_{n=0}^{\infty} b_n D^{u,(r)}_{z,(r_1,r_2)}[z^n]. \tag{24}$$

□

Theorem 3. *Assume that f(z) is a holomorphic function in the disc* $| z |< \delta$, *with the Taylor series expansion* $f(z) = \sum_{n=0}^{\infty} b_n z^n$. *Then*

$$D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta-1}f(z)] = \frac{\Gamma(\beta)}{\Gamma(\beta-u)} z^{\beta-u-1} \sum_{n=0}^{\infty} b_n \frac{(\beta)_n}{(\beta-m)_n} \frac{B^{(r)}_{(r_1,r_2)}(\beta-m+n, m-u)}{B(\beta-m, m-u)} z^n, \tag{25}$$

where $m-1 < \Re(u) < m < \Re(\beta)$.

Proof of Theorem 3. By application of Theorem 2, we have:

$$D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta-1}f(z)] = \sum_{n=0}^{\infty} b_n D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta+n-1}]. \tag{26}$$

Further, using Theorem 1, we get:

$$D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta-1}f(z)] = \sum_{n=0}^{\infty} b_n \frac{\Gamma(\beta+n)}{\Gamma(\beta+n-u)} \frac{B^{(r)}_{(r_1,r_2)}(\beta-m+n, m-u)}{B(\beta-m+n, m-u)} z^{\beta-u-1+n}. \tag{27}$$

Then, using the relation $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we get:

$$D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta-1}f(z)] = \frac{\Gamma(\beta)}{\Gamma(\beta-u)} z^{\beta-u-1} \sum_{n=0}^{\infty} b_n \frac{(\beta)_n}{(\beta-u)_n} \frac{B^{(r)}_{(r_1,r_2)}(\beta-m+n, m-u)}{B(\beta-m+n, m-u)} z^n. \tag{28}$$

Applying the identities $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we get the desired result of Theorem 3.

$$D^{u,(r)}_{z,(r_1,r_2)}[z^{\beta-1}f(z)] = \frac{\Gamma(\beta)}{\Gamma(\beta-u)} z^{\beta-u-1} \sum_{n=0}^{\infty} b_n \frac{(\beta)_n}{(\beta-m)_n} \frac{B^{(r)}_{(r_1,r_2)}(\beta-m+n, m-u)}{B(\beta-m, m-u)} z^n. \tag{29}$$

□

Theorem 4. *Consider* $m-1 < \Re(k-u) < m < \Re(k)$ *and* $| z |< 1$. *Then*

$$D^{k-u,(r)}_{z,(r_1,r_2)}[z^{k-1}(1-z)^{-l}] = \frac{\Gamma(k)}{\Gamma(u)} z^{u-1} {}_2F^{(r)}_{1,(r_1,r_2)}(l, k, u; z). \tag{30}$$

Proof of Theorem 4. By application of the identity $(1-z)^{-l} = \sum_{n=0}^{\infty}(l)_n \frac{z^n}{n!}$ and Theorem 1, we get:

$$\begin{aligned} D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-z)^{-l}] &= D_{z,(r_1,r_2)}^{k-u,(r)}\left[z^{k-1}\left(\sum_{n=0}^{\infty}(l)_n\frac{z^n}{n!}\right)\right] \\ &= \sum_{n=0}^{\infty}\frac{(l)_n}{n!}D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1+n}] \\ &= \sum_{n=0}^{\infty}\frac{(l)_n}{n!}\frac{\Gamma(k+n)}{\Gamma(u+n)}\frac{B_{(r_1,r_2)}^{(r)}(k-m+n,m-k+u)}{B(k-m+n,m-k+u)}z^{u-1+n}. \end{aligned} \tag{31}$$

Then, using identities $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we have:

$$D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-z)^{-l}] = \frac{\Gamma(k)}{\Gamma(u)}z^{u-1}\sum_{n=0}^{\infty}\frac{(k)_n(l)_n}{(k-m)_n}\frac{B_{(r_1,r_2)}^{(r)}(k-m+n,m-k+u)}{B(k-m,m-k+u)}\frac{z^n}{n!}. \tag{32}$$

From the definition of the extended Gauss hypergeometric function (11), we get our desired result of Theorem (4).

$$D_{z,(r_1,r_2)}^{k-u,(r)}[z^{k-1}(1-z)^{-l}] = \frac{\Gamma(k)}{\Gamma(u)}z^{u-1}{}_2F_{1,(r_1,r_2)}^{(r)}(l,k,u;z). \tag{33}$$

□

Theorem 5. *Consider $m-1 < \Re(k-u) < m < \Re(k)$. Then, the following generating relation for the extended hypergeometric function holds true:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_2F_{1,(r_1,r_2)}^{(r)}(\lambda+n,k,u;z)t^n = (1-t)^{-\lambda}{}_2F_{1,(r_1,r_2)}^{(r)}\left(\lambda,k,u;\frac{z}{(1-t)}\right), \tag{34}$$

provided that $\mid z \mid < \min(1, \mid 1-t \mid)$.

Proof of Theorem 5. Consider the series identity:

$$[(1-z)-t]^{-\lambda} = (1-t)^{-\lambda}\left(1-\frac{z}{(1-t)}\right)^{-\lambda}.$$

After re-arranging its terms, we recover:

$$(1-z)^{-\lambda}\left(1-\frac{t}{1-z}\right)^{-\lambda} = (1-t)^{-\lambda}[\left(1-\frac{z}{(1-t)}\right)^{-\lambda}.$$

Performing the binomial expansion of $(1-\frac{t}{1-z})^{-\lambda}$, we get:

$$(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n = (1-t)^{-\lambda}\left(1-\frac{z}{(1-t)}\right)^{-\lambda}. \tag{35}$$

Now, by multiplication of both sides by z^{k-1}, we get:

$$z^{k-1}(1-z)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z}\right)^n = z^{k-1}(1-t)^{-\lambda}\left(1-\frac{z}{(1-t)}\right)^{-\lambda}. \tag{36}$$

Then, applying the Caputo fractional derivative operator $D_{z,(r_1,r_2)}^{k-u,(r)}$ to both sides of the above equation, the following expression is found:

$$D_{z,(r_1,r_2)}^{k-u,(r)}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z)^{-\lambda}\left(\frac{t}{1-z}\right)^n z^{k-1}\right]=(1-t)^{-\lambda}D_{z,(r_1,r_2)}^{k-u,(r)}\left[z^{k-1}\left(1-\frac{z}{(1-t)}\right)^{-\lambda}\right]. \tag{37}$$

If the order of summation and operator $D_{z,(r_1,r_2)}^{k-u,(r)}$ is interchanged, we are left with:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_{z,(r_1,r_2)}^{k-u,(r)}\left[z^{k-1}(1-z)^{-\lambda-n}\right]t^n=(1-t)^{-\lambda}D_{z,(r_1,r_2)}^{k-u,(r)}\left[z^{k-1}\left(1-\frac{z}{(1-t)}\right)^{-\lambda}\right]. \tag{38}$$

Finally, applying Theorem 4, we get our desired result of Theorem 5.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_2F_{1,(r_1,r_2)}^{(r)}(\lambda+n,k,u;z)t^n=(1-t)^{-\lambda}{}_2F_{1,(r_1,r_2)}^{(r)}\left(\lambda,k,u;\frac{z}{(1-t)}\right). \tag{39}$$

□

Theorem 6. *Consider $\Re(\lambda)>m-1$ and $s>0$. Then, the Mellin transform for the extended Caputo fractional derivative operator defined as (12) is given by the following expression:*

$$\boldsymbol{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda];s\right]=\frac{\Gamma(\lambda+1)\Gamma_0^{(r_1,r_2)}(s)}{\Gamma(\lambda-m+1)\Gamma(m-u)}B(m-u+s,\lambda-m+s+1)z^{\lambda-u}. \tag{40}$$

Proof of Theorem 6. From the definition of Mellin transform, we have:

$$\mathbf{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda];s\right]=\int_0^\infty r^{s-1}D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda]dr. \tag{41}$$

Upon applying Theorem 1, we recover:

$$\mathbf{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda];s\right]=\int_0^\infty r^{s-1}\left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-u+1)}\frac{B_{(r_1,r_2)}^{(r)}(m-u,\lambda-m+1)}{B(m-u,\lambda-m+1)}z^{\lambda-u}\right]dr; \tag{42}$$

and, after some calculation:

$$\begin{gathered}\mathbf{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda];s\right]=\\ \frac{\Gamma(\lambda+1)z^{\lambda-u}}{\Gamma(\lambda-u+1)B(m-u,\lambda-m+1)}\int_0^\infty r^{s-1}B_{(r_1,r_2)}^{(r)}(m-u,\lambda-m+1)dr.\end{gathered} \tag{43}$$

Now, using the result from [10], we get our desired result of Theorem 6. Indeed, since

$$\int_0^\infty r^{(s-1)}B_{(r_1,r_2)}^{(r)}(x_1,x_2)dr=B(x_1+s,x_2+s)\Gamma_0^{(r_1,r_2)}(s), \tag{44}$$

where, $r\geq 0,\Re(x_1+s)>0,\Re(x_2+s)>0,\Re(r_1)>0,\Re(r_2)>0$ and $\Re(s)>0$, then:

$$\mathbf{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[z^\lambda];s\right]=\frac{\Gamma(\lambda+1)\Gamma_0^{(r_1,r_2)}(s)}{\Gamma(\lambda-m+1)\Gamma(m-u)}B(m-u+s,\lambda-m+s+1)z^{\lambda-u}. \tag{45}$$

□

Theorem 7. *Assume that $s>0$ and $\mid z\mid<1$. Then, another Mellin transform for the extended Caputo fractional derivative operator is defined by the following expression:*

$$\boldsymbol{M}\left[D_{z,(r_1,r_2)}^{u,(r)}[(1-z)^{-\alpha}];s\right]=\frac{\Gamma_0^{(r_1,r_2)}(s)z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B(m-u+s,n+s+1)(\alpha)_{n+m}\frac{z^n}{n!}. \tag{46}$$

Proof of Theorem 7. Using the identity $(1-z)^{-\alpha)} = \sum_{n=0}^{\infty}(\alpha)_n \frac{z^n}{n!}$ and taking $\lambda = n$ in Theorem 6, we have:

$$\begin{aligned}\mathbf{M}\big[D^{u,(r)}_{z,(r_1,r_2)}[(1-z)^{-\alpha}];s\big] &= \mathbf{M}\left[D^{u,(r)}_{z,(r_1,r_2)}\left(\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\right);s\right] \\ &= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathbf{M}\big[D^{u,(r)}_{z,(r_1,r_2)}[z^n];s\big] \qquad (47) \\ = \sum_{n=m}^{\infty}\frac{(\alpha)_n}{n!}&\left[\frac{\Gamma(n+1)\Gamma_0^{(r_1,r_2)}(s)}{\Gamma(n-m+1)\Gamma(m-u)}B(m-u+s,n-m+s+1)z^{n-u}\right].\end{aligned}$$

After some calculation, we recover:

$$\mathbf{M}\big[D^{u,(r)}_{z,(r_1,r_2)}[(1-z)^{-\alpha}];s\big] = \frac{\Gamma_0^{(r_1,r_2)}(s)z^{-u}}{\Gamma(m-u)}\sum_{n=m}^{\infty}\frac{B(m-u+s,n-m+s+1)}{\Gamma(n-m+1)}(\alpha)_n z^n. \qquad (48)$$

By substituting $n = n+m$, we get our desired result of Theorem 7:

$$\mathbf{M}\big[D^{u,(r)}_{z,(r_1,r_2)}[(1-z)^{-\alpha}];s\big] = \frac{\Gamma_0^{(r_1,r_2)}(s)z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B(m-u+s,n+s+1)(\alpha)_{n+m}\frac{z^n}{n!}. \qquad (49)$$

□

Theorem 8. *The following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}[e^z] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}, \qquad (50)$$

for all $z \in C$.

Proof of Theorem 8. By application of the power series of $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ and Theorems 1 and 2, we get:

$$D^{u,(r)}_{z,(r_1,r_2)}[e^z] = \sum_{n=0}^{\infty}\frac{1}{n!}D^{u,(r)}_{z,(r_1,r_2)}[z^n] = \sum_{n=m}^{\infty}\frac{1}{n!}\frac{\Gamma(n+1)}{\Gamma(n-u+1)}\frac{B^{(r)}_{(r_1,r_2)}(n-m+1,m-u)}{B(n-m+1,m-u)}z^{n-u}. \qquad (51)$$

By substituting $n = n+m$, we have:

$$D^{u,(r)}_{z,(r_1,r_2)}[e^z] = \sum_{n=0}^{\infty}\frac{1}{(n+m)!}\frac{\Gamma(n+m+1)}{\Gamma(n+m-u+1)}\frac{B^{(r)}_{(r_1,r_2)}(n+1,m-u)}{B(n+1,m-u)}z^{n+m-u}. \qquad (52)$$

Further applying some known identities, we get the desired result of Theorem (8).

$$D^{u,(r)}_{z,(r_1,r_2)}[e^z] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}. \qquad (53)$$

□

Theorem 9. *For the Prabhakar-type function, the following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[E^{\gamma}_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n+m}}{\Gamma(n\alpha+m\alpha+\beta)}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}, \qquad (54)$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the Prabhakar-type function given in [13].

Proof of Theorem 9. Using the *three*-parameter Mittag–Leffler function (Prabhakar-type function) defined in [13]:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\beta + n\alpha)} \frac{z^n}{n!}, \tag{55}$$

and reasoning along the same lines as the proof of Theorem 8, we get our desired result of Theorem 9:

$$D^{u,(r)}_{z,(r_1,r_2)}\left[E^{\gamma}_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+m}}{\Gamma(n\alpha + m\alpha + \beta)} B^{(r)}_{(r_1,r_2)}(m-u, n+1) \frac{z^n}{n!}. \tag{56}$$

□

Corollary 1. *The following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[E_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(n\alpha + m\alpha + \beta)} B^{(r)}_{(r_1,r_2)}(m-u, n+1) \frac{z^n}{n!}, \tag{57}$$

where $E_{\alpha,\beta}(z)$ is the Wiman's function given in [7].

Proof. Taking $\gamma = 1$ in Theorem 9, we get our desired result. □

Corollary 2. *The following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[E_{\alpha}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)} \sum_{n=0}^{\infty} \frac{(n+m)!}{\Gamma(n\alpha + m\alpha + 1)} B^{(r)}_{(r_1,r_2)}(m-u, n+1) \frac{z^n}{n!}, \tag{58}$$

where $E_{\alpha}(z)$ is the Mittag–Leffler function given in [8,14].

Proof. By substituting $\gamma = 1$ and $\beta = 1$ in Theorem 9, we get our desired result. □

Remark 4. *If we take $\gamma = 1$, $\alpha = 1$ and $\beta = 1$ in Theorem 9, we retrieve Theorem 8.*

Theorem 10. *For the generalized hypergeometric function, the following result holds true:*

$$\begin{aligned} &D^{u,(r)}_{z,(r_1,r_2)}\left[{}_pF_q(x_1, x_2 \ldots x_p; y_1, y_2 \ldots y_q; z)\right] = \\ &\quad \frac{z^{m-u}}{\Gamma(m-u)} \frac{\prod_{j=1}^{q} \Gamma(y_j)}{\prod_{i=1}^{p} \Gamma(x_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(x_i + n + m)}{\prod_{j=1}^{q} \Gamma(y_j + n + m)} B^{(r)}_{(r_1,r_2)}(m-u, n+1) \frac{z^n}{n!}. \end{aligned} \tag{59}$$

Here, ${}_pF_q(x_1, x_2..x_p; y_1, y_2..y_q; z)$ is the generalized hypergeometric function given in [11].

Proof of Theorem 10. Using the definition of the generalized hypergeometric function and reasoning along the same lines as Theorem 8, we obtain our desired result of Theorem 10. □

Corollary 3. *For the Gauss hypergeometric function, the following result holds true:*

$$\begin{aligned} &D^{u,(r)}_{z,(r_1,r_2)}\left[{}_2F_1(x_1, x_2; y_1; z)\right] = \\ &\frac{z^{m-u}}{\Gamma(m-u)} \frac{\Gamma(y_1)}{\Gamma(x_1)\Gamma(x_2)} \sum_{n=0}^{\infty} \frac{\Gamma(x_1 + n + m)\Gamma(x_2 + n + m)}{\Gamma(y_1 + n + m)} B^{(r)}_{(r_1,r_2)}(m-u, n+1) \frac{z^n}{n!}, \end{aligned} \tag{60}$$

provided that $|\, z \,| < 1$.

Proof. Take $p = 2$ and $q = 1$ in Theorem 10; then, the desired result is obtained. □

Corollary 4. *For the confluent hypergeometric function, the following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[{}_1F_1(x_1,y_1;z)\right] = \frac{z^{m-u}}{\Gamma(m-u)}\frac{\Gamma(y_1)}{\Gamma(x_1)}\sum_{n=0}^{\infty}\frac{\Gamma(x_1+n+m)}{\Gamma(y_1+n+m)}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}, \tag{61}$$

for all $z \in C$.

Proof. Take $p = 1$ and $q = 1$ in Theorem 10; then, the desired result is obtained. □

Theorem 11. *For the Wright–Fox function, the following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[{}_pW_q\left[z\left|\begin{matrix}(x_1,a_1)\dots(x_p,a_p)\\(y_1,b_1)\dots(x_q,b_q)\end{matrix}\right.\right].\right] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(x_i+a_i(n+m))}{\prod_{j=1}^{q}\Gamma(y_j+a_j(n+m)}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}. \tag{62}$$

Here, ${}_pW_q\left[z\left|\begin{matrix}(x_1,a_1)\dots(x_p,a_p)\\(y_1,b_1)\dots(y_q,b_q)\end{matrix}\right.\right]$ *is the Wright–Fox function given in [15].*

Proof of Theorem 11. Using the definition of the Wright–Fox function and reasoning along the same lines as Theorem 8, we obtain our desired result of Theorem 11. □

Theorem 12. *For the Le Roy-type function, the following result holds true:*

$$D^{u,(r)}_{z,(r_1,r_2)}\left[F^{\gamma}_{\alpha,\beta}(z)\right] = \frac{z^{m-u}}{\Gamma(m-u)}\sum_{n=0}^{\infty}\frac{(n+m)!}{[\Gamma(n\alpha+m\alpha+\beta)]^{\gamma}}B^{(r)}_{(r_1,r_2)}(m-u,n+1)\frac{z^n}{n!}, \tag{63}$$

where $F^{\gamma}_{\alpha,\beta}(z)$ *is the Le Roy-type function given in [16].*

Proof of Theorem 12. Using the definition of the Le Roy-type function and reasoning along the same lines as Theorem 8, we obtain our desired result of Theorem 12. □

Remark 5. *If we take* $\gamma = 1$ *in the above Theorem 12, then we get Corollary 1.*

4. Conclusions

We conclude our investigation by remarking that the results presented in this paper are easily retrieved by extension of the classical Caputo fractional derivative operator and some other special functions. Such results are new and very useful for the additional extension of other special functions in the field of the fractional calculus. It is easy to reduce extended hypergeometric functions into trigonometric functions, exponential function, beta function, and Gamma function, it will increase the rate to find out the solution of differential equations symmetrically. At present, we are trying to find possible applications of these results with other research areas. In the future, we will work on the computer algebra of these extended functions by using mathematical software.

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