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Several Integral Inequalities of Hermite–Hadamard Type Related to *k*-Fractional Conformable Integral Operators

Muhammad Tariq [1], Soubhagya Kumar Sahoo [2], Hijaz Ahmad [3,4,*] and Thanin Sitthiwirattham [5,*] and Jarunee Soontharanon [6]

[1] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan; 21phdamath05@students.muet.edu.pk
[2] Department of Mathematics, Institute of Technical Education and Research, Siksha O Anusandhan University, Bhubaneswar 751030, Odisha, India; soubhagyakumarsahoo@soa.ac.in
[3] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II 39, 00186 Roma, Italy
[4] Mathematics in Applied Sciences and Engineering Research Group, Scientific Research Center, Al-Ayen University, Nasiriyah 64001, Iraq
[5] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
[6] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jarunee.s@sci.kmutnb.ac.th
* Correspondence: f17ppbsi011@uetpeshawar.edu.pk (H.A.); thanin_sit@dusit.ac.th (T.S.)

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Abstract: In this paper, we present some ideas and concepts related to the *k*-fractional conformable integral operator for convex functions. First, we present a new integral identity correlated with the *k*-fractional conformable operator for the first-order derivative of a given function. Employing this new identity, the authors have proved some generalized inequalities of Hermite–Hadamard type via Hölder's inequality and the power mean inequality. Inequalities have a strong correlation with convex and symmetric convex functions. There exist expansive properties and strong correlations between the symmetric function and various areas of convexity, including convex functions, probability theory, and convex geometry on convex sets because of their fascinating properties in the mathematical sciences. The results of this paper show that the methodology can be directly applied and is computationally easy to use and exact.

Keywords: convexity; *k*-gamma function; Riemann–Liouville fractional integral; conformable integral; *k*-fractional conformable integral; E-beta functions; E-gamma functions

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

1. Introduction

Convex functions form a critical part in a few branches, such as mathematical inequalities, finance, engineering, statistics, and probability. Convex functions have a vital history and have been an intense topic of research for more than a century in sciences. Different speculations, expansions, and variations of the convex functions have been introduced by numerous researchers.

Definition 1 (See [1]). *A mapping $\Phi : \mathbb{J} \subseteq \mathbb{R} =]-\infty, \infty[\rightarrow \mathbb{R}$ is called convex if*

$$\Phi(\varrho \mathtt{b}_1 + (1-\varrho)\mathtt{b}_2) \leq \varrho\Phi(\mathtt{b}_1) + (1-\varrho)\Phi(\mathtt{b}_2) \tag{1}$$

holds, $\forall\, \mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ *and* $\varrho \in [0,1]$.

(Note: If the inequality in Equation (1) is reversed, then Φ is called to be concave.)

Employing convex functions, many inequalities or equalities have been proven by many authors; for example, Ostrowski-type inequality, Hardy-type inequality, Opial-type inequality, Simpson inequality, Fejer-type inequality, Cebysev-type inequalities. Among these inequalities, perhaps the one which takes the most consideration of researchers is the Hermite–Hadamard inequality on which many articles have been published.

Theorem 1 (See [2]). *A mapping $\Phi : \mathbb{J} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function in $\mathbb{J}$ and $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$, $\mathtt{b}_1 < \mathtt{b}_2$, then the $\mathrm{H}-\mathrm{H}$ inequality is expressed as follows:*

$$\Phi\left(\frac{\mathtt{b}_1+\mathtt{b}_2}{2}\right) \leq \frac{1}{\mathtt{b}_2-\mathtt{b}_1}\int_{\mathtt{b}_1}^{\mathtt{b}_2}\Phi(\varrho)d\varrho \leq \frac{\Phi(\mathtt{b}_1)+\Phi(\mathtt{b}_2)}{2}. \tag{2}$$

Since its discovery in 1883, Hermite–Hadamard's inequality has been considered as the most useful inequality in mathematical analysis. It is also known as traditional equation of $\mathrm{H}-\mathrm{H}$ Inequality. Several mathematicians have shown keen interest for the extensions and generalizations of Hermite–Hadamard inequality (2). A number of mathematicians in the field of pure and applied mathematics have devoted their work to extend, generalize, counterpart, and refine the Hermite–Hadamard inequality for different classes of convex functions and mappings. We refer the interested readers to [3–13]. Historically, the beginning of the idea of fractional calculus was credited to Leibnitz and L'Hospital (1695); however, critical improvements regarding the matter were presented later on by Riemann, Liouville, Grunwald-Letnikov, among others. Researchers are interested in the way that the speculation of fractional operator deciphers nature's existence in a grand and purposeful way [14–19]. Mubeen and Iqbal [20] added to the ongoing research by presenting the improved version of an integral representation for the Appell k-series.

The concept of fractional integral inequalities produced has a basic and important role in the mathematical field of sciences. Fractional calculus has a wide range of applications in various fields, such as magnetism, electricity, theory of viscoelasticity, lateral and longitudinal control of autonomous vehicles, numerical method, sound waves, heat transfer, fluid mechanics, propagation in human cancellous bone, propagation in rigid porous materials, cardiac tissue electrode interface, ultrasonic wave, wave propagation in viscoelastic horns, RLC electric circuit, etc. For the attraction of readers, see the references [21–24].

Symmetry, convexity and fractional operator have an extremely amazing association in light of their interesting properties. Whichever one we work on, it very well may be applied to the other one due to the strong relationship existing between them. The main aim of this work, excited and motivated by the ongoing research activities regarding this direction, is to investigate some Hermite–Hadamard type inequalities via the k-fractional conformable integral operator.

Definition 2 ([25]). *Suppose $\Phi \in \mathcal{L}_1[\mathtt{b}_1, \mathtt{b}_2]$ (sequence of all continuous spaces), then the left and right Riemann–Liouville fractional integrals $\mathfrak{J}^{\alpha}_{\mathtt{b}_1^+}\Phi$ and $\mathfrak{J}^{\alpha}_{\mathtt{b}_2^-}\Phi$ of order $\alpha \in \mathbb{C}$ $(\mathbb{R}(\alpha) > 0)$ with $\mathtt{b}_1, \mathtt{b}_2 \geq 0$ are defined by*

$$\mathfrak{J}^{\alpha}_{\mathtt{b}_1^+}\Phi(x) := \frac{1}{\Gamma(\alpha)}\int_{\mathtt{b}_1}^{x}(x-\varrho)^{\alpha-1}\Phi(\varrho)d\varrho, \qquad x > \mathtt{b}_1 \tag{3}$$

and

$$\mathfrak{J}^{\alpha}_{\mathtt{b}_2^-}\Phi(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathtt{b}_2}(\varrho-x)^{\alpha-1}\Phi(\varrho)d\varrho, \qquad x < \mathtt{b}_2 \tag{4}$$

respectively.

In addition,

$$\Gamma(\alpha) := \int_0^{\infty} e^{-\varrho}\varrho^{\alpha-1}\, d\varrho \ [\textit{Here } \alpha\Gamma(\alpha) = \Gamma(\alpha+1)],$$

where $\mathfrak{J}^{0}_{\mathtt{b}_1^+}\Phi(x) = \mathfrak{J}^{0}_{\mathtt{b}_2^-}\Phi(x) = \Phi(x)$.

Note: If $\alpha = 1$, then the Riemann–Liouville fractional integral reduces to the classical integral. The Beta function is defined as follows:

$$\beta(\mathtt{a}, \mathtt{b}) = \frac{\Gamma(\mathtt{a})\Gamma(\mathtt{b})}{\Gamma(\mathtt{a}+\mathtt{b})} = \int_0^1 \varrho^{\mathtt{a}-1}(1-\varrho)^{\mathtt{b}-1} d\varrho; \qquad \mathtt{a}, \mathtt{b} > 0.$$

Definition 3 ([26]). *If $k > 0$, then k-gamma function Γ_k is defined as*

$$\Gamma_k(\alpha) = \lim_{m\to\infty} \frac{m!\, k^m (mk)^{\frac{\alpha}{k}-1}}{(\alpha)_{m,k}}.$$

If $Re(\alpha) > 0$, the k-gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{\varrho^k}{k}} \varrho^{\alpha-1}\, d\varrho$$

with $\alpha\, \Gamma_k(\alpha) = \Gamma_k(\alpha + k)$. Here, $\Gamma_k(.)$ stands for the k-gamma function.

Sarikaya in [27] presented the Riemann–Liouville fractional version of the inequality 1. Soon after this article, many mathematicians generalized the Hermite–Hadamard inequality for various fractional operators employing several kinds of convexities (see [28–34]). In [30], Özdemir et al. proved a related identity and they found some new results by using this identity as follows:

Lemma 1. *Suppose a mapping $\Phi : \mathbb{J} = [\mathtt{b}_1, \mathtt{b}_2] \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}$ $\left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J}\right)$ where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ with $\mathtt{b}_1 < \mathtt{b}_2$. If $\Phi' \in \mathcal{L}[\mathtt{b}_1, \mathtt{b}_2]$, then for $\varrho \in [0, 1]$, and $\alpha > 0$, the following equality holds:*

$$\begin{aligned}
&\frac{(x-\mathtt{b}_1)^\alpha \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^\alpha \Phi(\mathtt{b}_2)}{\mathtt{b}_2 - \mathtt{b}_1} - \frac{\Gamma(\alpha+1)}{\mathtt{b}_2 - \mathtt{b}_1}\left\{\mathfrak{J}^\alpha_{x^-}\Phi(\mathtt{b}_1) + \mathfrak{J}^\alpha_{x^+}\Phi(\mathtt{b}_2)\right\} \\
&= \frac{(x-\mathtt{b}_1)^{\alpha+1}}{\mathtt{b}_2 - \mathtt{b}_1}\int_0^1 (\varrho^\alpha - 1)\Phi'(\varrho x + (1-\varrho)\mathtt{b}_1) d\varrho + \frac{(\mathtt{b}_2 - x)^{\alpha+1}}{\mathtt{b}_2 - \mathtt{b}_1}\int_0^1 (1-\varrho^\alpha)\Phi'(\varrho x + (1-\varrho)\mathtt{b}_2) d\varrho,
\end{aligned}$$

where $\Gamma(.)$ is defined as above.

Theorem 2. *Suppose a mapping $\Phi : [\mathtt{b}_1, \mathtt{b}_2] \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}$ $\left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J}\right)$ where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ with $\mathtt{b}_1 < \mathtt{b}_2$. If $\Phi' \in \mathcal{L}[\mathtt{b}_1, \mathtt{b}_2]$ and $|\Phi'|$ is s-convex function on $[\mathtt{b}_1, \mathtt{b}_2]$, then the following integral inequality holds:*

$$\begin{aligned}
&\left|\frac{(x-\mathtt{b}_1)^\alpha \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^\alpha \Phi(\mathtt{b}_2)}{\mathtt{b}_2 - \mathtt{b}_1} - \frac{\Gamma(\alpha+1)}{\mathtt{b}_2 - \mathtt{b}_1}\left\{\mathfrak{J}^\alpha_{x^-}\Phi(\mathtt{b}_1) + \mathfrak{J}^\alpha_{x^+}\Phi(\mathtt{b}_2)\right\}\right| \\
&\leq \frac{\alpha}{(s+1)(\alpha+s+1)}\left\{\frac{(x-\mathtt{b}_1)^{\alpha+1} + (\mathtt{b}_2 - x)^{\alpha+1}}{\mathtt{b}_2 - \mathtt{b}_1}\right\}\left|\Phi'(x)\right| \\
&+ \left(\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)\left\{\frac{(x-\mathtt{b}_1)^{\alpha+1}\left|\Phi'(\mathtt{b}_1)\right| + (\mathtt{b}_2 - x)^{\alpha+1}\left|\Phi'(\mathtt{b}_2)\right|}{\mathtt{b}_2 - \mathtt{b}_1}\right\}
\end{aligned} \tag{5}$$

where $\Gamma(.)$ is defined as above.

Theorem 3. *Suppose a mapping $\Phi : [\mathtt{b}_1, \mathtt{b}_2] \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}$ $\left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J}\right)$*

where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ *with* $\mathtt{b}_1 < \mathtt{b}_2$. *If* $\Phi' \in \mathcal{L}[\mathtt{b}_1, \mathtt{b}_2]$ *and* $|\Phi'|^q$ *is s-convex function on* $[\mathtt{b}_1, \mathtt{b}_2]$, *then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{(x-\mathtt{b}_1)^\alpha \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^\alpha \Phi(\mathtt{b}_2)}{\mathtt{b}_2 - \mathtt{b}_1} - \frac{\Gamma(\alpha+1)}{\mathtt{b}_2 - \mathtt{b}_1} \left\{ \mathfrak{J}^\alpha_{x^-} \Phi(\mathtt{b}_1) + \mathfrak{J}^\alpha_{x^+} \Phi(\mathtt{b}_2) \right\} \right| \\
&\quad \leq \left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}+1\right)}{\Gamma\left(\frac{1}{\alpha}+1+p\right)} \right)^{\frac{1}{p}} \left\{ \frac{(x-\mathtt{b}_1)^{\alpha+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left(\frac{\left|\Phi'(x)\right|^q + \left|\Phi'(\mathtt{b}_1)\right|^q}{s+1} \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \frac{(\mathtt{b}_2 - x)^{\alpha+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left(\frac{\left|\Phi'(x)\right|^q + \left|\Phi'(\mathtt{b}_2)\right|^q}{s+1} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\alpha > 0$ *and* Γ *is the Euler gamma function.*

In [20], the Riemann–Liouville k-fractional integrals are defined as

$$\mathfrak{J}^\alpha_{k,\mathtt{b}_1^+} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{\mathtt{b}_1}^x (x-\varrho)^{\frac{\alpha}{k}-1} \Phi(\varrho) d\varrho, \qquad x > \mathtt{b}_1 \tag{6}$$

and

$$\mathfrak{J}^\alpha_{k,\mathtt{b}_2^-} g(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{\mathtt{b}_2} (\varrho - x)^{\frac{\alpha}{k}-1} \Phi(\varrho) d\varrho, \qquad x < \mathtt{b}_2. \tag{7}$$

For $k = 1$, the k-fractional integrals gives the Riemann–Liouville integral. If we take $\alpha = k = 1$, the k-fractional integrals give the 'classical integrals'.

Theorem 4. *[27] Suppose a mapping* $\Phi : \mathbb{J} \subseteq \mathbb{R} \to \mathbb{R}$ *with* $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$, $\mathtt{b}_1 < \mathtt{b}_2$. *If* $\Phi \in \mathcal{L}[\mathtt{b}_1, \mathtt{b}_2]$ *is convex on the closed interval* $[\mathtt{b}_1, \mathtt{b}_2]$, *then the following inequality holds:*

$$\frac{\Phi(\mathtt{b}_1) + \Phi(\mathtt{b}_2)}{2} \geq \frac{\Gamma(\alpha+1)}{2(\mathtt{b}_2 - \mathtt{b}_1)^\alpha} \left[\mathfrak{J}^\alpha_{\mathtt{b}_1^+} \Phi(\mathtt{b}_2) + \mathfrak{J}^\alpha_{\mathtt{b}_2^-} \Phi(\mathtt{b}_1) \right] \geq \Phi\left(\frac{\mathtt{b}_1 + \mathtt{b}_2}{2} \right). \tag{8}$$

Note: If $\alpha = 1$, *then the inequality* (8) *reduces to* (2).

Theorem 5 ([29]). *Suppose a mapping* $\Phi : \mathbb{J} \subseteq \mathbb{R} \to \mathbb{R}$ *with* $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$, $\mathtt{b}_1 < \mathtt{b}_2$.*If* $\Phi \in \mathcal{L}[\mathtt{b}_1, \mathtt{b}_2]$ *is convex on the closed interval* $[\mathtt{b}_1, \mathtt{b}_2]$, *then the following inequality holds:*

$$\frac{\Phi(\mathtt{b}_1) + \Phi(\mathtt{b}_2)}{2} \geq \frac{k\Gamma_k(\lambda+k)\alpha^{\frac{\lambda}{k}}}{2(\mathtt{b}_2 - \mathtt{b}_1)^{\frac{\alpha\lambda}{k}}} \left[{}^{\lambda}_{k}\mathfrak{J}^\alpha_{\mathtt{b}_1^+} \Phi(\mathtt{b}_2) + {}^{\lambda}_{k}\mathfrak{J}^\alpha_{\mathtt{b}_2^-} \phi(\mathtt{b}_1) \right] \geq \Phi\left(\frac{\mathtt{b}_1 + \mathtt{b}_2}{2} \right), \tag{9}$$

for $\alpha, \lambda, k > 0$.
Note: If $\lambda = k = 1$, *then the inequality* (9) *reduces to* (2).

In this paper, we discuss some new ideas and develop several related inequalities of the Hadamard-type for such mapping whose differentiation is k-fractional conformable.

The most important definitions of the left and right FCIO (Fractional Conformable Integral Operators) are defined respectively in [35] as

$${}^{\lambda}\mathfrak{J}^\alpha_{\mathtt{b}_1^+} \Phi(x) = \frac{1}{\Gamma(\lambda)} \int_{\mathtt{b}_1}^x \left(\frac{(x-\mathtt{b}_1)^\alpha - (\varrho - \mathtt{b}_1)^\alpha}{\alpha} \right)^{\lambda-1} \frac{\Phi(\varrho)}{(\varrho - \mathtt{b}_1)^{1-\alpha}} d\varrho, x > \mathtt{b}_1 \tag{10}$$

and

$$ {}^{\lambda}\mathfrak{J}^{\alpha}_{\mathtt{b}_2-}\Phi\,(x)\;=\frac{1}{\Gamma(\lambda)}\int_x^{\mathtt{b}_2}\left(\frac{(\mathtt{b}_2-x)^{\alpha}-(\mathtt{b}_2-\varrho)^{\alpha}}{\alpha}\right)^{\lambda-1}\frac{\Phi(\varrho)}{(\mathtt{b}_2-\varrho)^{1-\alpha}}\,d\varrho, x<\mathtt{b}_2 \tag{11}$$

for $\alpha>0$ and $\mathbb{R}(\lambda)>0$. It is obvious that if we take $\mathtt{b}_1=0$ and $\alpha=1$, then (10) and (11) reduce to (3) and (4), respectively.

Now, the k-fractional conformable integral of generalized expression is defined in [28] as

$$ {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{\mathtt{b}_1+}\Phi(x)=\frac{1}{k\Gamma_k(\lambda)}\int_{\mathtt{b}_1}^{x}\left(\frac{(x-\mathtt{b}_1)^{\alpha}-(\varrho-\mathtt{b}_1)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}-1}\frac{\Phi(\varrho)}{(\varrho-\mathtt{b}_1)^{1-\alpha}}d\varrho,\ x>\mathtt{b}_1 \tag{12}$$

and

$$ {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{\mathtt{b}_2-}\Phi(x)=\frac{1}{k\Gamma_k(\lambda)}\int_{x}^{\mathtt{b}_2}\left(\frac{(\mathtt{b}_2-x)^{\alpha}-(\mathtt{b}_2-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}-1}\frac{\Phi(\varrho)}{(\mathtt{b}_2-\varrho)^{1-\alpha}}\,d\varrho,\ x<\mathtt{b}_2 \tag{13}$$

where $k>0$, $\alpha\in\mathbb{R}\setminus\{0\}$ and $\mathbb{R}(\lambda)>0$.

2. Main Results

Firstly, we propose a new lemma and, then, employing this lemma, some new integral inequalities of the Hadamard-type for k-fractional conformable integrals are apprehended.

Lemma 2. *Suppose a mapping $\Phi:\mathbb{J}=[\mathtt{b}_1,\mathtt{b}_2]\subseteq\mathbb{R}\to\mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}\left(\overset{o}{\mathbb{J}}\text{ is the interior of }\mathbb{J}\right)$ of an interval $\mathbb{J}$ in $\mathbb{R}$, where $\mathtt{b}_1,\mathtt{b}_2\in\mathbb{J}$, $\mathtt{b}_1<\mathtt{b}_2$. If $\Phi'\in\mathcal{L}_1[\mathtt{b}_1,\mathtt{b}_2]$ (sequence of all continuous spaces), then for $\varrho\in[0,1]$, $\alpha,\lambda\in\mathbb{R}^+$, and $k>0$, the following identity holds true.*

$$\begin{aligned}
&\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_1)+(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}}(\mathtt{b}_2-\mathtt{b}_1)}-\frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1}\left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x-}\Phi(\mathtt{b}_1)+{}^{\lambda}_{k}\,\mathfrak{J}^{\ \alpha}_{x+}\Phi(\mathtt{b}_2)\right]\\
&=\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\int_0^1\left[\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}}-\frac{1}{\alpha^{\frac{\lambda}{k}}}\right]\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)d\varrho\\
&+\frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\int_0^1\left[\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}}\right]\Phi'(\varrho x+(1-\varrho)\mathtt{b}_2)d\varrho.
\end{aligned} \tag{14}$$

Proof. It suffices to note that

$$\begin{aligned}
I_1&=\int_0^1\left[\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}}-\frac{1}{\alpha^{\frac{\lambda}{k}}}\right]\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)d\varrho\\
&=\int_0^1\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}}\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)d\varrho-\frac{1}{\alpha^{\frac{\lambda}{k}}}\int_0^1\ \Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)d\varrho.
\end{aligned}$$

By using integration by parts, we have

$$= \left(\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \frac{\Phi(\varrho x + (1-\varrho)\mathtt{b}_1)}{x-\mathtt{b}_1} \Bigg|_0^1 \right.$$

$$- \int_0^1 \frac{\Phi(\varrho x + (1-\varrho)\mathtt{b}_1)}{x-\mathtt{b}_1} \frac{\lambda}{k} \left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}-1} (1-\varrho)^{\alpha-1} d\varrho - \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} . \frac{\Phi(\varrho x + (1-\varrho)\mathtt{b}_1)}{x-\mathtt{b}_1} \Bigg|_0^1 \right.$$

$$I_1 = \frac{1}{\alpha^{\frac{\lambda}{k}}} \frac{\Phi(\mathtt{b}_1)}{x-\mathtt{b}_1} - \frac{\Gamma_k(\lambda+k)}{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}} {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1). \tag{15}$$

Similarly, one has

$$I_2 = \frac{1}{\alpha^{\frac{\lambda}{k}}} \frac{\Phi(\mathtt{b}_2)}{\mathtt{b}_2-x} - \frac{\Gamma_k(\ \lambda+k)}{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}} {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2). \tag{16}$$

Now by multiplying (15) by $\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}$ and (16) by $\frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}$ and adding these equations, we have the results of (14). □

Remark 1. *For k=1, in the above Lemma 2, the equality becomes Lemma 2.1 in [36]. If we choose $\alpha = k = 1$, then the above Lemma 2 becomes Lemma 1 in [30].*

Theorem 6. *Suppose a mapping $\Phi : [\mathtt{b}_1, \mathtt{b}_2] \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}} \left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J} \right)$ of the interval $\mathbb{J}$ in $\mathbb{R}$, where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ with $\mathtt{b}_1 < \mathtt{b}_2$. If $\Phi' \in \mathcal{L}_1[\mathtt{b}_1, \mathtt{b}_2]$ (sequence of all continuous spaces) and $|\Phi'|$ is convex function on $[\mathtt{b}_1, \mathtt{b}_2]$, then the following inequality for k–fractional conformable integral holds:*

$$\begin{aligned} &\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_1) + (\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2-\mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1} \left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1) + {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2) \right] \right| \\ &\le \left[\frac{1}{2} - \frac{1}{\alpha} \left\{ \beta\left(\frac{1}{\alpha}, \frac{\lambda}{k}+1 \right) - \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k}+1 \right) \right\} \right] \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1} + (\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2-\mathtt{b}_1)} \left| \Phi'(x) \right| \\ &+ \left[\frac{1}{2} - \frac{1}{\alpha} \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k}+1 \right) \right] \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1} \left| \Phi'(\mathtt{b}_1) \right| + (\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1} \left| \Phi'(\mathtt{b}_2) \right|}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2-\mathtt{b}_1)}, \end{aligned} \tag{17}$$

where $\beta(.\,,.)$ is the beta function, $k > 0$ and $\alpha, \lambda \in \mathbb{R}^+$.

Proof. By using Lemma 2 and the convexity of $|\Phi'|$, we obtain

$$\begin{aligned} &\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_1) + (\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2-\mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1} \left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1) + {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2) \right] \right| \\ &\le \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1} \int_0^1 \left[\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \right] \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_1) \right| d\varrho \\ &+ \frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1} \int_0^1 \left[\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \right] \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_2) \right| d\varrho. \end{aligned} \tag{18}$$

By using the convexity of $|\Phi'|$, we obtain

$$\int_0^1 \left[\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \right] \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_1) \right| d\varrho$$
$$\leq \frac{1}{\alpha^{\frac{\lambda}{k}}} \int_0^1 \left[1 - (1-(1-\varrho)^{\alpha})^{\frac{\lambda}{k}} \right] \left(\varrho \left| \Phi'(x) \right| + (1-\varrho) \left| \Phi'(\mathtt{b}_1) \right| \right) d\varrho.$$

After a change of variable with $y = (1-\varrho)^{\alpha}$, we obtain

$$\int_0^1 \varrho d\varrho - \int_0^1 \varrho (1-(1-\varrho)^{\alpha})^{\frac{\lambda}{k}} d\varrho = \frac{1}{2} - \frac{1}{\alpha} \left\{ \beta\left(\frac{1}{\alpha}, \frac{\lambda}{k} + 1 \right) - \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right\} \tag{19}$$

and

$$\int_0^1 (1-\varrho) d\varrho - \int_0^1 (1-\varrho)(1-(1-\varrho)^{\alpha})^{\frac{\lambda}{k}} d\varrho = \frac{1}{2} - \frac{1}{\alpha} \left\{ \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right\}. \tag{20}$$

Now, from Equations (19), (20) with (18), we obtain the desired result. □

Remark 2. *If $\alpha = k = 1$, then Theorem 6 becomes Theorem 7 in [30], when $s = 1$.*

Theorem 7. *Suppose a mapping $\Phi : [\mathtt{b}_1, \mathtt{b}_2] \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}$ $\left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J} \right)$ of the interval $\mathbb{J}$ in $\mathbb{R}$, where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ with $\mathtt{b}_1 < \mathtt{b}_2$. If $\Phi' \in \mathcal{L}_1[\mathtt{b}_1, \mathtt{b}_2]$ (sequence of all continuous spaces) and $|\Phi'|^q$ is convex function on $[\mathtt{b}_1, \mathtt{b}_2]$, where $q > 1$ and $q^{-1} + p^{-1} = 1$; $\alpha, \lambda \in \mathbb{R}^+$, then the following inequality for k-fractional conformable integral holds:*

$$\begin{aligned} &\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2 - \mathtt{b}_1)} - \frac{\Gamma_k(\lambda + k)}{\mathtt{b}_2 - \mathtt{b}_1} \left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1) + {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2) \right] \right| \\ &\leq \left[\frac{1}{\alpha^{\frac{\lambda p}{k}}} - \frac{1}{\alpha^{\frac{\lambda p}{k}}} \beta\left(\frac{1}{\alpha}, 1 + \frac{\lambda p}{k} \right) \right]^{\frac{1}{p}} \\ &\times \left\{ \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left(\frac{\left|\Phi'(x)\right|^q + \left|\Phi'(\mathtt{b}_1)\right|^q}{2} \right)^{\frac{1}{q}} + \frac{(\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left(\frac{\left|\Phi'(x)\right|^q + \left|\Phi'(v)\right|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned} \tag{21}$$

where β(. , .) is the Beta function, $k > 0$ and $\alpha, \lambda \in \mathbb{R}^+$.

Proof. By using Lemma 2 and the well-known Hölder inequality with convexity of $|\Phi'|^q$, we obtain

$$\begin{aligned} &\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2 - \mathtt{b}_1)} - \frac{\Gamma_k(\lambda + k)}{\mathtt{b}_2 - \mathtt{b}_1} \left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1) + {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2) \right] \right| \\ &\leq \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left[\int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \right)^p d\varrho \right]^{\frac{1}{p}} \left[\int_0^1 \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_1) \right|^q d\varrho \right]^{\frac{1}{q}} \\ &+ \frac{(\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left[\int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^{\alpha}}{\alpha} \right)^{\frac{\lambda}{k}} \right)^p d\varrho \right]^{\frac{1}{p}} \left[\int_0^1 \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_2) \right|^q d\varrho \right]^{\frac{1}{q}} \end{aligned} \tag{22}$$

Now, by the convexity of $|\Phi'|$

$$\int_0^1 \left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)\right|^q d\varrho \leq \frac{|\Phi'(x)|^q+|\Phi'(\mathtt{b}_1)|^q}{2} \tag{23}$$

and

$$\int_0^1 \left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_2)\right|^q d\varrho \leq \frac{|\Phi'(x)|^q+|\Phi'(\mathtt{b}_2)|^q}{2}. \tag{24}$$

Using the fact that $|m-n|^c \leq m^c - n^c$ and $c>1$; *as for* $m,n>0;\ m>n$.

$$\begin{aligned} \int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda}{k}}\right)^p d\varrho &\leq \frac{1}{\alpha^{\frac{\lambda p}{k}}} \int_0^1 1\, d\varrho - \int_0^1 \left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda p}{k}} d\varrho \\ &= \frac{1}{\alpha^{\frac{\lambda p}{k}}} - \frac{1}{\alpha^{\frac{\lambda p}{k}+1}}\beta\left(1+\frac{\lambda p}{k},\frac{1}{\alpha}\right). \end{aligned} \tag{25}$$

Now, substituting Equations (23)–(25) in (22), we obtain the desired result (21). □

Remark 3. *If $\alpha=k=1$, then Theorem 7 becomes Theorem 8 in [30], when $s=1$.*

Theorem 8. *Suppose a mapping $\Phi:[\mathtt{b}_1,\mathtt{b}_2]\to\mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}}$ $\left(\overset{o}{\mathbb{J}}\text{ is the interior of }\mathbb{J}\right)$ of the interval $\mathbb{J}$ in $\mathbb{R}$, where $\mathtt{b}_1,\mathtt{b}_2\in\mathbb{J}$ with $\mathtt{b}_1<\mathtt{b}_2$. If $\Phi'\in\mathcal{L}_1[\mathtt{b}_1,\mathtt{b}_2]$ (sequence of all continuous spaces) and $|\Phi'|^q$ is convex function on $[\mathtt{b}_1,\mathtt{b}_2]$, where $q\geq 1$ and $\alpha,\lambda\in\mathbb{R}^+$, then the following inequality for k-fractional conformable integral holds:*

$$\begin{aligned} &\left|\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}}\ \Phi(\mathtt{b}_1)+(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}}\ \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}}(\mathtt{b}_2-\mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1}\left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-}\ \Phi(\mathtt{b}_1)+{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+}\ \Phi(\mathtt{b}_2)\right]\right| \\ &\leq \left[\frac{1}{\alpha^{\frac{\lambda}{k}}} - \frac{1}{\alpha^{1+\frac{\lambda}{k}}}\beta\left(\frac{1}{\alpha},1+\frac{\lambda}{k}\right)\right]^{1-\frac{1}{q}} \\ &\times\left(\left\{\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left\{\frac{1}{\alpha^{\frac{\lambda}{k}}}\left(\left\{\frac{1}{2}-\frac{1}{\alpha}\left(\beta\left(\frac{1}{\alpha},\frac{\lambda}{k}+1\right)-\beta\left(\frac{2}{\alpha},\frac{\lambda}{k}+1\right)\right)\right\}\left|\Phi'(x)\right|^q\right.\right.\right.\right. \\ &\left.\left.\left.+\left\{\frac{1}{2}-\frac{1}{\alpha}\left(\beta\left(\frac{2}{\alpha},\frac{\lambda}{k}+1\right)\right)\right\}\left|\Phi'(\mathtt{b}_1)\right|^q\right)\right\}^{\frac{1}{q}} \\ &+\frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left\{\frac{1}{\alpha^{\frac{\lambda}{k}}}\left(\left\{\frac{1}{2}-\frac{1}{\alpha}\left(\beta\left(\frac{1}{\alpha},\frac{\lambda}{k}+1\right)-\beta\left(\frac{2}{\alpha},\frac{\lambda}{k}+1\right)\right)\right\}\left|\Phi'(x)\right|^q\right.\right. \\ &\left.\left.+\left\{\frac{1}{2}-\frac{1}{\alpha}\left(\beta\left(\frac{2}{\alpha},\frac{\lambda}{k}+1\right)\right)\right\}\left|\Phi'(\mathtt{b}_2)\right|^q\right)\right\}^{\frac{1}{q}}, \end{aligned} \tag{26}$$

where $\beta(.\,,.)$ is the beta function, $k>0$ and $\alpha,\lambda\in\mathbb{R}^+$.

Proof. Suppose that $q\geq 1$ and using Lemma 2 with the well-known power mean inequality, we obtain

$$\begin{aligned} &\left|\frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_1)+(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}}(\mathtt{b}_2-\mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1}\left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-}\Phi(\mathtt{b}_1)+{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+}\Phi(\mathtt{b}_2)\right]\right| \\ &\leq \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left[\int_0^1\left|\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda}{k}}\right|d\varrho\right]^{1-\frac{1}{q}} \\ &\times\left[\int_0^1\left|\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda}{k}}\right|\left|\Phi'(\varrho x+(1-\varrho)u)\right|^q d\varrho\right]^{\frac{1}{q}} \\ &+\frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left[\int_0^1\left|\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda}{k}}\right|d\varrho\right]^{1-\frac{1}{q}} \\ &\times\left[\int_0^1\left|\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^\alpha}{\alpha}\right)^{\frac{\lambda}{k}}\right|\left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_2)\right|^q d\varrho\right]^{\frac{1}{q}}. \end{aligned} \tag{27}$$

Using the convexity of $|\Phi'|^q$, we have

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^\alpha}{\alpha} \right)^{\frac{\lambda}{k}} \right) \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_1) \right|^q d\varrho \\
&\le \frac{1}{\alpha^{\frac{\lambda}{k}}} \int_0^1 \left(1 - (1-(1-\varrho)^\alpha)^{\frac{\lambda}{k}} \right) \left(\varrho \left| \Phi'(x) \right|^q + (1-\varrho) \left| \Phi'(\mathtt{b}_1) \right|^q \right) d\varrho \\
&\le \frac{1}{\alpha^{\frac{\lambda}{k}}} \left[\left\{ \frac{1}{2} - \frac{1}{\alpha} \left(\beta\left(\frac{1}{\alpha}, \frac{\lambda}{k} + 1 \right) - \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right) \right\} \left| \Phi'(x) \right|^q \right. \\
&\left. \quad + \left\{ \frac{1}{2} - \frac{1}{\alpha} \left(\beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right) \right\} \left| \Phi'(\mathtt{b}_1) \right|^q \right].
\end{aligned} \tag{28}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^\alpha}{\alpha} \right)^{\frac{\lambda}{k}} \right) \left| \Phi'(\varrho x + (1-\varrho)\mathtt{b}_2) \right|^q d\varrho \\
&\le \frac{1}{\alpha^{\frac{\lambda}{k}}} \left[\left\{ \frac{1}{2} - \frac{1}{\alpha} \left(\beta\left(\frac{1}{\alpha}, \frac{\lambda}{k} + 1 \right) - \beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right) \right\} \left| \Phi'(x) \right|^q \right. \\
&\left. \quad + \left\{ \frac{1}{2} - \frac{1}{\alpha} \left(\beta\left(\frac{2}{\alpha}, \frac{\lambda}{k} + 1 \right) \right) \right\} \left| \Phi'(\mathtt{b}_2) \right|^q \right]
\end{aligned} \tag{29}$$

and

$$\int_0^1 \left(\frac{1}{\alpha^{\frac{\lambda}{k}}} - \left(\frac{1-(1-\varrho)^\alpha}{\alpha} \right)^{\frac{\lambda}{k}} \right) d\varrho = \frac{1}{\alpha^{\frac{\lambda}{k}}} - \frac{1}{\alpha^{\frac{\lambda}{k}+1}} \, \beta\left(\frac{1}{\alpha}, \frac{\lambda}{k} + 1 \right). \tag{30}$$

Now, from Equations (28), (29) and (30) with (27), we obtain the desired result (26). □

Remark 4. *If we choose $\alpha = k = 1$, then Theorem 8 becomes Theorem 9 in [30], when $s = 1$.*

Theorem 9. *Suppose a mapping $\Phi : [\mathtt{b}_1, \mathtt{b}_2] \to \mathbb{R}$ is differentiable on $\overset{o}{\mathbb{J}} \left(\overset{o}{\mathbb{J}} \text{ is the interior of } \mathbb{J} \right)$ of the interval $\mathbb{J}$ in $\mathbb{R}$, where $\mathtt{b}_1, \mathtt{b}_2 \in \mathbb{J}$ with $\mathtt{b}_1 < \mathtt{b}_2$. If $\Phi' \in \mathcal{L}_1[\mathtt{b}_1, \mathtt{b}_2]$ and $|\Phi'|^q$ is concave function on $[\mathtt{b}_1, \mathtt{b}_2]$, where $q, p \ge 1$ and $\alpha, \lambda \in \mathbb{R}^+$, then the following inequality for k-fractional conformable integral holds:*

$$\begin{aligned}
&\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_1) + (\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}} \Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}} (\mathtt{b}_2 - \mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2 - \mathtt{b}_1} \left[{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^-} \Phi(\mathtt{b}_1) + {}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+} \Phi(\mathtt{b}_2) \right] \right| \\
&\le \left[\frac{1}{\alpha^{\frac{\lambda p}{k}}} - \frac{1}{\alpha^{1+\frac{\lambda p}{k}}} \beta\left(\frac{1}{\alpha}, 1 + \frac{\lambda p}{k} \right) \right]^{\frac{1}{p}} \\
&\times \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left| \Phi'\left(\frac{x+\mathtt{b}_1}{2} \right) \right| + \frac{(\mathtt{b}_2 - x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2 - \mathtt{b}_1} \left| \Phi'\left(\frac{x+\mathtt{b}_2}{2} \right) \right|,
\end{aligned} \tag{31}$$

where $\beta(.\,,.)$ is the beta function, $k > 0$ and $\alpha, \lambda \in \mathbb{R}^+$.

Proof. By using Lemma 2 and the well-known Hölder inequality with concavity of $|\Phi'|^q$, we obtain

$$\begin{aligned}
&\left| \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_1)+(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}}\Phi(\mathtt{b}_2)}{\alpha^{\frac{\lambda}{k}}(\mathtt{b}_2-\mathtt{b}_1)} - \frac{\Gamma_k(\lambda+k)}{\mathtt{b}_2-\mathtt{b}_1}\left[{}^{\lambda}_{k}\mathfrak{J}^{\ \alpha}_{x^-}\ \Phi(\mathtt{b}_1)+{}^{\lambda}_{k}\mathfrak{J}^{\alpha}_{x^+}\ \Phi(\mathtt{b}_2)\right]\right| \\
&\leq \frac{(x-\mathtt{b}_1)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left(\int_0^1\left(\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda p}{k}}\right)d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)\right|^q d\varrho\right)^{\frac{1}{q}} \\
&+\frac{(\mathtt{b}_2-x)^{\frac{\alpha\lambda}{k}+1}}{\mathtt{b}_2-\mathtt{b}_1}\left(\int_0^1\left(\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda p}{k}}\right)d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_2)\right|^q d\varrho\right)^{\frac{1}{q}}.
\end{aligned} \tag{32}$$

Using the fact that $|m-n|^c \leq m^c-n^c$ and $c>1\,;\,for\ m,n>0,\ m>n.$

$$\begin{aligned}
\int_0^1\left(\frac{1}{\alpha^{\frac{\lambda}{k}}}-\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda}{k}}\right)^{p}d\varrho \quad &\leq \frac{1}{\alpha^{\frac{\lambda p}{k}}}\int_0^1 1\,d\varrho-\int_0^1\left(\frac{1-(1-\varrho)^{\alpha}}{\alpha}\right)^{\frac{\lambda p}{k}}d\varrho \\
&= \frac{1}{\alpha^{\frac{\lambda p}{k}}}-\frac{1}{\alpha^{\frac{\lambda p}{k}+1}}\beta\left(1+\frac{\lambda p}{k},\frac{1}{\alpha}\right).
\end{aligned} \tag{33}$$

By the concavity of $|\Phi'|^q$, it can be observed that

$$\int_0^1\left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_1)\right|^q d\varrho \leq \left|\Phi'\left(\frac{x+\mathtt{b}_1}{2}\right)\right|^q$$

and

$$\int_0^1\left|\Phi'(\varrho x+(1-\varrho)\mathtt{b}_2)\right|^q d\varrho \leq \left|\Phi'\left(\frac{x+\mathtt{b}_2}{2}\right)\right|^q. \tag{34}$$

Now, substituting Equations (33) and (34) in (32), we obtain the desired result (31). □

3. Conclusions

On the advancement of this article, we have presented the idea of a k-fractional conformable integral operator. Embracing this original methodology, we inferred another equality that corresponds with some novel and notable trapezoid type inequalities in the literature. For right and proper selection of definitions, we are attaining several new k-fractional conformable integral inequalities. We derived the Hermite–Hadamard type inequalities and also designed some new equations. It is probable that notions and skills of this article may invite concerned readers. If we take $\alpha = k = 1$, we have similar results to those reported by Özdemir et al. [30]. This work demonstrates strategies that can be used to analyze many challenges in the area of conformable fractional calculus.

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