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Refinements of Ostrowski Type Integral Inequalities Involving Atangana–Baleanu Fractional Integral Operator

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Abstract: In this article, first, we deduce an equality involving the Atangana–Baleanu (AB)-fractional integral operator. Next, employing this equality, we present some novel generalization of Ostrowski type inequality using the Hölder inequality, the power-mean inequality, Young's inequality, and the Jensen integral inequality for the convexity of $|Y|$. We also deduced some new special cases from the main results. There exists a solid connection between fractional operators and convexity because of their fascinating properties in the mathematical sciences. Scientific inequalities of this nature and, particularly, the methods included have applications in different fields in which symmetry plays a notable role. It is assumed that the results presented in this article will show new directions in the field of fractional calculus.

Keywords: Ostrowski inequality; Hölder inequality; power mean inequality; Young's inequality; Atangana–Baleanu fractional integral operator; convex function

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

1. Introduction

Recently, fractional derivatives and fractional integrals have received significant interest among researchers. In numerous applications, fractional derivatives and fractional integrals provide more exact models of the frameworks than classical derivative and integrals do. Numerous utilizations of fractional calculus in bioengineering, electrochemical processes, modeling of viscoelastic damping, dielectric polarization, and various branches of sciences could be found in [1–4].

Over the past several years, fractional derivative and fractional integration has kept the attention of high level mathematicians, and it has become an extraordinarily significant idea for dealing with the components of complex systems from various areas of science. Fractional calculus began to be utilized as an integral tool by numerous scientists working in different directions of theory of inequalities, for example, [5–11].

In this short manuscript, we momentarily audit the gigantic impact that the AB fractional calculus has on establishing Ostrowski inequality. The fundamental objective of this article is to set up the Ostrowski-type inequalities for convex functions involving

the Atangana–Baleanu fractional operator. By a wide margin, the majority of the results introduced are refinements of the overall composition of the current results for new and classical convex functions.

This article is coordinated as follows: In Section 2, we review some fundamental and essential definitions and results. In Section 3, we demonstrate Atangana–Baleanu fractional integral inequalities of the Ostrowski type and related results for convex functions. In Section 4, we present our final comments.

2. Preliminaries

It is clearly a fact that the convex function is extremely important in the exploration of mathematical inequalities since it has many applications in pure and applied mathematics, mechanics, probability and statistics theory, economics, engineering and optimization theory. Lately, a few mathematicians have worked on the theories, generalizations, augmentations, variations and refinements of the convexity. It is a useful technique for cognizance and showing various issues in different branches of science and mathematics, for example, (see [12–16]).

There exist many famous inequalities, such as the Hermite–Hadamard inequality, the Ostrowski inequality, the Simpson inequality, the Bullen type inequality, the Opial type inequality, and many more, which can be generalized using the convexity property. Among them, the Ostrowski type inequality is one of the most extensively discussed results involving different kinds of convexities such as convex functions, s -convex functions, h -convex functions, (h, m) -convex functions, (s, m) -convex functions, and so forth. In 1938, Ostrowski inequality was established as the following useful and interesting integral inequality (see [17], p. 468).

Let $Y : J \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ be a differentiable mapping such that $Y \in \mathcal{L}[b_1, b_2]$, where $b_1, b_2 \in J$ with $b_2 > b_1$. If $|Y'(z)| \leq \mathbb{K}$, for all $z \in [b_1, b_2]$, then the following inequality holds:

$$\left| Y(z) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} Y(z) dz \right| \leq \mathbb{K}(b_2 - b_1) \left[\frac{1}{4} + \frac{\left(z - \frac{b_1 + b_2}{2} \right)^2}{(b_2 - b_1)^2} \right]. \quad (1)$$

Here, the constant $\frac{1}{4}$ is the least possible value.

This integral inequality has elegant and effective importance for numerical integration, optimization theory, integral operator theory, information, probability, statistics and stochastic process. During the last few years, numerous mathematicians and researchers focused their incredible commitment and consideration on the investigation of this inequality. In 1997, this inequality was investigated by Dragomir and Wang [18,19] in terms of the lower and upper bounds of the first derivative. Barnett et al. and Cerone et al. [20,21] worked on this inequality involving twice differentiable convex functions. For some articles concerning the Ostrowski inequality, one can refer to [22–28] and the references cited therein. This inequality yields an upper bound for the approximation of the integral average $\frac{1}{b_2 - b_1} \int_{b_1}^{b_2} Y(z) dz$ by the value of $Y(z)$ at the point $z \in [b_1, b_2]$.

Definition 1 ([29]). A function $Y : J \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$ is said to be a convex function if

$$Y(\sigma b_1 + (1 - \sigma)b_2) \leq \sigma Y(b_1) + (1 - \sigma)Y(b_2) \quad (2)$$

holds for all $[b_1, b_2] \in J$ and $\sigma \in [0, 1]$. We say that Y is concave if $(-Y)$ is convex.

The Hermite–Hadamard (H–H) inequality (see [30]) asserts that, if a mapping $Y : J \subset \mathfrak{R} \rightarrow \mathfrak{R}$ is convex on J with $b_1, b_2 \in J$ and $b_2 > b_1$, then:

$$Y\left(\frac{b_1 + b_2}{2}\right) \leq \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} Y(\sigma) d\sigma \leq \frac{Y(b_1) + Y(b_2)}{2}. \quad (3)$$

One can see the evolution of fractional integral and derivative operators across time by looking at the few selected papers [31–34] and the references therein. The latest compact review about fractional calculus is by two eminent Professors, D. Baleanu and R. P. Agrawal in their review article “Fractional calculus in the sky” [35].

The fractional derivative operators with non-singular kernels are very effective in solving the non-locality of real world problems in an appropriate way. Now, we recall the notion of the Caputo–Fabrizio integral operator:

Definition 2 ([36]). Let $Y \in \mathcal{H}^1(0, b_2)$, $b_2 > b_1$, $\xi \in [0, 1]$, then the definition of the new Caputo fractional derivative is:

$${}^{\text{CF}}D^\xi Y(t) = \frac{\mathcal{M}(\xi)}{1-\xi} \int_{b_1}^t Y'(s) \exp\left[-\frac{\xi}{(1-\xi)}(t-s)\right] ds,$$

where $\mathcal{M}(\xi)$ is a normalization function.

Moreover, the corresponding Caputo–Fabrizio fractional integral operator is given as:

Definition 3 ([37]). Let $Y \in \mathcal{H}^1(0, b_2)$, $b_2 > b_1$, $\xi \in [0, 1]$.

$$\left({}^{\text{CF}}I_{b_1}^\xi\right)(t) = \frac{1-\xi}{\mathcal{M}(\xi)} Y(t) + \frac{\xi}{\mathcal{M}(\xi)} \int_{b_1}^t Y(y) dy,$$

and

$$\left({}^{\text{CF}}I_{b_2}^\xi\right)(t) = \frac{1-\xi}{\mathcal{M}(\xi)} Y(t) + \frac{\xi}{\mathcal{M}(\xi)} \int_t^{b_2} Y(y) dy,$$

where $\mathcal{M}(\xi)$ is a normalization function.

As of late, Atangana and Baleanu presented another fractional operator involving the special Mittag–Leffler function, which tackles the issue of recovering the original function. It is seen that Mittag–Leffler’s function is more reasonable than a power law in demonstrating the physical phenomenon around us. This made the operator more powerful and accommodating. Thus, numerous researchers have shown a keen fascination for using this special operator. Atangana and Baleanu presented the derivative in both the Caputo and the Reimann–Liouville sense:

Definition 4 ([38]). Let $b_2 > b_1$, $\xi \in [0, 1]$ and $Y \in \mathcal{H}^1(b_1, b_2)$. The new fractional derivative is given by:

$${}^{\text{ABC}}D_t^\xi[Y(t)] = \frac{\mathcal{M}(\xi)}{1-\xi} \int_{b_1}^t Y'(z) \mathbb{E}_\xi\left[-\xi \frac{(t-x)^\xi}{(1-\xi)}\right] dx.$$

Definition 5 ([38]). Let $Y \in \mathcal{H}^1(b_1, b_2)$, $b_1 > b_2$, $\xi \in [0, 1]$. The new fractional derivative is given by:

$${}^{\text{ABR}}D_t^\xi[Y(t)] = \frac{\mathcal{M}(\xi)}{1-\xi} \frac{d}{dt} \int_{b_1}^t Y(z) \mathbb{E}_\xi\left[-\xi \frac{(t-x)^\xi}{(1-\xi)}\right] dx.$$

However, in the same paper they provide the corresponding Atangana–Baleanu (AB)–fractional integral operator as:

Definition 6 ([38]). The fractional integral operator with the non-local kernel of a function $Y \in \mathcal{H}^1(b_1, b_2)$ is defined as:

$${}^{\text{AB}}I_t^\xi\{Y(t)\} = \frac{1-\xi}{\mathcal{M}(\xi)} Y(t) + \frac{\xi}{\mathcal{M}(\xi)\Gamma(\xi)} \int_{b_1}^t Y(y)(t-y)^{\xi-1} dy,$$

where $b_2 > b_1$, $\xi \in [0, 1]$.

In [39], the right hand side of **AB**-fractional integral operator is written as follows:

$${}^{\mathbf{AB}}I_{\mathfrak{t}}^{\xi}\{Y(\mathfrak{t})\} = \frac{1-\xi}{\mathcal{M}(\xi)}Y(\mathfrak{t}) + \frac{\xi}{\mathcal{M}(\xi)\Gamma(\xi)}\int_{\mathfrak{t}}^{b_2}Y(y)(y-\mathfrak{t})^{\xi-1}dy,$$

where $\Gamma(\xi)$ is the Gamma function.

The positivity of the $\mathcal{M}(\xi)$ implies that the Atangana–Baleanu **AB** fractional integral of a positive function is positive. It is worth noting that the case in which the order $\xi \rightarrow 1$, it yields the classical integral and the case when $\xi \rightarrow 0$, it provides the initial function. For some recent papers on fractional calculus, interested readers can see [40–44].

In this article, we set up an equality and applied it to present new Ostrowski-type inequalities. Further, results for the Hölder inequality, the power-mean inequality, the Young inequality, and the Jensen integral inequality for functions with a bounded first derivative are presented as well.

Definition 7 (Hölder’s inequality [45]). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If Y and Ψ be real functions defined on $[b_1, b_2]$ and if $|Y|^p$ and $|\Psi|^q$ are integrable on $[b_1, b_2]$, then the following inequality holds:

$$\int_{b_1}^{b_2}|Y(x)\Psi(x)|dx \leq \left(\int_{b_1}^{b_2}|Y|^p dx\right)^{\frac{1}{p}}\left(\int_{b_1}^{b_2}|\Psi|^q dx\right)^{\frac{1}{q}}. \quad (4)$$

Definition 8 (Power-mean inequality [45]). Let $q \geq 1$. If Y and Ψ are real functions defined on $[b_1, b_2]$ and if $|Y|$, $|Y||\Psi|^q$ are integrable on $[b_1, b_2]$, then the following inequality holds:

$$\int_{b_1}^{b_2}|Y(x)\Psi(x)|dx \leq \left(\int_{b_1}^{b_2}|Y(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{b_1}^{b_2}|Y(x)||\Psi(x)|^q dx\right)^{\frac{1}{q}}. \quad (5)$$

3. Main Results

In order to present our main results, we need the following vital lemma in fractional settings involving Atangana–Baleanu integral operators as follows:

Lemma 1. Suppose a mapping $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is differentiable on (b_1, b_2) with $b_2 > b_1$. If $Y' \in \mathcal{L}_1[b_1, b_2]$, then for all $z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following identity for Atangana–Baleanu fractional integral holds:

$$\begin{aligned} & \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \\ & - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}I_{b_1}^{\xi} Y(b_1) + {}^{\mathbf{AB}}I_{b_2}^{\xi} Y(b_2) \right\} \\ & = \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_1) d\sigma - \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_2) d\sigma, \end{aligned} \quad (6)$$

where $\mathcal{M}(\xi)$ is normalization function.

Proof. For easier manipulations, let us write

$$\begin{aligned} \mathbf{I} &= \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_1) d\sigma - \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_2) d\sigma \\ &= \frac{(z-b_1)^{\xi+1}}{b_2-b_1} I_1 - \frac{(b_2-z)^{\xi+1}}{b_2-b_1} I_2, \end{aligned} \quad (7)$$

where

$$\begin{aligned} I_1 &= \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_1) d\sigma \\ &= \left. \frac{\sigma^{\xi} Y(\sigma z + (1-\sigma)b_1)}{z - b_1} \right|_0^1 - \frac{\xi}{z - b_1} \int_0^1 \sigma^{\xi-1} Y'(\sigma z + (1-\sigma)b_1) d\sigma \\ &= \frac{Y(z)}{z - b_1} - \frac{\xi}{z - b_1} \int_0^1 \sigma^{\xi-1} Y'(\sigma z + (1-\sigma)b_1) d\sigma. \end{aligned}$$

By changing the variables, we have:

$$I_1 = \frac{Y(z)}{z - b_1} - \frac{\xi}{(z - b_1)^{\xi+1}} \int_{b_1}^z (u - b_1)^{\xi-1} Y(u) du.$$

Similarly, we can find:

$$\begin{aligned} I_2 &= \int_0^1 \sigma^{\xi} Y'(\sigma z + (1-\sigma)b_2) d\sigma \\ &= -\frac{Y(z)}{b_2 - z} + \frac{\xi}{(b_2 - z)^{\xi+1}} \int_z^{b_2} (b_2 - v)^{\xi-1} Y(v) dv. \end{aligned}$$

By putting the values of I_1 and I_2 in (7), we get (6), which completes the proof of the theorem. $\square$

Theorem 1. Suppose $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|$ is a convex function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} &\left| \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^{\xi} Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^{\xi} Y(b_2) \right\} \right| \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \frac{|Y'(z)|}{\xi+2} + \frac{|Y'(b_1)|}{(\xi+1)(\xi+2)} \right\} + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \frac{|Y'(z)|}{\xi+2} + \frac{|Y'(b_2)|}{(\xi+1)(\xi+2)} \right\}. \end{aligned} \quad (8)$$

Proof. By using the identity that is given in Lemma 1, and $|Y'|$ being a convex function, we have:

$$\begin{aligned} &\left| \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^{\xi} Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^{\xi} Y(b_2) \right\} \right| \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} |Y'(\sigma z + (1-\sigma)b_2)| d\sigma \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} \{ \sigma |Y'(z)| + (1-\sigma) |Y'(b_1)| \} d\sigma \\ &\quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} \{ \sigma |Y'(z)| + (1-\sigma) |Y'(b_2)| \} d\sigma \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \frac{|Y'(z)|}{\xi+2} + \frac{|Y'(b_1)|}{(\xi+1)(\xi+2)} \right\} + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \frac{|Y'(z)|}{\xi+2} + \frac{|Y'(b_2)|}{(\xi+1)(\xi+2)} \right\}, \end{aligned}$$

which ends the Theorem. $\square$

Corollary 1. If we choose $|Y'| \leq \mathbb{K}$, $\mathbb{K} > 0$ in Theorem 1, then we have the following inequality:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{\mathbb{K}}{b_2-b_1} \left(\frac{1}{\xi+2} + \frac{1}{(\xi+1)(\xi+2)} \right) \left\{ (z-b_1)^{\xi+1} + (b_2-z)^{\xi+1} \right\}. \end{aligned}$$

Corollary 2. If we choose $z = \frac{b_1+b_2}{2}$, in Corollary 1, then we get the following mid point inequality:

$$\begin{aligned} & \left| \frac{(b_2-b_1)^{\xi-1}}{2^{\xi-1}} Y\left(\frac{b_1+b_2}{2}\right) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_2) \right\} \right| \\ & \leq \mathbb{K} \left(\frac{1}{\xi+2} \right) \frac{(b_2-b_1)^\xi}{2^\xi}. \end{aligned}$$

Theorem 2. Suppose $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|^q$ is a convex function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi p+1} \right)^{\frac{1}{p}} \left(\frac{|Y'(z)|^q + |Y'(b_1)|^q}{2} \right)^{\frac{1}{q}} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi p+1} \right)^{\frac{1}{p}} \left(\frac{|Y'(z)|^q + |Y'(b_2)|^q}{2} \right)^{\frac{1}{q}}, \end{aligned} \quad (9)$$

where $\frac{1}{q} = 1 - \frac{1}{p}$ and $q > 1$.

Proof. Let $p > 1$. From Lemma 1 and using the Hölder inequality, one has:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_2)| d\sigma \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi p} d\sigma \right)^{\frac{1}{p}} \left(\int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma \right)^{\frac{1}{q}} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi p} d\sigma \right)^{\frac{1}{p}} \left(\int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma \right)^{\frac{1}{q}}. \end{aligned} \quad (10)$$

Since $|Y'|^q$ is convex function, one has:

$$\begin{aligned} \int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma & \leq \int_0^1 \{ \sigma |Y'(z)|^q + (1-\sigma) |Y'(b_1)|^q \} d\sigma \\ & = \frac{|Y'(z)|^q + |Y'(b_1)|^q}{2} \end{aligned} \quad (11)$$

and

$$\begin{aligned} \int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma &\leq \int_0^1 \{\sigma |Y'(z)|^q + (1-\sigma) |Y'(b_2)|^q\} d\sigma \\ &= \frac{|Y'(z)|^q + |Y'(b_2)|^q}{2}. \end{aligned} \quad (12)$$

Combining (11) and (12) with (10), we get (9). $\square$

Corollary 3. If we take $|Y'| \leq \mathbb{K}$, such that $\mathbb{K} > 0$ in Theorem 2, then we get:

$$\begin{aligned} &\left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ &\leq \frac{\mathbb{K}}{b_2-b_1} \left(\frac{1}{\xi p + 1} \right)^{\frac{1}{p}} \left\{ (z-b_1)^{\xi+1} + (b_2-z)^{\xi+1} \right\}. \end{aligned}$$

Corollary 4. In Corollary 3, if we choose $z = \frac{b_1+b_2}{2}$, then we obtain the following mid point inequality:

$$\begin{aligned} &\left| \frac{(b_2-b_1)^{\xi-1}}{2^{\xi-1}} Y\left(\frac{b_1+b_2}{2}\right) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_2) \right\} \right| \\ &\leq \mathbb{K} \left(\frac{1}{\xi p + 1} \right)^{\frac{1}{p}} \frac{(Y-b_1)^\xi}{2^\xi}. \end{aligned}$$

Theorem 3. Suppose $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|^q$ is a convex function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} &\left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \frac{1}{(\xi p + 1)p} + \frac{|Y'(z)|^q + |Y'(b_1)|^q}{2q} \right\} \\ &\quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \frac{1}{(\xi p + 1)p} + \frac{|Y'(z)|^q + |Y'(b_2)|^q}{2q} \right\}, \end{aligned} \quad (13)$$

where $\frac{1}{q} = 1 - \frac{1}{p}$ and $q > 1$.

Proof. From Lemma 1, we have:

$$\begin{aligned} &\left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ &\quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ &\leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_2)| d\sigma. \end{aligned}$$

By using the Young's inequality,

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \frac{1}{p} \int_0^1 \sigma^{\xi p} d\sigma + \frac{1}{q} \int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma \right\} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \frac{1}{p} \int_0^1 \sigma^{\xi p} d\sigma + \frac{1}{q} \int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma \right\}. \end{aligned} \quad (14)$$

Since $|Y'|^q$ is a convex function, we have:

$$\begin{aligned} \int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma & \leq \int_0^1 \{\sigma |Y'(z)|^q + (1-\sigma) |Y'(b_1)|^q\} d\sigma \\ & = \frac{|Y'(z)|^q + |Y'(b_1)|^q}{2} \end{aligned} \quad (15)$$

and

$$\begin{aligned} \int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma & \leq \int_0^1 \{\sigma |Y'(z)|^q + (1-\sigma) |Y'(b_2)|^q\} d\sigma \\ & = \frac{|Y'(z)|^q + |Y'(b_2)|^q}{2}. \end{aligned} \quad (16)$$

Combining (15) and (16) with (14), we get (13), which ends the Theorem. $\square$

Corollary 5. For $|Y'| \leq \mathbb{K}$, $\mathbb{K} > 0$ in Theorem 3, we have the following inequality:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \left(\frac{1}{(\xi p + 1)p} + \frac{\mathbb{K}^q}{q} \right) \left\{ \frac{(z-b_1)^{\xi+1}}{b_2-b_1} + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \right\}. \end{aligned}$$

Corollary 6. For $z = \frac{b_1+b_2}{2}$, in Corollary 5, then we obtain the following mid point inequality:

$$\begin{aligned} & \left| \frac{(b_2-b_1)^{\xi-1}}{2^{\xi-1}} Y\left(\frac{b_1+b_2}{2}\right) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{\frac{b_1+b_2}{2}}^\xi Y(b_2) \right\} \right| \\ & \leq \left(\frac{1}{(\xi p + 1)p} + \frac{\mathbb{K}^q}{q} \right) \frac{(b_2-b_1)^\xi}{2^\xi}. \end{aligned}$$

Theorem 4. Suppose $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|^q$ is a convex function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \left(\frac{1}{\xi+1} \right)^{1-\frac{1}{q}} \left(\frac{|Y'(z)|^q}{\xi+2} + \frac{|Y'(b_1)|^q}{(\xi+1)(\xi+2)} \right)^{\frac{1}{q}} \right\} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \left(\frac{1}{\xi+1} \right)^{1-\frac{1}{q}} \left(\frac{|Y'(z)|^q}{\xi+2} + \frac{|Y'(b_2)|^q}{(\xi+1)(\xi+2)} \right)^{\frac{1}{q}} \right\}, \end{aligned} \quad (17)$$

where $\frac{1}{q} = 1 - \frac{1}{p}$ and $q \geq 1$.

Proof. From the identity presented in Lemma 1 and using the power mean inequality, we have:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_2)| d\sigma \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^\xi d\sigma \right)^{1-\frac{1}{q}} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^\xi d\sigma \right)^{1-\frac{1}{q}} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^\xi d\sigma \right)^{1-\frac{1}{q}} \int_0^1 \sigma^\xi \left\{ \sigma |Y'(z)|^q + (1-\sigma) |Y'(b_1)|^q \right\} d\sigma \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^\xi d\sigma \right)^{1-\frac{1}{q}} \int_0^1 \sigma^\xi \left\{ \sigma |Y'(z)|^q + (1-\sigma) |Y'(b_2)|^q \right\} d\sigma \\ & = \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left\{ \left(\frac{1}{\xi+1} \right)^{1-\frac{1}{q}} \left(\frac{|Y'(z)|^q}{\xi+2} + \frac{|Y'(b_1)|^q}{(\xi+1)(\xi+2)} \right)^{\frac{1}{q}} \right\} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left\{ \left(\frac{1}{\xi+1} \right)^{1-\frac{1}{q}} \left(\frac{|Y'(z)|^q}{\xi+2} + \frac{|Y'(b_2)|^q}{(\xi+1)(\xi+2)} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

which ends the Theorem. $\square$

Corollary 7. For $|Y'| \leq \mathbb{K}$, $\mathbb{K} > 0$ in Theorem 4, we have the following inequality:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{\mathbb{K}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left\{ (z-b_1)^{\xi+1} + (b_2-z)^{\xi+1} \right\}. \end{aligned}$$

Corollary 8. For $z = \frac{b_1+b_2}{2}$, In Corollary 7, we obtain the following mid point inequality:

$$\begin{aligned} & \left| \frac{(b_2 - b_1)^{\xi-1}}{2^{\xi-1}} Y\left(\frac{b_1 + b_2}{2}\right) + \frac{1 - \xi}{b_2 - b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2 - b_1} \left\{ \frac{\mathbf{AB}}{\frac{b_1+b_2}{2}} I_{b_1}^{\xi} Y(b_1) + \frac{\mathbf{AB}}{\frac{b_1+b_2}{2}} I_{b_2}^{\xi} Y(b_2) \right\} \right| \\ & \leq \mathbb{K} \left(\frac{1}{\xi + 1} \right) \frac{(b_2 - b_1)^{\xi}}{2^{\xi}}. \end{aligned}$$

Theorem 5. Suppose $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|$ is a concave function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ \frac{\mathbf{AB}}{z} I_{b_1}^{\xi} Y(b_1) + \frac{\mathbf{AB}}{z} I_{b_2}^{\xi} Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left| Y' \left(\frac{(\xi+1)z+b_1}{\xi+2} \right) \right| + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left| Y' \left(\frac{(\xi+1)z+b_2}{\xi+2} \right) \right|. \end{aligned} \quad (18)$$

Proof. From Lemma 1 and using the Jensen integral inequality with the concavity of $|Y'|$, we have:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ \frac{\mathbf{AB}}{z} I_{b_1}^{\xi} Y(b_1) + \frac{\mathbf{AB}}{z} I_{b_2}^{\xi} Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^{\xi} |Y'(\sigma z + (1-\sigma)b_2)| d\sigma \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi} d\sigma \right) \left| Y' \left(\frac{\int_0^1 \sigma^{\xi} (\sigma z + (1-\sigma)b_1) d\sigma}{\int_0^1 \sigma^{\xi} d\sigma} \right) \right| \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi} d\sigma \right) \left| Y' \left(\frac{\int_0^1 \sigma^{\xi} (\sigma z + (1-\sigma)b_2) d\sigma}{\int_0^1 \sigma^{\xi} d\sigma} \right) \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left| Y' \left(\frac{(\xi+1)z+b_1}{\xi+2} \right) \right| + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left| Y' \left(\frac{(\xi+1)z+b_2}{\xi+2} \right) \right|, \end{aligned}$$

which ends the proof. $\square$

Corollary 9. For $|Y'| \leq \mathbb{K}$, $\mathbb{K} > 0$ in Theorem 5, then we have the following inequality:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^{\xi} + (b_2-z)^{\xi}}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ \frac{\mathbf{AB}}{z} I_{b_1}^{\xi} Y(b_1) + \frac{\mathbf{AB}}{z} I_{b_2}^{\xi} Y(b_2) \right\} \right| \\ & \leq \frac{\mathbb{K}}{b_2-b_1} \left(\frac{1}{\xi+1} \right) \left\{ (z-b_1)^{\xi+1} + (b_2-z)^{\xi+1} \right\}. \end{aligned}$$

Theorem 6. Suppose a mapping $Y : [b_1, b_2] \rightarrow \mathfrak{R}$ is a differentiable mapping on (b_1, b_2) with $b_2 > b_1$ and $Y' \in \mathcal{L}_1[b_1, b_2]$. If $|Y'|^q$ is a concave function, then $\forall z \in [b_1, b_2]$ and $\xi \in [0, 1]$, the following inequality for the Atangana–Baleanu fractional integral exists:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi^{p+1}} \right)^{\frac{1}{p}} \left| Y' \left(\frac{z+b_1}{2} \right) \right| + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\frac{1}{\xi^{p+1}} \right)^{\frac{1}{p}} \left| Y' \left(\frac{z+b_2}{2} \right) \right|, \end{aligned} \quad (19)$$

where $\frac{1}{q} = 1 - \frac{1}{p}$ and $q > 1$.

Proof. Using the identity given Lemma 1 and the Hölder inequality, we have:

$$\begin{aligned} & \left| \left[\frac{(z-b_1)^\xi + (b_2-z)^\xi}{b_2-b_1} \right] Y(z) + \frac{1-\xi}{b_2-b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2-b_1} \left\{ {}^{\mathbf{AB}}\mathbf{I}_{b_1}^\xi Y(b_1) + {}^{\mathbf{AB}}\mathbf{I}_{b_2}^\xi Y(b_2) \right\} \right| \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_1)| d\sigma + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \int_0^1 \sigma^\xi |Y'(\sigma z + (1-\sigma)b_2)| d\sigma \\ & \leq \frac{(z-b_1)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi p} d\sigma \right)^{\frac{1}{p}} \left(\int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma \right)^{\frac{1}{q}} \\ & \quad + \frac{(b_2-z)^{\xi+1}}{b_2-b_1} \left(\int_0^1 \sigma^{\xi p} d\sigma \right)^{\frac{1}{p}} \left(\int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma \right)^{\frac{1}{q}}. \end{aligned} \quad (20)$$

Using the concavity of $|Y'|^q$ and the Jensen integral inequality,

$$\int_{b_1}^{b_2} Y(\Psi(x)) d\mu \leq \left(\int_{b_1}^{b_2} d\mu \right) Y \left(\frac{\int_{b_1}^{b_2} \Psi(x) d\mu}{\int_{b_1}^{b_2} d\mu} \right),$$

we have

$$\begin{aligned} & \int_0^1 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma = \int_0^1 \sigma^0 |Y'(\sigma z + (1-\sigma)b_1)|^q d\sigma \\ & \leq \left(\int_0^1 \sigma^0 d\sigma \right) \left| Y' \left(\frac{\int_0^1 (\sigma z + (1-\sigma)b_1) d\sigma}{\int_0^1 \sigma^0 d\sigma} \right) \right|^q \\ & \leq \left| Y' \left(\frac{z+b_1}{2} \right) \right|^q \end{aligned} \quad (21)$$

and, similarly,

$$\int_0^1 |Y'(\sigma z + (1-\sigma)b_2)|^q d\sigma \leq \left| Y' \left(\frac{z+b_2}{2} \right) \right|^q, \quad (22)$$

combining the above numbered Equations (21) and (22) with the (20), we get (19). This completes proof of the theorem. $\square$

Corollary 10. For $z = \frac{b_1+b_2}{2}$, in Theorem 6, we obtain the following mid point inequality:

$$\begin{aligned} & \left| \frac{(b_2 - b_1)^{\xi-1}}{2^{\xi-1}} Y\left(\frac{b_1 + b_2}{2}\right) + \frac{1 - \xi}{b_2 - b_1} \Gamma(\xi) \left[Y(b_1) + Y(b_2) \right] \right. \\ & \quad \left. - \frac{\mathcal{M}(\xi)\Gamma(\xi)}{b_2 - b_1} \left\{ \frac{\mathbf{AB}}{\frac{b_1+b_2}{2}} I_{b_1}^{\xi} Y(b_1) + \frac{\mathbf{AB}}{\frac{b_1+b_2}{2}} I_{b_2}^{\xi} Y(b_2) \right\} \right| \\ & \leq \left(\frac{1}{\xi p + 1} \right)^{\frac{1}{p}} \frac{1}{b_2 - b_1} \left\{ (z - b_1)^{\xi+1} \left| Y'\left(\frac{3b_1 + b_2}{4}\right) \right| + (b_2 - z)^{\xi+1} \left| Y'\left(\frac{b_1 + 3b_2}{4}\right) \right| \right\}. \end{aligned}$$

4. Conclusions

In this article, we build up Ostrowski-type inequalities for convex functions involving the Atangana–Baleanu fractional integral operator. As far as our knowledge is concerned, the results presented in this article are unique. Due to the notable applications convex functions have in numerous scientific branches, it can be believed that our new improvements can be generalized to some special functions involving convexity, interval analysis, quantum calculus, fractional calculus, and coordinates. The presented results might invigorate further exploration in the field of mathematical inequalities. We envision that one of the keys for the achievement of future speculative and applied points of view is to ponder the possibility of various classes of fractional operators.

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