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On q-Starlike Functions Defined by q-Ruscheweyh Differential Operator in Symmetric Conic Domain

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Abstract: Motivated by q-analogue theory and symmetric conic domain, we study here the q-version of the Ruscheweyh differential operator by applying it to the starlike functions which are related with the symmetric conic domain. The primary aim of this work is to first define and then study a new class of holomorphic functions using the q-Ruscheweyh differential operator. A new class $k-ST_q^{\tau}[C,D]$ of k-Janowski starlike functions associated with the symmetric conic domain, which are defined by the generalized Ruscheweyh derivative operator in the open unit disk, is introduced. The necessary and sufficient condition for a function to be in the class $k-ST_q^{\tau}[C,D]$ is established. In addition, the coefficient bound, partial sums and radii of starlikeness for the functions from the class of k-Janowski starlike functions related with symmetric conic domain are included.

Keywords: holomorphic functions; starlike functions; q-starlike functions; Ruscheweyh q-differential operator; symmetric conic domain

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1. Introduction

The wide range of applications of q-analysis has attracted the considerable attention of researchers in this area, as can be seen in the current literature. The range of its applications covers several categories of research in mathematics. Jackson [1,2] was the first mathematician to use the idea of q-calculus. He first proposed the well-known q-derivative and also the q-integral in a composed manner. After that, since the early 1980s, geometrical specifications of q-analysis have been discussed and analyzed through investigations on quantum groups. This investigation additionally proposes a connection between appropriate frameworks and q-analysis. In [3–5], the q-version of the famous Baskakov Durrmeyer operator was introduced, which relies on the q-beta function. Two more significant q-speculations of complex operators are the q-Picard integral operator and the q-Gauss–Weierstrass integral operator (see [6,7]). These operators were studied and analyzed in terms of their geometric specifications for a few subclasses of holomorphic functions. Currently, many operators are studied in terms of their q-analogues; see [8–10]. The q-symmetric differential operator and its applications can be seen in [11–15]. The concept of the convolution of the standardized holomorphic functions and q-versions of hypergeometric functions were utilized to define these q-operators, and numerous amazing outcomes have been observed. This series of transformations of differential as well as integral operators made this common in recent research work, which has consequently opened a wide range of research in the space of holomorphic functions. The class of k-Janowski starlike functions summed up by the q-derivative operator, denoted by $k-ST_q[C,D]$, was presented recently (see [16]). The

Janowski functions and their related materials can be found in [17]. Special functions are vital in various branches of applied sciences and mathematics. The geometric characteristics of some extraordinary special functions were investigated by numerous researchers; see [18–23].

After a careful study of the relevant literature, it was observed that the q-version of the well-known and most cited differential operator, named the Ruscheweyh differential operator, was introduced in [24]. However, this has not been studied for starlike functions defined in the symmetric conic domain. This was the main motivation behind the following Definition 4 and its related results. In this paper, our focus is the presentation of some comparable outcomes for the standardized types of the k-Janowski starlike function summed up by the q-Ruscheweyh derivative operator, which has vital applications in different zones of mathematics. In this article, we focus primarily on the partial sums and existence of the radius of the starlikeness of the k-Janowski starlike functions related to the generalized q-Ruscheweyh derivative. Furthermore, our aim is also to give a sufficient condition and coefficient bound for the class $k - ST_q^{\tau}[C, D]$. The paper is organized into two sections. After the brief review of the literature about the ongoing research, the remaining part of this section is devoted to some essential definitions that are required for the proof of our fundamental outcomes. The second section contains some preliminary outcomes, which are important in giving a concrete base to our main theorems. The section contains the sufficient condition and coefficient bound, followed by the ratios of partial sums of the functions from the class $k - ST_q^{\tau}[C, D]$. We now begin with a pair of terms.

Let Λ represent the class of functions having the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

which are holomorphic in $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we represent the class $\mathcal{S}$ of all functions in Λ which are univalent in $\mathcal{U}$; for details, see [25].

A function $h(z) \in \mathcal{S}$ is called starlike of order α if it satisfies

$$\Re \frac{zh'(z)}{h(z)} > \alpha \qquad (z \in \mathcal{U})$$

for $\alpha \in [0, 1)$. We represent by $S^*(\alpha)$ the subclass of $\mathcal{S}$ containing the functions which are starlike of order α in $\mathcal{U}$.

If h_1 and h_2 are holomorphic functions in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| \leq 1, \forall z \in \mathcal{U}$ so that $h_1(z) = h_2(w(z))$, then we say that h_2 is subordinated by h_1, denoted symbolically as $h_1 \prec h_2$. If h_2 is univalent, then $h_1 \prec h_2$ iff $h_1(0) = h_2(0)$ and $h_1(\mathcal{U}) \subseteq h_2(\mathcal{U})$.

For two holomorphic functions,

$$h_1(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad h_2(z) = \sum_{k=0}^{\infty} b_k z^k \qquad (z \in \mathcal{U}),$$

the Hadamard product of $h_1(z)$ and $h_2(z)$ is defined as

$$h_1(z) * h_2(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

We give a few notations and the theory of q-calculus utilized as a part of this article; see [1,2,10]. For $q \in (0, 1)$ and $n \in \mathbb{N}$, the q-hypothesis starts with the q-analogues of the positive integers. The following expression

$$\lim_{q \to 1} \frac{1 - q^n}{1 - q} = n.$$

gives the q-analogue of n, also known as the q-number of n; that is,

$$[n, q] = \frac{1-q^n}{1-q}.$$

We include the q-factorial, which is defined by

$$[n, q]! = \begin{cases} [n, q][n-1, q] \cdots [1, q], & n = 1, 2, \cdots, \\ 1, & n = 0. \end{cases}$$

The q-difference operator for $h \in \Lambda$ is defined as

$$\partial_q h(z) = \frac{h(qz) - h(z)}{z(q-1)}, \qquad (z \in \mathcal{U}).$$

For $n \in \mathbb{N}$ and $z \in \mathcal{U}$, consider

$$\partial_q z^n = [n, q] z^{n-1}$$

and

$$\partial_q \left\{ \sum_{n-1}^{\infty} a_n z^n \right\} = \sum_{n-1}^{\infty} [n, q] a_n z^{n-1}.$$

Definition 1. *For a function $h(z) \in \Lambda$, the q-analog of the Ruscheweyh differential operator is defined as*

$$\varsigma_q^\tau h(z) = \varphi(q, \tau+1; z) * h(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n, \ (z \in \mathcal{U} \text{ and } \tau > -1) \tag{2}$$

where

$$\varphi(q, \tau+1; z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n,$$

and

$$\psi_{n-1} = \frac{\Gamma_q(\tau+n)}{[n-1, q]! \Gamma_q(\tau+1)} = \frac{[\tau+1, q]_{n-1}}{[n-1, q]!}, \qquad (\psi_0 = 1), \tag{3}$$

where $[\tau+1, q]_{n-1}$ is a Pochhammer symbol, which is defined as follows.

$$[\tau+1, q]_{n-1} = \begin{cases} 1, & n = 1, \\ [\tau+1, q][\tau+2, q][\tau+3, q][\tau+4, q] \cdots [\tau+n-1, q], & n = 2, 3, 4, \cdots. \end{cases}$$

It is evident from (2) that

$$\varsigma_q^0 h(z) = h(z) \quad \text{and} \quad \varsigma_q^1 h(z) = z \partial_q h(z),$$

and

$$\varsigma_q^n h(z) = \frac{z \partial_q^n \left(z^{n-1} h(z) \right)}{[n, q]!}, \quad (n \in \mathbb{N}).$$

$$\lim_{q \to 1-} \varphi(q, \tau+1; z) = \frac{z}{(1-z)^{\tau+1}},$$

and

$$\lim_{q \to 1-} \varsigma_q^\tau h(z) = h(z) * \frac{z}{(1-z)^{\tau+1}}.$$

This reveals that, for $q \to 1-$, the q-Ruscheweyh differential operator changes into the Ruscheweyh differential operator $D^{\delta}(h(z))$, see [26]. The following is an obvious well-known derivation from (2).

$$z\partial \varsigma_q^{\tau} h(z) = \left(1 + \frac{[\tau, q]}{q^{\tau}}\right) \varsigma_q^{\tau+1} h(z) - \frac{[\tau, q]}{q^{\tau}} \varsigma_q^{\tau} h(z).$$

If $q \to 1-$, then it reduces to

$$z(\varsigma^{\tau} h(z))' = (1+\tau)\varsigma^{\tau+1} h(z) - \tau \varsigma^{\tau} h(z).$$

Definition 2. *The function $p(z) \in k - P_q[C, D]$, iff,*

$$p(z) \prec \frac{((3-q) + C(1+q))\widetilde{p}_k(z) + ((3-q) - C(1+q))}{((3-q) + D(1+q))\widetilde{p}_k(z) + ((3-q) - D(1+q))}, \quad k \geq 0,$$

where $-1 \leq D < C \leq 1$, $k \geq 0$, $q \in (0, 1)$ and

$$\widetilde{p}_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{2}{1-k^2} \sinh^2\left[\left(\frac{2}{\pi}\cos^{-1} k\right) \tan^{-1} h\sqrt{z}\right], & 0 < k < 1, \\ 1 + \frac{1}{k^2-1} \sin\left(\frac{\pi}{2Q(s)} \int_0^{\frac{u(z)}{\sqrt{s}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(sx)^2}} dx\right) + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{4}$$

The function $\widetilde{p}_k(z)$ gives the image of $\mathcal{U}$ as a conic domain which is symmetric about the real axis. For more details, see [27,28]. If $\widetilde{p}_k(z) = 1 + \delta_k z + \cdots$, then it is shown in [29] that from (4), one can have

$$\delta_k = \begin{cases} \frac{8(\cos^{-1} k)^2}{\pi^2(1-k^2)}, & 0 \leq k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{s}(1+s)Q^2(s)}, & k > 1. \end{cases}$$

Geometrically, the values of the function $p(z) \in k - P[C, D]$ belong to the q-symmetric conic domain $\Omega_{k,q}[C, D]$, $-1 \leq D < C \leq 1$, $k \geq 0$ which is defined as

$$\Omega_{k,q}[C, D] = \left\{ g : \Re(\lambda) > k|\lambda - 1|, \ \lambda = \frac{((q-3) + D(1+q))g(z) + ((3-q) - C(1+q))}{((3-q) + D(1+q))g(z) - ((3-q) + C(1+q))} \right\}.$$

Definition 3. *The function $h(z) \in \Lambda$ will be in the class $k - ST_q[C, D]$, $k \geq 0$, $-1 \leq D < C \leq 1$, iff*

$$\Re\left[\frac{((q-3) + D(1+q))\frac{zD_q h(z)}{h(z)} + ((3-q) - C(1+q))}{((3-q) + D(1+q))\frac{zD_q h(z)}{h(z)} - ((3-q) + C(1+q))}\right] > k\left|\frac{((q-3) + D(1+q))\frac{zD_q h(z)}{h(z)} + ((3-q) - C(1+q))}{((3-q) + D(1+q))\frac{zD_q h(z)}{h(z)} - ((3-q) + C(1+q))} - 1\right|$$

or equivalently

$$\frac{z\partial_q h(z)}{h(z)} \in k - P_q[C, D].$$

For more details of the above classes, we refer to [16]. It is noted that $0 - ST_q[C, D] = S_q^*[C, D]$, the class of q-starlike functions, was given by Srivastava et al. [19]

and $\lim\limits_{q \longrightarrow 1-} \left(k - ST_q[C,D]\right) = k - ST[C,D]$, the class of Janowski k-starlike functions, was presented by Noor and Malik [18].

The detailed study of the above-mentioned classes motivated us to define the much generalized class of functions with the q-Ruscheweyh differential operator related with the symmetric conic domain defined by Janowski functions. This class is denoted by $k - ST_q^\tau[C,D]$ and is defined as follows.

Definition 4. *A function $h \in \Lambda$ will be in the class $k - ST_q^\tau[C,D]$, $k \geq 0$, $-1 \leq D < C \leq 1$, iff*

$$\Re\left[\frac{((q-3)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} + ((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} - ((3-q)+C(1+q))}\right] > k\left|\frac{((q-3)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} + ((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} - ((3-q)+C(1+q))} - 1\right|$$

Or equivalently,

$$\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} \in k - P[C,D].$$

Lemma 1 ([30])**.** *Let $h(z) = 1 + \sum\limits_{n=1}^{\infty} v_n z^n$ be subordinate to $H(z) = 1 + \sum\limits_{n=1}^{\infty} V_n z^n$. If $H(z)$ is univalent in $\mathcal{U}$ and $H(\mathcal{U})$ is convex, then*

$$|v_n| \leq |V_1|, \ \ n \geq 1. \tag{5}$$

2. Main Results

The following theorem gives a condition which is sufficient for functions to be in $k - ST_q^\tau[C,D]$.

Theorem 1. *A function $h \in \Lambda$ and with the form (1) will belong to the class $k - ST_q^\tau[C,D]$, $k \geq 0$, $1 \leq D < C \leq 1$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} \frac{g_n}{(1+q)|D-C|}|a_n| < 1, \tag{6}$$

where

$$g_n = \{2q(3-q)(k+1)[n-1,q] + |((3-q)+D(1+q))[n,q] - ((3-q)+C(1+q))|\}\psi_{n-1} \tag{7}$$

with ψ, as defined by (3).

Proof. Let us assume (6) holds; then, it is enough to show that

$$k\left|\frac{((q-3)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} + ((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} - ((3-q)+C(1+q))} - 1\right| - \Re\left[\frac{((q-3)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} + ((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q\varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)} - ((3-q)+C(1+q))} - 1\right] < 1$$

We assume for our convenience

$$
\begin{aligned}
& k\left|\frac{((q-3)+D(1+q))\frac{z\partial_q \varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)}+((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q \varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)}-((3-q)+C(1+q))}-1\right| \\
& \qquad -\Re\left[\frac{((q-3)+D(1+q))\frac{z\partial_q \varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)}+((3-q)-C(1+q))}{((3-q)+D(1+q))\frac{z\partial_q \varsigma_q^\tau h(z)}{\varsigma_q^\tau h(z)}-((3-q)+C(1+q))}-1\right]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & (k+1)\left|\frac{((q-3)+D(1+q))z\partial_q \varsigma_q^\tau h(z)+((3-q)-C(1+q))\varsigma_q^\tau h(z)}{((3-q)+D(1+q))z\partial_q \varsigma_q^\tau h(z)-((3-q)+C(1+q))\varsigma_q^\tau h(z)}-1\right| \\
=\ & 2(3-q)(k+1)\left|\frac{\varsigma_q^\tau h(z)-z\partial_q \varsigma_q^\tau h(z)}{((3-q)+D(1+q))z\partial_q \varsigma_q^\tau h(z)-((3-q)+C(1+q))\varsigma_q^\tau h(z)}\right| \\
=\ & 2(3-q)(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-[n,q])\psi_{n-1}a_n z^n}{z(D-C)(1+q)+\sum_{n=2}^{\infty}\begin{pmatrix} ((3-q)+D(1+q))[n,q] \\ -((3-q)+C(1+q)) \end{pmatrix}\psi_{n-1}a_n z^n}\right| \\
=\ & 2(3-q)(k+1)\left|\frac{-q\sum_{n=2}^{\infty}[n-1,q]\psi_{n-1}a_n z^n}{z(D-C)(1+q)+\sum_{n=2}^{\infty}\begin{pmatrix} ((3-q)+D(1+q))[n,q] \\ -((3-q)+C(1+q)) \end{pmatrix}\psi_{n-1}a_n z^n}\right| \\
\leq\ & \frac{2q(3-q)(k+1)\sum_{n=2}^{\infty}[n-1,q]\psi_{n-1}|a_n|}{(1+q)|D-C|-\sum_{n=2}^{\infty}|((3-q)+D(1+q))[n,q]-((3-q)+C(1+q))|\psi_{n-1}|a_n|}. \\
=\ & \frac{2q(3-q)(k+1)\sum_{n=2}^{\infty}[n-1,q]|a_n|}{(1+q)|D-C|\frac{1}{\psi_{n-1}}-\sum_{n=2}^{\infty}|((3-q)+D(1+q))[n,q]-((3-q)+C(1+q))||a_n|}
\end{aligned}
$$

The last expression is bounded above by 1 if

$$
\begin{aligned}
2q(3-q)(k+1)\sum_{n=2}^{\infty}[n-1,q]|a_n| &< (1+q)|D-C|\frac{1}{\psi_{n-1}} \\
&\quad -\sum_{n=2}^{\infty}|((3-q)+D(1+q))[n,q]-((3-q)+C(1+q))||a_n|,
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\{2q(3-q)(k+1)[n-1,q]+|((3-q)+D(1+q))[n,q]-((3-q)+C(1+q))|\}\psi_{n-1}|a_n| \\
&\qquad < (1+q)|D-C|.
\end{aligned}
$$

and the proof is complete. □

As a special case, taking $\tau=0$, the following result is obtained, which is already proved in [16].

Corollary 1. *A function $h \in \Lambda$ will be said to belong to the class $k - ST_q[C, D]$, $k \geq 0$, $-1 \leq D < C \leq 1$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} \{2q(3-q)[n-1,q](k+1) + |((3-q)+D(1+q))[n,q] - ((3-q)+C(1+q))|\}|a_n| < (1+q)|D-C|.$$

Theorem 2. *Let $h(z) \in k - ST_q^{\tau}[C, D]$, $k \geq 0$, $-1 \leq D < C \leq 1$ and is of the form (1), then for $n \geq 2$,*

$$|a_n| \leq \prod_{j=0}^{n-2} \frac{|(C-D)\delta_k \psi_j - 2qD[j,q]\psi_j|}{2q[j+1,q]\psi_{j+1}}, \tag{8}$$

where ψ is defined by (3).

Proof. By definition, for $h(z) \in k - ST_q^{\tau}[C, D]$, we have

$$\frac{z\partial_q \varsigma_q^{\tau} h(z)}{\varsigma_q^{\tau} h(z)} = p(z), \tag{9}$$

where

$$p(z) \prec \frac{((3-q)+C(1+q))\widetilde{p}_k(z) + ((3-q)-C(1+q))}{((3-q)+D(1+q))\widetilde{p}_k(z) + ((3-q)-D(1+q))}.$$

If $\widetilde{p}_k(z) = 1 + \delta_k z + \ldots$, then

$$\frac{((3-q)+C(1+q))\widetilde{p}_k(z) + ((3-q)-C(1+q))}{((3-q)+D(1+q))\widetilde{p}_k(z) + ((3-q)-D(1+q))} = 1 + \frac{1}{4}(C-D)(q+1)\delta_k + \frac{1}{4}\left[\left(-\frac{1}{4}Cq - \frac{1}{4}C + \frac{1}{4}Dq + \frac{1}{4}D\right)((D+1)(1+q)+2-2q)\right]\delta_k^2 + \ldots. \tag{10}$$

Now, if $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then by (5) and (10), we get

$$|p_n| \leq \frac{1}{4}(C-D)(q+1)|\delta_k|, \quad n \geq 1. \tag{11}$$

Now, from (9), we have

$$z\partial_q \varsigma_q^{\tau} h(z) = p(z)\varsigma_q^{\tau} h(z).$$

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$

$$\begin{aligned} z + \sum_{n=2}^{\infty}[n,q]\psi_{n-1}a_n z^n &= \left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty}\psi_{n-1}a_n z^n\right). \\ \sum_{n=1}^{\infty}[n,q]\psi_{n-1}a_n z^n &= \left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\left(\sum_{n=1}^{\infty}\psi_{n-1}a_n z^n\right), \ p_0 = 1. \end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}[n,q]\psi_{n-1}a_n z^n = \sum_{n=1}^{\infty}\psi_{n-1}a_n z^n + \left(\sum_{n=1}^{\infty} p_n z^n\right)\left(\sum_{n=1}^{\infty}\psi_{n-1}a_n z^n\right).$$

By using the Cauchy product formula, we get

$$\sum_{n=1}^{\infty}([n,q]-1)\psi_{n-1}a_n z^n = \sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\psi_{j-1}a_j p_{n-j} z^n.$$

Comparing the coefficients of z^n, we have

$$([n-1,q])q\psi_{n-1}a_n = \sum_{j=1}^{n-1}\psi_{j-1}a_j p_{n-j},$$

which implies that

$$a_n = \frac{1}{[n-1,q]q\psi_{n-1}}\sum_{j=1}^{n-1}\psi_{j-1}a_j p_{n-j}.$$

Using (11), we have

$$|a_n| \leq \frac{|\delta_k|(C-D)(q+1)}{4[n-1,q]q\psi_{n-1}}\sum_{j=1}^{n-1}\psi_{j-1}|a_j|. \tag{12}$$

Now, we prove that

$$\frac{|\delta_k|(C-D)(q+1)}{4[n-1,q]q\psi_{n-1}}\sum_{j=1}^{n-1}\psi_{j-1}|a_j| \leq \prod_{j=0}^{n-2}\frac{|(C-D)(q+1)\delta_k\psi_j - 4Dq[j,q]\psi_j|}{4q[j+1,q]\psi_{j+1}},$$

To proceed for this proof, one may use the induction method.
For $n=2$, from (12), we get

$$|a_2| \leq \frac{|\delta_k|(C-D)(q+1)}{4[1,q]q\psi_1}\sum_{j=1}^{2-1}\psi_{j-1}|a_j|,$$

which reduces to

$$|a_2| \leq \frac{|\delta_k|(C-D)(q+1)}{4[1,q]q\psi_1},\ \psi_0 = 1.$$

From (8)

$$|a_2| \leq \frac{(C-D)(q+1)|\delta_k|}{4q[1,q]\psi_1}.$$

For $n=3$, from (12), we have

$$\begin{aligned}
|a_3| &\leq \frac{|\delta_k|(C-D)(q+1)}{4[2,q]q\psi_2}\sum_{j=1}^{2}\psi_{j-1}|a_j| \\
&= \frac{|\delta_k|(C-D)(q+1)}{4[2,q]q\psi_2}(\psi_0|a_1| + \psi_1|a_2|) \\
&\leq \frac{|\delta_k|(q+1)(C-D)}{4[2,q]q\psi_2}\left(1 + \frac{(q+1)(C-D)|\delta_k|}{4q[1,q]}\right).
\end{aligned}$$

From (8), we have

$$
\begin{aligned}
|a_3| &\leq \prod_{j=0}^{1} \frac{\left|(C-D)(q+1)\delta_k\psi_j - 4Dq[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}} \\
&= \frac{(C-D)(q+1)|\delta_k|}{4q[1,q]\psi_1}\left(\frac{(C-D)(q+1)|\delta_k|\psi_1 + 4q[1,q]\psi_1}{4q[2,q]\psi_2}\right) \\
&= \frac{|\delta_k|(q+1)(C-D)}{4[2,q]q\psi_2}\left(1 + \frac{(q+1)(C-D)|\delta_k|}{4q[1,q]}\right).
\end{aligned}
$$

Consider that the hypothesis is correct for $n = m+1$. Then, from (12), one may have

$$
|a_m| \leq \frac{|\delta_k|(C-D)(q+1)}{4[m-1,q]q\psi_{m-1}} \sum_{j=1}^{m-1} \psi_{j-1}|a_j|.
$$

From (8), we have

$$
|a_m| \leq \prod_{j=0}^{m-2} \frac{\left|(C-D)(q+1)\delta_k\psi_j - 4qD[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}}.
$$

By induction hypothesis,

$$
\prod_{j=0}^{m-2} \frac{\left|(C-D)(q+1)\delta_k\psi_j - 4qD[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}} \geq \frac{|\delta_k|(C-D)(q+1)}{4[m-1,q]q\psi_{m-1}} \sum_{j=1}^{m-1} \psi_{j-1}|a_j|,
$$

Multiplying both sides by $\frac{(C-D)(q+1)|\delta_k|\psi_{m-1} + 4q[m-1,q]\psi_{m-1}}{4q[m,q]\psi_m}$, we have

$$
\begin{aligned}
\prod_{j=0}^{m-2} \frac{\left|(C-D)(q+1)\delta_k\psi_j - 4qD[j,q]\psi_j\right|}{4q[j+1,q]\psi_{j+1}} &\geq \left(\frac{(C-D)(q+1)|\delta_k|\psi_{m-1} + 4q[m-1,q]\psi_{m-1}}{4q[m,q]\psi_m}\right) \\
&\quad \cdot \left(\frac{|\delta_k|(C-D)(q+1)}{4[m-1,q]q\psi_{m-1}} \sum_{j=1}^{m-1} \psi_{j-1}|a_j|\right) \\
&= \frac{|\delta_k|(C-D)(q+1)}{4[m,q]q\psi_m}\left(\psi_{m-1}\frac{|\delta_k|(C-D)(q+1)}{4[m-1,q]q\psi_{m-1}}\right. \\
&\quad \left.\sum_{j=1}^{m-1} \psi_{j-1}|a_j| + \sum_{j=1}^{m-1} \psi_{j-1}|a_j|\right) \\
&\geq \frac{|\delta_k|(C-D)(q+1)}{4[m,q]q\psi_m}\left(\psi_{m-1}|a_m| + \sum_{j=1}^{m-1} \psi_{j-1}|a_j|\right) \\
&= \frac{|\delta_k|(C-D)((q+1))}{4[m,q]q\psi_m} \sum_{j=1}^{m} \psi_{j-1}|a_j|.
\end{aligned}
$$

That is,

$$
\frac{|\delta_k|(C-D)}{2[m-1,q]q\psi_{m-1}} \sum_{j=1}^{m-1} \psi_{j-1}|a_j| \leq \prod_{j=0}^{m-2} \frac{\left|(C-D)\delta_k\psi_j - 2qD[j,q]\psi_j\right|}{2q[j+1,q]\psi_{j+1}}.
$$

Thus, the result holds for $n = m+1$. Consequently, by the induction principle, it is proved that (8) holds for all $n \geq 2$. □

As a special case, taking $\tau = 0$ gives the following already proved result (see [16]).

Corollary 2. *Let the function $h \in k - ST_q[C,D]$ be of the form* (1), *then*

$$|a_n| \leq \prod_{j=0}^{n-2} \frac{\left|(C-D)(q+1)\delta_k - 4qD[j]_q\right|}{4q[j+1]_q}, (n \in \mathbb{N}\backslash\{1\}). \tag{13}$$

3. Partial Sums

In this section, using the already proven results of Silverman [31] and Silvia [32] on partial sums of holomorphic functions, we examine the ratio of a function with the form (1) to its sequence of partial sums $h_n(z) = z + \sum_{j=2}^{n} a_j z^j$ when the function $h(z)$ has coefficients that are small enough to satisfy the inequality (6). We investigate sharp lower bounds for $\frac{h(z)}{h_n(z)}, \frac{h'(z)}{h_n'(z)}, \frac{h_n(z)}{h(z)}$ and $\frac{h_n'(z)}{h'(z)}$ for the functions of the class $k - ST_q^{\tau}[C,D]$.

Theorem 3. *If $h(z) \in k - ST_q^{\tau}[C,D]$, then*

$$\Re\left\{\frac{h(z)}{h_n(z)}\right\} \geq 1 - \frac{\varepsilon}{g_{n+1}}, \tag{14}$$

where g_{n+1} is defined by (7) *and $\varepsilon = (1+q)|D-C|$. The extremal function*

$$h(z) = z + \frac{\varepsilon}{g_{n+1}} z^{n+1}. \tag{15}$$

gives the sharp result.

Proof. Define a function $w(z)$ as

$$\begin{aligned} w(z) &= \frac{g_{n+1}}{\varepsilon}.\left[\frac{h(z)}{h_n(z)} - \left(1 - \frac{\varepsilon}{g_{n+1}}\right)\right] \\ &= \frac{g_{n+1}}{\varepsilon}\frac{h(z)}{h_n(z)} - \frac{g_{n+1}}{\varepsilon} + 1, \end{aligned}$$

which reduces to

$$\begin{aligned} w(z) &= \frac{g_{n+1}\left(1 + \sum_{j=2}^{\infty} a_j z^{j-1}\right)}{\varepsilon\left(1 + \sum_{j=2}^{n} a_j z^{j-1}\right)} - \frac{g_{n+1}}{\varepsilon} + 1 \\ &= \frac{1 + \sum_{j=2}^{n} a_j z^{j-1} + \frac{g_{n+1}}{\varepsilon}\sum_{j=n+1}^{\infty} a_j z^{j-1}}{1 + \sum_{j=2}^{n} a_j z^{j-1}}. \end{aligned}$$

Using this, one may have

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\frac{g_{n+1}}{\varepsilon}\sum_{j=n+1}^{\infty}|a_j|}{2 - 2\sum_{j=2}^{n}|a_j| - \frac{g_{n+1}}{\varepsilon}\sum_{j=n+1}^{\infty}|a_j|}.$$

Now

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq 1,$$

if

$$\sum_{j=2}^{n}|a_j| + \frac{g_{n+1}}{\varepsilon}\sum_{j=n+1}^{\infty}|a_j| \leq 1. \tag{16}$$

It would be sufficient to show that the left side of (16) has an upper bound $\sum_{j=2}^{\infty} \frac{g_j}{\varepsilon} |a_j|$, if

$$\sum_{j=2}^{n} |a_j| + \frac{g_{n+1}}{\varepsilon} \sum_{j=n+1}^{\infty} |a_j| \leq \sum_{j=2}^{\infty} \frac{g_j}{\varepsilon} |a_j|,$$

which leads to the following expression

$$\sum_{j=2}^{n} \left(\frac{g_j - \varepsilon}{\varepsilon} \right) |a_j| + \sum_{j=n+1}^{\infty} \left(\frac{g_j - g_{n+1}}{\varepsilon} \right) |a_j| \geq 0.$$

To justify the sharpness of the result, we see from the function given by (15) that for $z = re^{\frac{i\pi}{n}}$

$$\begin{aligned} \frac{h(z)}{h_n(z)} &= 1 + \frac{\varepsilon}{g_{n+1}} z^n \\ &= 1 + \frac{\varepsilon}{g_{n+1}} r^n e^{i\pi} \\ &= 1 - \frac{\varepsilon r^n}{g_{n+1}} \\ &= \frac{g_{n+1} - \varepsilon}{g_{n+1}} \text{ when } r \to 1. \end{aligned}$$

□

The following results are due to certain values of parameters, as proved in [31].

Corollary 3. *If* $h(z) \in 0 - ST_{1^-}^{0}[1 - 2\alpha, -1] \equiv S^*(\alpha), 0 \leq \alpha < 1$, *then*

$$\Re\left\{ \frac{h(z)}{h_n(z)} \right\} \geq \frac{n}{n - \alpha + 1}.$$

This bound is sharp and the following function gives the sharp bound:

$$h(z) = z + \frac{1 - \alpha}{n - \alpha + 1} z^{n+1}. \tag{17}$$

Now, setting $\alpha = 0$ and $n = 1$, we get the following result.

Corollary 4. *If* $h(z) \in 0 - ST_{1^-}^{0}[1, -1] \equiv S^*$, *then*

$$\Re\left\{ \frac{h(z)}{z} \right\} \geq \frac{1}{2}.$$

This bound is sharp and the following function gives the sharp bound:

$$h(z) = z + \frac{1}{2} z^2. \tag{18}$$

Theorem 4. *If* $h(z) \in k - ST_q^{\tau}[C, D]$, *then*

$$\Re\left\{ \frac{h_n(z)}{h(z)} \right\} \geq \frac{g_{n+1}}{g_{n+1} + \varepsilon}, \tag{19}$$

where g_{n+1} *is defined by* (7) *and* $\varepsilon = (1 + q)|D - C|$. *The bound* (19) *is best possible for the function, represented in* (15).

Proof. Define a function $w(z)$ as

$$\begin{aligned} w(z) &= \frac{g_{n+1}+\varepsilon}{\varepsilon}.\left[\frac{h_n(z)}{h(z)} - \frac{g_{n+1}}{g_{n+1}+\varepsilon}\right] \\ &= \frac{(g_{n+1}+\varepsilon)h_n(z)}{\varepsilon h(z)} - \frac{g_{n+1}}{\varepsilon}. \end{aligned}$$

This implies that

$$\begin{aligned} w(z) &= \frac{(g_{n+1}+\varepsilon)\left(1+\sum_{j=2}^{n} a_j z^{j-1}\right)}{\varepsilon\left(1+\sum_{j=2}^{\infty} a_j z^{j-1}\right)} - \frac{g_{n+1}}{\varepsilon} \\ &= \frac{1+\sum_{j=2}^{n} a_j z^{j-1} - \frac{g_{n+1}}{\varepsilon}\sum_{j=n+1}^{\infty} a_j z^{j-1}}{1+\sum_{j=2}^{\infty} a_j z^{j-1}}. \end{aligned}$$

This leads us to the following:

$$\frac{w(z)-1}{w(z)+1} = \frac{-\left(1+\frac{g_{n+1}}{\varepsilon}\right)\sum_{j=n+1}^{\infty} a_j z^{j-1}}{2+2\sum_{j=2}^{n} a_j z^{j-1} + \left(1-\frac{g_{n+1}}{\varepsilon}\right)\sum_{j=n+1}^{\infty} |a_j| z^{j-1}},$$

which implies that

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\left(1+\frac{g_{n+1}}{\varepsilon}\right)\sum_{j=n+1}^{\infty} |a_j|}{2-2\sum_{j=2}^{n} |a_j| - \left(1-\frac{g_{n+1}}{\varepsilon}\right)\sum_{j=n+1}^{\infty} |a_j|}.$$

Now

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq 1,$$

if

$$\sum_{j=2}^{n} |a_j| + \sum_{j=n+1}^{\infty} |a_j| \leq 1. \tag{20}$$

It would be enough to show that the left side of (20) has the upper bound $\sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|$, if

$$\sum_{j=2}^{n} |a_j| + \sum_{j=n+1}^{\infty} |a_j| \leq \sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|,$$

which leads to the following expression:

$$\sum_{j=2}^{n} \left(\frac{g_j}{\varepsilon} - 1\right)|a_j| + \sum_{j=n+1}^{\infty} \left(\frac{g_j}{\varepsilon} - 1\right)|a_j| \geq 0$$

That is,

$$\sum_{j=2}^{\infty} \left(\frac{g_j}{\varepsilon} - 1\right)|a_j| \geq 0.$$

Finally, equality holds for function $h(z)$, as presented in (15). □

The following results are due to certain values of parameters, as proved in [31].

Corollary 5. *If* $h(z) \in 0 - ST_{1^-}^{0}[1-2\alpha, -1] \equiv S^*(\alpha)$, $0 \leq \alpha < 1$, *then*

$$\Re\left\{\frac{h_n(z)}{h(z)}\right\} \geq \frac{n-\alpha+1}{n-2\alpha+2}.$$

This bound is sharp and the function defined by (17) gives the sharp bound.

Now, setting $n = 1$ and $\alpha = 0$, the following result is obtained.

Corollary 6. *If $h(z) \in 0 - ST^{0}_{1^-}[1, -1] \equiv S^*$, then*

$$\Re\left\{\frac{z}{h(z)}\right\} \geq \frac{2}{3}.$$

This bound is sharp and the function defined by (18) gives the sharp bound.

$$h(z) = z + \frac{1}{2}z^2.$$

Theorem 5. *If $h(z) \in k - ST^{\tau}_{q}[C, D]$, then*

$$\Re\left\{\frac{h'(z)}{h'_n(z)}\right\} \geq 1 - \frac{\varepsilon(n+1)}{g_{n+1}}, \tag{21}$$

where g_{n+1} is defined by (7) and $\varepsilon = (1+q)|D - C|$. The bound (21) is best possible for function, presented in (15).

Proof. Consider the function $w(z)$ as follows.

$$w(z) = \frac{g_{n+1}}{\varepsilon(n+1)} \cdot \left[\frac{h'(z)}{h'_n(z)} - \frac{g_{n+1} - \varepsilon(n+1)}{g_{n+1}}\right],$$

which becomes

$$\begin{aligned} w(z) &= \frac{g_{n+1}\left(1 + \sum_{j=2}^{\infty} j a_j z^{j-1}\right)}{\varepsilon(n+1)\left(1 + \sum_{j=2}^{n} j a_j z^{j-1}\right)} - \frac{(g_{n+1} - \varepsilon(n+1))}{\varepsilon(n+1)} \\ &= \frac{1 + \sum_{j=2}^{n} j a_j z^{j-1} + \frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j a_j z^{j-1}}{1 + \sum_{j=2}^{n} j a_j z^{j-1}}. \end{aligned}$$

This leads us to

$$\frac{w(z) - 1}{w(z) + 1} = \frac{\frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j a_j z^{j-1}}{2 + 2\sum_{j=2}^{n} j a_j z^{j-1} + \frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j a_j z^{j-1}},$$

which reduces to

$$\left|\frac{w(z) - 1}{w(z) + 1}\right| \leq \frac{\frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j|a_j|}{2 - 2\sum_{j=2}^{n} j|a_j| - \frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j|a_j|}.$$

Now

$$\left|\frac{w(z) - 1}{w(z) + 1}\right| \leq 1,$$

if

$$\sum_{j=2}^{n} j|a_j| + \frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j|a_j| \leq 1. \tag{22}$$

It would be enough to show that the left side of (22) has the upper bound $\sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|$, if

$$\sum_{j=2}^{n} j|a_j| + \frac{g_{n+1}}{\varepsilon(n+1)} \sum_{j=n+1}^{\infty} j|a_j| \leq \sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|,$$

which leads to the following expression

$$\sum_{j=2}^{n}\left(\frac{g_j}{\varepsilon}-j\right)|a_j|+\sum_{j=n+1}^{\infty}\left(\frac{g_j}{\varepsilon}-\frac{jg_{n+1}}{\varepsilon(n+1)}\right)|a_j|\geq 0.$$

□

The following results are due to certain values of parameters, as proved in [31].

Corollary 7. *If $h(z)\in 0-ST_{1^-}^{0}[1-2\alpha,-1]\equiv S^*(\alpha), 0\leq\alpha<1$, then*

$$\Re\left\{\frac{h'(z)}{h_n'(z)}\right\}\geq\frac{n\alpha}{n-\alpha+1}.$$

This bound is sharp and the function defined by (17) gives the sharp bound.

Now setting $n=1$ and $\alpha=0$, the following result is obtained:

Corollary 8. *If $h(z)\in 0-ST_{1^-}^{0}[1,-1]\equiv S^*$, then*

$$\Re\{h'(z)\}\geq 0.$$

This bound is sharp and the function defined by (18) gives the sharp bound.

$$h(z)=z+\frac{1}{2}z^2.$$

Theorem 6. *If $h(z)\in k-ST_q^{\tau}[C,D]$, then*

$$\Re\left\{\frac{h_n'(z)}{h'(z)}\right\}\geq\frac{g_{n+1}}{\varepsilon(n+1)+g_{n+1}}, \tag{23}$$

where g_{n+1} is defined by (7) and $\varepsilon=(1+q)|D-C|$. The bound (23) is sharp for the function presented by (15).

Proof. Consider $w(z)$ as

$$w(z)=\frac{\varepsilon(n+1)+g_{n+1}}{\varepsilon(n+1)}.\left[\frac{h_n'(z)}{h'(z)}-\frac{g_{n+1}}{\varepsilon(n+1)+g_{n+1}}\right],$$

which takes the form

$$\begin{aligned} w(z) &= \frac{(\varepsilon(n+1)+g_{n+1})\left(1+\sum_{j=2}^{n}ja_jz^{j-1}\right)}{\varepsilon(n+1)\left(1+\sum_{j=2}^{\infty}ja_jz^{j-1}\right)}-\frac{g_{n+1}}{\varepsilon(n+1)} \\ &= \frac{1+\sum_{j=2}^{n}ja_jz^{j-1}-\frac{g_{n+1}}{\varepsilon(n+1)}\sum_{j=n+1}^{\infty}ja_jz^{j-1}}{\left(1+\sum_{j=2}^{\infty}ja_jz^{j-1}\right)}. \end{aligned}$$

This leads us to

$$\begin{aligned} \frac{w(z)-1}{w(z)+1} &= \frac{\sum_{j=2}^{n}ja_jz^{j-1}-\sum_{j=2}^{\infty}ja_jz^{j-1}-\frac{g_{n+1}}{\varepsilon(n+1)}\sum_{j=n+1}^{\infty}ja_jz^{j-1}}{2+\sum_{j=2}^{n}ja_jz^{j-1}+\sum_{j=2}^{\infty}ja_jz^{j-1}-\frac{g_{n+1}}{\varepsilon(n+1)}\sum_{j=n+1}^{\infty}ja_jz^{j-1}} \\ &= \frac{-\sum_{j=n+1}^{\infty}\left(1+\frac{g_{n+1}}{\varepsilon(n+1)}\right)ja_jz^{j-1}}{2+2\sum_{j=2}^{n}ja_jz^{j-1}+\sum_{j=n+1}^{\infty}\left(1-\frac{g_{n+1}}{\varepsilon(n+1)}\right)ja_jz^{j-1}}. \end{aligned}$$

That is,

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{\left(1+\frac{g_{n+1}}{\varepsilon(n+1)}\right)\sum_{j=n+1}^{\infty} j|a_j|}{2-2\sum_{j=2}^{n} j|a_j| - \left(1-\frac{g_{n+1}}{\varepsilon(n+1)}\right)\sum_{j=n+1}^{\infty} j|a_j|}.$$

Now

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq 1,$$

if

$$\sum_{j=2}^{n} j|a_j| + \sum_{j=n+1}^{\infty} j|a_j| \leq 1. \tag{24}$$

Since the left side of (24) would be bounded above by $\sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|$, if

$$\sum_{j=2}^{n} j|a_j| + \sum_{j=n+1}^{\infty} j|a_j| \leq \sum_{j=2}^{\infty} \frac{g_j}{\varepsilon}|a_j|,$$

which can be written as

$$\sum_{j=2}^{n} j|a_j| + \sum_{j=n+1}^{\infty} j|a_j| \leq \sum_{j=2}^{n} \frac{g_j}{\varepsilon}|a_j| + \sum_{j=n+1}^{\infty} \frac{g_j}{\varepsilon}|a_j|,$$

which leads to the following expression:

$$\sum_{j=2}^{n}\left(\frac{g_j}{\varepsilon}-j\right)|a_j| + \sum_{j=n+1}^{\infty}\left(\frac{g_j}{\varepsilon}-j\right)|a_j| \geq 0.$$

That is,

$$\sum_{j=2}^{\infty}\left(\frac{g_j}{\varepsilon}-j\right)|a_j| \geq 0.$$

□

The following results are due to certain values of parameters, as proved in [31].

Corollary 9. *If* $h(z) \in 0 - ST_{1^-}^{0}[1-2\alpha, -1] \equiv S^*(\alpha), 0 \leq \alpha < 1$, *then*

$$\Re\left\{\frac{h_n'(z)}{h'(z)}\right\} \geq \frac{n-\alpha+1}{2n-2\alpha+2-n\alpha}.$$

This bound is sharp and the function defined by (17) gives the sharp bound.

Now, setting $n = 1$ and $\alpha = 0$, the following result is obtained.

Corollary 10. *If* $h(z) \in 0 - ST_{1^-}^{0}[1, -1] \equiv S^*$, *then*

$$\Re\left\{\frac{1}{h'(z)}\right\} \geq \frac{1}{2}.$$

This bound is sharp, and the function defined by (18) gives the sharp bound.

$$h(z) = z + \frac{1}{2}z^2.$$

In the next theorems, we will find the radii of starlikeness of order α for the class $k - ST_q^{\tau}[C, D]$.

Theorem 7. *Let $h(z) \in k - ST_q^{\tau}[C,D]$. Then $h(z)$ is a starlike of order $\alpha \in [0,1)$ in $|z| < r = r_1(\alpha)$, where*

$$r_1(\alpha) = \left(\frac{g_n(1-\alpha)}{\varepsilon\Big(q[n-1]_q + (1-\alpha)\Big)} \right)^{\frac{1}{n-1}}, \; n = 2.3, \ldots,$$

where g_n is defined by (7) and $\varepsilon = (1+q)|D-C|$.

Proof. Let $h(z) \in k - ST_q^{\tau}[C,D]$ be of the form (1). Then, from Theorem 16 gives

$$\sum_{n=2}^{\infty} \frac{g_n \psi_{n-1}}{\varepsilon} |a_n| < 1 \tag{25}$$

where g_n is defined by (7) and $\varepsilon = (1+q)|D-C|$. For $\alpha \in [0,1)$, we need to show that

$$\left| \frac{z\partial_q \varsigma_q^{\tau} h(z)}{\varsigma_q^{\tau} h(z)} - 1 \right| < 1 - \alpha;$$

that is,

$$\begin{aligned} \left| \frac{z\partial_q \varsigma_q^{\tau} h(z) - \varsigma_q^{\tau} h(z)}{\varsigma_q^{\tau} h(z)} \right| &= \left| \frac{-\sum_{n=2}^{\infty} \psi_{n-1} q[n-1]_q a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \psi_{n-1} a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} \psi_{n-1} q[n-1]_q |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \psi_{n-1} |a_n| |z|^{n-1}} \\ &< 1 - \alpha. \end{aligned}$$

Thus $\left| \frac{z\partial_q \varsigma_q^{\tau} h(z) - \varsigma_q^{\tau} h(z)}{\varsigma_q^{\tau} h(z)} \right| < 1 - \alpha$, if

$$\left(\frac{q[n-1]_q}{1-\alpha} + 1 \right) \psi_{n-1} |a_n| |z|^{n-1} < 1. \tag{26}$$

According to Theorem 16, (26) will be true if

$$\left(\frac{q[n-1]_q}{1-\alpha} + 1 \right) |z|^{n-1} < \frac{g_n}{\varepsilon}. \tag{27}$$

Now, solving (27) for $|z|$, we obtain

$$|z|^{n-1} < \frac{g_n(1-\alpha)}{\varepsilon\Big(q[n-1]_q + (1-\alpha)\Big)}. \tag{28}$$

Setting $|z| = r(\alpha)$ in (28), we get

$$r(\alpha) = \left(\frac{g_n(1-\alpha)}{\varepsilon\Big(q[n-1]_q + (1-\alpha)\Big)} \right)^{\frac{1}{n-1}}$$

as required. □

4. Conclusions

We have studied the q-version of the famous Ruscheweyh differential operator and applied it to define and study a new class $k - ST_q^{\tau}[C,D]$ of q-starlike functions related to the symmetric conic domain. This class generalizes the class $k - ST_q[C,D]$ which is defined in [16]. The study in [16] covers certain coefficient inequalities for q-starlike functions including coefficient bounds and sufficient conditions, which are obtained as a special case from the results, as proved above in this article. Using the same analogy of special cases, the results related to partial sums for the functions of class $k - ST_q^{\tau}[C, D]$ also give similar bounds for functions of class $k - ST_q[C, D]$, which have not been investigated to date.

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References

1. Jackson, F.H. On q-functions and certain difference operator. *Trans. Roy. Soc. Edinb.* **1908**, *46*, 253–281. [CrossRef]
2. Jackson, F.H. On q-definite integrals. *Q. J. Pure Appl. Maths.* **1910**, *41*, 193–203.
3. Aral, A.; Gupta, V. On q-Baskakov type operators. *Demonstr. Math.* **2009**, *42*, 109–122.
4. Aral, A.; Gupta, V. On the Durrmeyer type modification of the q-Baskakov type operators. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1171–1180. [CrossRef]
5. Aral, A.; Gupta, V. Generalized q-Baskakov operators. *Math. Slovaca* **2011**, *61*, 619–634. [CrossRef]
6. Anastassiou, G.A.; Gal, S.G. Geometric and approximation properties of some singular integrals in the unit disk. *J. Inequal. Appl.* **2006**, 17231. [CrossRef]
7. Aral, A. On the generalized Picard and Gauss weierstrass singular integrals. *J. Comput. Anal. Appl.* **2006**, *8*, 249–261.
8. Mohammad, A.; Darus, M. A generalized operator involving the q-hypergeometric function. *Mat. Vesnik* **2013**, *65*, 454–465.
9. Aldweby, H.; Darus, M. A subclass of harmonic univalent functions associated with q-analogue of Dziok-Srivastava operator. *Int. Schol. Res. Not.* **2013**, *2013*, 1–6. [CrossRef]
10. Aldweby, H.; Darus, M. On harmonic meromorphic functions associated with basic hypergeometric functions. *Sci. World J.* **2013**, *2013*, 1–7. [CrossRef]
11. Ahmad, B.; Khan, M.G.; Frasin, B.A.; Aouf, M.K.; Abdeljawad, T.; Mashwani, W.K.; Arif, M. On q-analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain. *AIMS Math.* **2021**, *6*, 3037–3052. [CrossRef]
12. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some General Classes of q-Starlike Functions Associated with the Janowski Functions. *Symmetry* **2019**, *11*, 292. [CrossRef]
13. Da Cruz, A.M.; Martins, N. The q-symmetric variational calculus. *Comput. Math. Appl.* **2012**, *64*, 2241–2250. [CrossRef]
14. Kamel, B.; Yosr, S. On some symmetric q-special functions. *Matematiche* **2013**, *68*, 107–122.
15. Khan, S.; Hussain, S.; Naeem, M.; Darus, M.; Rasheed, A. A Subclass of q-Starlike Functions Defined by Using a Symmetric q-Derivative Operator and Related with Generalized Symmetric Conic Domains. *Mathematics* **2021**, *9*, 917. [CrossRef]
16. Mahmood, S.; Jabeen, M.; Malik, S.N.; Srivastava, H.M.; Manzoor, R.; Riaz, S.M.J. Some Coefficient Inequalities of q-Starlike Functions Associated with Conic Domain Defined by q-Derivative. *J. Funct. Spaces* **2018**, *2018*, 1–13. doi: 10.1155/2018/8492072. [CrossRef]
17. Janowski, W. Some extremal problems for certain families of analytic functions. *Ann. Polon. Math.* **1973**, *28*, 297–326. [CrossRef]
18. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Maths. Appl.* **2011**, *62*, 2209–2217. [CrossRef]
19. Srivastava, H.M.; Khan, B.; Khan, N.; Zahoor, Q. Coefficients inequalities for q-starlike functions associated with Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407-425. [CrossRef]
20. Aktaş, İ.; Baricz, Á.; Orhan, H. Bounds for radii of starlikeness and convexity of some special functions. *Turkish J. Maths.* **2018**, *42*, 211–226. [CrossRef]
21. Aktaş, I.; Baricz, Á.; Yağmur, N. Bounds for the radii of univalence of some special functions. *Math. Inequal. Appl.* **2017**, *20*, 825–843.

22. Baricz, Á.; Dimitrov, D.; Orhan, H.; Yağmur, N. Radii of starlikeness of some special functions. *Proc. Am. Math. Soc.* **2016**, *144*, 3355–3367. [CrossRef]
23. Baricz, Á.; Kupan, P.A.; Szasz, R. The radius of starlikeness of normalized Bessel functions of the first kind. *Proc. Am. Math. Soc.* **2014**, *142*, 2019–2025. [CrossRef]
24. Aldweby, H.; Darus, M. Some Subordination Results on q-Analogue of Ruscheweyh Differential Operator. *Abstr. Appl. Anal.* **2014**, *2014*, 958563. [CrossRef]
25. Goodman, A.W. *Univalent Functions*; Polygonal Publishing House: Washington, NJ, USA, 1983; Volumes I and II.
26. Ruscheweyh, S. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
27. Kanas, S.; Wiśniowska, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
28. Kanas, S.; Wiśniowska, A. Conic domains and starlike functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
29. Kanas, S. Coefficient estimates in subclasses of the Caratheodory class related to conical domains. *Acta Math. Univ. Comen.* **2005**, *74*, 149–161.
30. Rogosinski, W. On the coefficients of subordinate functions. *Proc. Lond. Math. Soc.* **1943**, *48*, 48–82. [CrossRef]
31. Silverman, H. Partial sums of starlike and convex functions. *J. Math. Anal. Appl.* **1997**, *209*, 221–227. [CrossRef]
32. Silvia, E.M. Partial sums of convex functions of order α. *Houston J. Math.* **1985**, 11, 397–404.