symmetry

MDPI

Article

Conditional Lie–Bäcklund Symmetries and Functionally Generalized Separation of Variables to Quasi-Linear Diffusion Equations with Source

Rui Wang and Lina Ji *

Department of Information and Computational Science, He'nan Agricultural University, Zhengzhou 450002, China; rhoda3221@henau.edu.cn

* Correspondence: jilina@henau.edu.cn

Received: 19 January 2020; Accepted: 6 May 2020; Published: 21 May 2020

Abstract: The conditional Lie–Bäcklund symmetry method is applied to investigate the functionally generalized separation of variables for quasi-linear diffusion equations with a source. The equations and the admitted conditional Lie–Bäcklund symmetries related to invariant subspaces are identified. The exact solutions possessing the form of the functionally generalized separation of variables are constructed for the resulting equations due to the corresponding symmetry reductions.

Keywords: conditional Lie–Bäcklund symmetry; separation of variables; invariant subspace; symmetry reduction; diffusion equation

1. Introduction

The separation of variables is the most widely used method for solving linear heat- and mass-transfer equations and other linear equations of mathematical physics. The productive separable solutions were presented in [1,2]. The additive separable solutions were constructed for some classes of nonlinear first-order partial differential equations (PDEs) in [3,4], and for some nonlinear heat- and mass-transfer and wave equations of the second and higher orders in [5,6]. The functional separation of variables (FSV)

$$f(u) = \phi(x) + \psi(t) \tag{1}$$

is sought for some nonlinear wave and heat equations. The solutions of this form were also obtained [7] for the nonlinear diffusion equation $u_t = [D(u)u_x]_x$.

The nonlinear separation of variables (NSV)

$$u = \phi(x)\psi(t) + \eta(t) \tag{2}$$

was introduced [8,9] for the study of the nonlinear parabolic equation. The generalized separation of variables (GSV)

$$u = \widetilde{C}_1(t)\widetilde{f}_1(x) + \widetilde{C}_2(t)\widetilde{f}_2(x) + \cdots + \widetilde{C}_n(t)\widetilde{f}_n(x) \ \ (n \geq 2) \tag{3}$$

can also provide exact solutions involving more summands, which was extensively discussed by the invariant subspace method [10].

The additive separable solutions and the productive separable solutions are both the particular case of the FSV (1) and the GSV (3). A further extension to the separable solutions is of the form

$$f(u) = C_1(t)f_1(x) + C_2(t)f_2(x) + \cdots + C_n(t)f_n(x) \ \ (n \geq 2), \tag{4}$$

which we call the functionally generalized separation of variables (FGSV). It can be regarded as a natural generalization of GSV in a similar way to how FSV is a generalization of the additive separation of variables. The more general separable solutions

$$f(u) = \phi_0(t) + \phi_1(t)g_1(x,t) + \phi_2(t)g_2(x,t) + \cdots + \phi_n(t)g_n(x,t)$$

are proposed due to the additional generating condition method in [11]. All these different types of separable solutions are very useful to study different kinds of PDEs. This is particularly true for nonlinear diffusion equations [10–13].

Conditional Lie–Bäcklund symmetry (CLBS) [14]

$$\rho = [f(u)]_{xt} \tag{5}$$

is introduced to study FSV (1). The key point is due to the compatibility of $\rho = 0$ and the governing equation. As a consequence, CLBS

$$\sigma = [f(u)]_{nx} + a_1(x)\,[f(u)]_{(n-1)x} + \cdots + a_n(x)f(u) \tag{6}$$

can be used to study FGSV (4). It is noted that $[f(u)]_{kx} = \partial^k f(u)/\partial x^k$ and $k = 1, 2, \cdots, n$.

A great number of publications have been devoted to study the quasi-linear diffusion equation with a source

$$u_t = [D(u)u_x]_x + Q(u),\ x \in R,\ t > 0, \tag{7}$$

where $D(u)$ and $Q(u)$ are respectively the diffusion and source term. The involved methods include the Lie-point symmetry method [15], the nonclassical symmetry method [16,17], the CLBS method [18–21], the nonlocal symmetry method [22], the truncated Painlevé approach [23], the differential constraints method [24], the sign-invariant and invariant subspace method [25,26], the transformation method [27,28], the ansatz-based method [29], the spectral volume method [30], etc.

The studies about FSV (1), NSV (2), and FGSV (4) of Equation (7) can respectively refer to [14,31], [31,32], and [10,31]. CLBS (6) is the key point to consider FGSV (4) of Equation (7). Equation (7) admits CLBS (6) is equivalent to saying that the equation

$$v_t = A(v)v_{xx} + B(v)v_x^2 + C(v) \tag{8}$$

admits the CLBS

$$\eta = v_{nx} + a_1(x)v_{(n-1)x} + \cdots + a_n(x)v. \tag{9}$$

In fact, Equation (8) can be derived from Equation (7) due to the transformation $v = f(u)$, and these two equations are related as

$$A(v) = D[g(v)],\ B(v) = A'(v) + \frac{g''(v)}{g'(v)}A(v),\ C(v) = \frac{Q[g(v)]}{g'(v)}, \tag{10}$$

where $u = g(v)$ is the inverse function of $v = f(u)$.

CLBS (9) is the key point to give a symmetry interpretation to the invariant subspace method proposed by Galaktionov and Svirshchevskii [10]. Since Equation (8) admits CLBS (9), it is easy to know that the corresponding group invariant solutions of Equation (8) are exactly defined on the linear solution space

$$S_n = S\{f_1(x), f_2(x), \cdots, f_n(x)\} \equiv \{\Sigma_{i=1}^n C_i f_i(x),\ C_i \in R\} \tag{11}$$

determined by the linear ordinary differential equation (ODE) $\eta = 0$. It is proved that the maximal dimension of the linear solution space is five for the second-order diffusion Equation (8) [33]. This confines oneself to study CLBS (9) with $2 \leq n \leq 5$ of Equation (8).

The classifications and reductions of the nonlinear diffusion equations with convection and the source

$$u_t = [D(u)u_x]_x + P(u)u_x + Q(u)$$

due to CLBS (6) were studied in [34], where the studies were concerned with the case of $P(u) \neq 0$. Indeed, some results can degenerate to the case of $P(u) = 0$. However, most of these degenerated cases correspond to x-independent $a_1(x)$ and x-independent $a_2(x)$. Here, we are mainly concerned with x-dependent $a_i(x)$ $(i = 1, 2)$.

The remainder of this paper is arranged as follows. In Section 2, the classification of Equation (7) due to CLBS (6) is presented. In the subsequent section, the FGSV (4) of Equation (7) is constructed. The last section is devoted to the conclusions.

2. Conditional Lie-Bäcklund Symmetry (6) of Equation (7)

Consider a nonlinear evolution equation

$$u_t = F(t, x, u, u_x, u_{2x}, \cdots, u_{kx}) \tag{12}$$

with Lie–Bäcklund vector field (LBVF)

$$V = \eta \frac{\partial}{\partial u} + D_x \eta \frac{\partial}{\partial u_x} + D_t \eta \frac{\partial}{\partial u_t} + D_x^2 \eta \frac{\partial}{\partial u_{2x}} + D_x D_t \eta \frac{\partial}{\partial u_{xt}} \cdots, \tag{13}$$

where $\eta = \eta(x, t, u, u_x, u_t, u_{2t}, u_{xt}, \cdots)$ is the characteristic of LBVF (13) and the total differentiation operators respectively denote

$$\begin{aligned} D_t &= \tfrac{\partial}{\partial t} + u_t \tfrac{\partial}{\partial u} + u_{2t} \tfrac{\partial}{\partial u_t} + u_{xt} \tfrac{\partial}{\partial u_x} + \cdots, \\ D_x &= \tfrac{\partial}{\partial x} + u_x \tfrac{\partial}{\partial u} + u_{xt} \tfrac{\partial}{\partial u_t} + u_{2x} \tfrac{\partial}{\partial u_x} + \cdots. \end{aligned}$$

Definition 1. *The evolutionary vector field (13) is said to be a Lie–Bäcklund symmetry of the evolution Equation (12) if the condition*

$$V\,(u_t - F)\,|_M = 0$$

holds, where M denotes the set of all differential consequences of the Equation (12).

Definition 2. *The evolutionary vector field (13) is said to be a CLBS of (12) if the following condition*

$$V\,(u_t - F)\,|_{M \cap L_x} = 0$$

holds, where L_x denotes the set of all differential consequences of equation $\eta = 0$ with respect to x.

The following proposition is useful for further study.

Proposition 1 ([18,19])**.** *Equation (8) admits the CLBS (9) if there exists a function $W(t, x, v, \eta)$ such that*

$$\frac{\partial \eta}{\partial t} = [F, \eta] + W(t, x, v, \eta), \quad W(t, x, v, 0) = 0,$$

where $[F,\eta] = F'\eta - \eta' F$ *and* $F[v] = A(v)v_{xx} + B(v)v_x^2 + C(v)$, *the prime denotes the Gateaux derivative, and W is an analytic function of* $t, x, v, v_x, \cdots$ *and* $\eta, D_x\eta, D_x^2\eta, \cdots$.

A direct conclusion of this proposition is that Equation (8) admits the CLBS with the characteristic (9) if

$$D_t\eta|_{L\cap M} = 0, \tag{14}$$

where M is the set of all differential consequences of the equation, that is,

$$v_t - F[v] = 0, \quad D_x^j D_t^k (v_t - F[v]) = 0, \quad j,k = 0,1,2,\cdots$$

and L denotes the set of all differential consequences of equation $\eta = 0$ with respect to x, that is,

$$D_x^j\eta = 0,\ j = 0,1,2,\cdots.$$

(14) is equivalent to

$$(F[v])_{nx} + a_1(x)\,(F[v])_{(n-1)x} + \cdots + a_n(x)\,(F[v])\,|_M = 0, \tag{15}$$

which is the right sufficient condition that the linear space (11) is invariant with respect to the operator $F[\cdot]$ [10].

Different cases will be considered respectively according to different cases of $n = 2,\ 3,\ 4,\ 5$. For $n = 2$, a direct computation from (15) gives a polynomial about v_x. The vanishing of all the coefficients yields the following over-determined system about the unknown functions in Equation (8) and the CLBS (9), which is listed as

$$\begin{aligned}
&B'' = 0,\\
&a_1(4B' + A'') = 0,\\
&2(a_1^2 - a_1' - a_2)A' - a_2[v(A'' + 5B') + B] + C'' + 2(a_1^2 - a_1')B = 0,\\
&(3a_1a_2 - 2a_2')vA' - (a_1'' - 2a_1a_1' + 2a_2')A + 2(2a_1a_2 - a_2')vB = 0,\\
&a_2^2v^2A' + (2a_1'a_2 - a_2'')vA + 2a_2^2v^2B + a_2(C - vC') = 0.
\end{aligned} \tag{16}$$

We can not give the general solutions of the determining system (16) since it is a coupled nonlinear system of PDEs. The workable way is to find as many particular solutions as possible.

The first equation of the determining system (16) gives $B(v) = b_1v + b_2$. As a consequence, the second one of (16) can be simplified as

$$a_1(4b_1 + A'') = 0.$$

Thus, we consider two cases for further study.

Case 1. $a_1(x) = 0$.

Substituting $B(v) = b_1v + b_2$ and $a_1(x) = 0$ into the system (16) and solving the resulting system, we can finally determine $A(v),\ B(v),\ C(v)$, and $a_2(x)$. The corresponding results are listed as

$$\begin{aligned}
&(i)\ A(v) arbitrary,\ B(v) = b_1v + b_2,\ C(v) = c_1v + c_2,\ a_2(x) = 0;\\
&(ii)\ A(v) arbitrary,\ B(v) = b_1v,\ C(v) = [s(b_1v^2 + A(v)) + c_1]v,\ a_2(x) = s(s \neq 0);\\
&(iii)\ A(v) = -\frac{1}{2}b_2v,\ B(v) = b_2,\ C(v) = c_1v,\ a_2''(x) + 3a_2^2(x) = 0(a_2(x) \neq s);\\
&(iv)\ A(v) = \frac{k_1}{v},\ B(v) = 0,\ C(v) = c_1v + c_2,\ k_1[a_2''(x) + a_2^2(x)] - c_2a_2(x) = 0(a_2(x) \neq s).
\end{aligned}$$

In [10], it is shown that the IS(11) is also invariant with respect to the operator $F[v] + pv$ if (11) is invariant with the operator $F[v]$. Thus, $v_t = F[v] + pv$ admits linear CLBS (9) if Equation (8) admits the linear CLBS (9). Hence, we can omit the linear term about v in $C(v)$. For the case (i), the second one of (10) becomes

$$\frac{g''(v)}{g'(v)} = \frac{b_1 v + b_2 - A'(v)}{A(v)}.$$

It is well known that the diffusion coefficient $D(u)$ is a power function or exponential function. In general, we consider the following five cases to identify the transformation $u = g(v)$, which are presented as

$$\begin{aligned}
&(1)\ A(v) = v^2,\ B(v) = kv\ (k \neq 1);\\
&(2)\ A(v) = v^2,\ B(v) = v;\\
&(3)\ A(v) = v^k\ (k \neq 1),\ B(v) = 0;\\
&(4)\ A(v) = v,\ B(v) = 0;\\
&(5)\ A(v) = v,\ B(v) = k\ (k \neq 0).
\end{aligned}$$

For the case of $A(v) = v^2,\ B(v) = kv\ (k \neq 1)$, the second one of (10) is simplified as

$$\frac{g''(v)}{g'(v)} = \frac{k-2}{v}.$$

$g(v) = v^{k-1}$ can be obtained by solving this equation, which is of the canonical coordinate form without considering the translation and scaling transformations. Consequently, $D(u) = u^{\frac{2}{k-1}}$ and $Q(u) = (k-1)qu^{\frac{k-2}{k-1}}$ are given from the first one and third one of (10) due to the transformation $u = v^{k-1}$. A similar discussion for Cases 2–5 will identify the corresponding governing Equation (7) and the admitted CLBS (6). The procedure of solving the system (16) for the case of (ii) is almost the same as that for the case of (i). Since the obtained $a_1(x)$ and $a_2(x)$ for these two cases are both x-independent and the resulting CLBS (6) and the governing Equation (7) are all special cases of the ones in [34], we do not consider these special cases for further study. So is the case for all the similar ones appearing below.

For the case of (iii), Equation (8) is invariant with respect to scaling transformation about x. Therefore, we consider $A(v) = v,\ B(v) = -2$ without loss of generality. The transformation $u = 1/v^2$ is obtained by solving the second one of (10). Consequently, $D(u) = u^{-\frac{1}{2}}$ and $Q(u) = 0$ are given. $a_2(x)$ satisfy the nonlinear ODE

$$a_2''(x) + 3a_2^2(x) = 0.$$

Now that the general solution of this ODE is in terms of the Weierstrass function, the solution of second-order linear ODE $v_{xx} + a_2(x)v = 0$ is not analytic. Hence, the analytic solution of Equation (8) can not be constructed due to the corresponding symmetry reductions. However, the particular CLBS (9) will lead to symmetry reductions of the governing Equation (8). Thus, we can satisfy our study by finding CLBS (9) in particular cases. Considering the form of the equation about $a_2(x)$, we apply the power law function constraint. $a_2(x) = -\frac{2}{x^2}$ is determined, which will yield the analytic solution of Equation (8). The corresponding result is listed as Case 9 for $s = -2$ in Table 1. The corresponding results for Case (iv) is also presented as Case 1 of Table 1.

Table 1. Conditional Lie–Bäcklund symmetry (CLBS) (6) for Equation (7).

No.	Equation (7)	CLBS (6)
1	$u_t = \left(u^{-\frac{1}{2}}u_x\right)_x$	$\sigma = \left(u^{\frac{1}{2}}\right)_{xx} - \frac{6}{x^2}u^{\frac{1}{2}}$
2	$u_t = \left(u^{\frac{1}{k}}u_x\right)_x + kqu^{\frac{k-1}{k}}$	$\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \frac{1}{x}\left(u^{\frac{1}{k}}\right)_x$
3	$u_t = [\exp(u)u_x]_x + q\exp(-u)$	$\sigma = [\exp(u)]_{xx} - \frac{1}{x}[\exp(u)]_x$
4	$u_t = \left(u^{-\frac{4}{3}}u_x\right)_x - \frac{3}{2}qu^{\frac{5}{3}}$	$\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} - \frac{1}{x}\left(u^{-\frac{2}{3}}\right)_x$
5	$u_t = \left(u^{\frac{1}{k}}u_x\right)_x + k(k+1)su^{\frac{k+1}{k}} + kqu^{\frac{k-1}{k}}$	$\sigma = \left(u^{\frac{1}{k}}\right)_{xx} + \sqrt{s}\tan\left(\sqrt{s}x\right)\left(u^{\frac{1}{k}}\right)_x$ $\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \sqrt{s}\cot\left(\sqrt{s}x\right)\left(u^{\frac{1}{k}}\right)_x$ $\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \sqrt{-s}\tanh\left(\sqrt{-s}x\right)\left(u^{\frac{1}{k}}\right)_x$ $\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \sqrt{-s}\coth\left(\sqrt{-s}x\right)\left(u^{\frac{1}{k}}\right)_x$
6	$u_t = [\exp(u)u_x]_x + s\exp(u) + q\exp(-u)$	$\sigma = [\exp(u)]_{xx} + \sqrt{s}\tan\left(\sqrt{s}x\right)[\exp(u)]_x$ $\sigma = [\exp(u)]_{xx} - \sqrt{s}\cot\left(\sqrt{s}x\right)[\exp(u)]_x$ $\sigma = [\exp(u)]_{xx} - \sqrt{-s}\tanh\left(\sqrt{-s}x\right)[\exp(u)]_x$ $\sigma = [\exp(u)]_{xx} - \sqrt{-s}\coth\left(\sqrt{-s}x\right)[\exp(u)]_x$
7	$u_t = \left(u^{-\frac{4}{3}}u_x\right)_x - \frac{3}{4}su^{-\frac{1}{3}} - \frac{3}{2}qu^{\frac{5}{3}}$	$\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} + \sqrt{s}\tan\left(\sqrt{s}x\right)\left(u^{-\frac{2}{3}}\right)_x$ $\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} - \sqrt{s}\cot\left(\sqrt{s}x\right)\left(u^{-\frac{2}{3}}\right)_x$ $\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} - \sqrt{-s}\tanh\left(\sqrt{-s}x\right)\left(u^{-\frac{2}{3}}\right)_x$ $\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} - \sqrt{-s}\coth\left(\sqrt{-s}x\right)\left(u^{-\frac{2}{3}}\right)_x$
8	$u_t = \left(u^{-\frac{4}{3}}u_x\right)_x$	$\sigma = \left(u^{-\frac{2}{3}}\right)_{xx} - \frac{2}{x}\left(u^{-\frac{2}{3}}\right)_x + \frac{2}{x^2}u^{-\frac{2}{3}}$ $\sigma = \left(u^{-\frac{4}{3}}\right)_{xx} - \frac{6}{x}\left(u^{-\frac{4}{3}}\right)_x + \frac{12}{x^2}u^{-\frac{4}{3}}$
9	$u_t = \left(u^{\frac{s}{2-s}}u_x\right)_x$	$\sigma = \left(u^{\frac{s}{2-s}}\right)_{xx} - \frac{s+2}{2x}\left(u^{\frac{s}{2-s}}\right)_x + \frac{s}{x^2}u^{\frac{s}{2-s}}$
10	$u_t = [\exp(u)u_x]_x$	$\sigma = [\exp(u)]_{xx} - \frac{2}{x}[\exp(u)]_x + \frac{2}{x^2}\exp(u)$
11	$u_t = (u^{-\frac{4}{3}}u_x)_x - \frac{3}{4}qu^{\frac{7}{3}}$	$\sigma = (u^{-\frac{4}{3}})_{xxx} - \frac{3}{x}(u^{-\frac{4}{3}})_{xx} + \frac{3}{x^2}(u^{-\frac{4}{3}})_x$
12	$u_t = (u^{-\frac{4}{3}}u_x)_x$	$\sigma = (u^{-\frac{4}{3}})_{xxx} - \frac{6}{x}(u^{-\frac{4}{3}})_{xx} + \frac{18}{x^2}(u^{-\frac{4}{3}})_x - \frac{24}{x^3}u^{-\frac{4}{3}}$
13	$u_t = (u^{-\frac{4}{3}}u_x)_x - \frac{3}{4}su^{-\frac{1}{3}} - \frac{3}{4}qu^{\frac{7}{3}}$	$\sigma = (u^{-\frac{4}{3}})_{xxx} + 3\sqrt{s}\tan\left(\sqrt{s}x\right)(u^{-\frac{4}{3}})_{xx} + \left[1 + 3\tan^2\left(\sqrt{s}x\right)\right]s(u^{-\frac{4}{3}})_x$ $\sigma = (u^{-\frac{4}{3}})_{xxx} - 3\sqrt{s}\cot\left(\sqrt{s}x\right)(u^{-\frac{4}{3}})_{xx} + \left[1 + 3\cot^2\left(\sqrt{s}x\right)\right]s(u^{-\frac{4}{3}})_x$ $\sigma = (u^{-\frac{4}{3}})_{xxx} - 3\sqrt{-s}\tanh\left(\sqrt{-s}x\right)(u^{-\frac{4}{3}})_{xx} + \left[1 - 3\tanh^2\left(\sqrt{-s}x\right)\right]s(u^{-\frac{4}{3}})_x$ $\sigma = (u^{-\frac{4}{3}})_{xxx} - 3\sqrt{-s}\coth\left(\sqrt{-s}x\right)(u^{-\frac{4}{3}})_{xx} + \left[1 - 3\coth^2\left(\sqrt{-s}x\right)\right]s(u^{-\frac{4}{3}})_x$

Case 2. $a_1(x) \neq 0$.

$A(v) = -2b_1v^2 + k_1v + k_2$ is given by solving $4b_1 + A'' = 0$. Since the power or exponential law diffusion is the usual phenomenon, $A(v)$ and $B(v)$ take the form of

$$(1)\ A(v) = v,\ B(v) = k\ (k \neq 0);$$
$$(2)\ A(v) = v,\ B(v) = 0;$$
$$(3)\ A(v) = v^2,\ B(v) = -\frac{1}{2}v.$$

Accordingly,

$$
\begin{aligned}
&(1)\ D(u) = u^{\frac{1}{k}},\ g(v) = v^k;\\
&(2)\ D(u) = \exp(u),\ g(v) = \ln(u);\\
&(3)\ D(u) = u^{-\frac{4}{3}},\ g(v) = v^{-\frac{3}{2}}
\end{aligned}
$$

will be derived due to the the first two equation of (10). For the case of $A(v) = v,\ B(v) = k\ (k \neq 0)$, $C(v) = c_1v^2 + c_2 + c_3$ can be derived from the third one of the determining system (16). Then, the last one of the system becomes

$$\left[-a_2''(x) + (2k+1)a_2^2(x) + 2a_2(x)a_2'(x) - c_1a_2(x)\right] v^2 + c_3a_2(x) = 0.$$

$a_2(x) = 0$ and $a_2(x) \neq 0$ will be respectively considered.

For the case of $a_2(x) = 0$, the determining system (16) is finally simplified as

$$
\begin{aligned}
&(k+1)\left[a_1^2(x) - a_1'(x)\right] + c_1 = 0,\\
&a_1''(x) - 2a_1(x)a_1'(x) = 0.
\end{aligned}
$$

It is easy to know that $a_1(x) = s$ and $c_1 = -(k+1)s^2$ satisfy the system. In addition, this system can be rewritten as $c_1 = -(k+1)s$ and

$$a_1(x)^2 - a_1'(x) = s.$$

The form of $a_1(x)$ can be represented as

- $a_1(x) = -\dfrac{1}{x}\ for\ s = 0;$
- $a_1(x) = \sqrt{s}\tan\left(\sqrt{s}x\right)\ or\ a_1(x) = -\sqrt{s}\cot\left(\sqrt{s}x\right)\ for\ s > 0;$
- $a_1(x) = -\sqrt{-s}\tanh\left(\sqrt{-s}x\right)\ or\ a_1(x) = -\sqrt{-s}\coth\left(\sqrt{-s}x\right)\ for\ s < 0.$

The corresponding Equation (7) and the admitted CLBS (6) are listed in Table 1.

For the case of $a_2(x) \neq 0$, $c_3 = 0$ must hold. The third one of (16) is simplified as

$$-(k+2)a_2(x) - 2(k+1)\left[a_1'(x) - a_1^2(x)\right] + 2c_1 = 0.$$

It becomes $c_1 + a_1'(x) - a_1^2(x) = 0$ in view of $k = -2$. $a_1(x)a_2(x) = 0$ can be derived from the fourth one of (16), which is contrary to $a_1(x) =\neq 0$ and $a_2(x) =\neq 0$. For the case of $k \neq -2$, the fourth equation of (16) becomes

$$(4k+3)\left[(k+2)a_1''(x) + 2(k+1)a_1^3(x) - 2(2k+3)a_1(x)a_1'(x) + 2c_1a_1(x)\right] = 0$$

due to

$$a_2(x) = \frac{-2(k+1)\left[a_1'(x) - a_1^2(x)\right] + 2c_1}{k+2}.$$

Thus, the fourth one of (16) holds for the case of $k = -\frac{3}{4}$. In addition, the last one becomes

$$
\begin{aligned}
&-5a_1'''(x) + 10a_1(x)a_1''(x) + 21a_1'^2(x) - 12a_1'(x)a_1^2(x)\\
&-53c_1a_1'(x) + 13c_1a_1^2(x) + a_1^4(x) = 0.
\end{aligned}
$$

Although it is impossible to present the general solution of this nonlinear ODE, we can find that

- $a_1(x) = l,\ a_2(x) = \dfrac{2}{9}l^2;$
- $a_1(x) = -\dfrac{5}{x}\ a_2(x) = \dfrac{8}{x^2}$ *and* $c_1 = 0;$
- $a_1(x) = -\dfrac{6}{x}\ a_2(x) = \dfrac{12}{x^2}$ *and* $c_1 = 0$

satisfy the system (16). For the case of $k \neq -\frac{3}{4}$, $a_1(x) = -\frac{s+2}{2x}$, $a_2(x) = \frac{s}{x^2}$ and $k = \frac{2-s}{s}$ can be obtained by solving the determining system (16). The corresponding Equation (7) and the admitted CLBS (6) for these cases are all listed in Table 1. We omit the tedious computational procedure for Cases (2) and (3) and just list the corresponding results in Table 1. It is noted that c, k, s, and q in Table 1 are arbitrary constants hereafter.

Although the determining system (16) is exactly the special case of the one in [34], we find many particular solutions of System (16) that are not special cases of the results listed in [34]. As a consequence, new forms of Equation (7) and the admitted CLBS (6) are obtained. A similar procedure as for $n = 2$ will yield CLBS (6) with $n = 3,\ 4,\ 5$ of Equation (7), and these results are also listed in Table 1.

It is interesting to note that some particular equations (including the ones in Cases 1, 8, and 10 of Table 1) admit several CLBSs. These different CLBSs of the same governing equation will naturally lead to different forms of exact solutions.

3. Exact Solutions of Equation (7)

In this section, we construct exact solutions of Equation (7) due to the compatibility of $\sigma = 0$ and the governing equation (7). $\sigma = 0$ is exactly the invariant surface condition of the corresponding CLBS (6) admitted by Equation (7). Since $\sigma = 0$ can be linearized due to the transformation $f(u) = v$, the corresponding solution is exactly defined on the space (11) and possesses the form

$$u = g\left[C_1(t)f_1(x) + C_2(t)f_2(x) + \cdots + C_n(t)f_n(x)\right] \quad (2 \leq n \leq 5),$$

which is known as FGSV. Substituting the resulting solution into (7) will finally yield that the t-dependent coefficients $C_i(t)$ $(i = 1, 2, \cdots, n)$ satisfy the finite-dimensional dynamical system. Here, we just present several examples to illustrate the reduction procedure.

Example 1. *Equation*

$$u_t = \left(u^{\frac{1}{k}}u_x\right)_x + kqu^{\frac{k-1}{k}}$$

admits the CLBS

$$\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \frac{1}{x}\left(u^{\frac{1}{k}}\right)_x.$$

The corresponding solutions are given by

$$u(x,t) = \left[\alpha(t) + \beta(t)x^2\right]^k,$$

where $\alpha(t)$ and $\beta(t)$ are listed as below.
(i) For $k \neq -1$,

$$\alpha(t) = \frac{[2(2k+1)t - c_1]q}{4(k+1)} + c_2[2(2k+1)t - c_1]^{-\frac{1}{2k+1}},\ \beta(t) = \frac{1}{c_1 - 2(2k+1)t}.$$

(ii) For $k = -1$,

$$\alpha(t) = \left[\frac{1}{2} q \ln(2t + c_1)\right] (2t + c_1),\ \beta(t) = \frac{1}{2t + c_1}.$$

For the case of $k < 0$, the solutions have the asymptotical behavior $u \longrightarrow 0$ as $x \longrightarrow \infty$ and exhibit singularity along the curves $x = \pm \left[(-\alpha/\beta)_+\right]^{1/2}$.

Example 2. *Equation*

$$u_t = \left(u^{\frac{1}{k}} u_x\right)_x + k(k+1) s u^{\frac{k+1}{k}} + k q u^{\frac{k-1}{k}}$$

admits the CLBS

$$\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \sqrt{s} \cot\left(\sqrt{s} x\right) \left(u^{\frac{1}{k}}\right)_x$$

for the case of $s > 0$ *and*

$$\sigma = \left(u^{\frac{1}{k}}\right)_{xx} - \sqrt{-s} \coth\left(\sqrt{-s} x\right) \left(u^{\frac{1}{k}}\right)_x$$

for the case of $s < 0$.

The corresponding separable solutions are respectively

$$u(x,t) = \left[\alpha(t) + \beta(t) \cos\left(\sqrt{s} x\right)\right]^k$$

and

$$u(x,t) = \left[\alpha(t) + \beta(t) \cosh\left(\sqrt{-s} x\right)\right]^k,$$

where $\alpha(t)$ and $\beta(t)$ satisfy two-dimensional dynamical system

$$\alpha' = (k+1) s \alpha^2 + k s \beta^2 + q,\ \beta' = (2k+1) s \alpha \beta.$$

For $s > 0$, the solutions are x-periodic with the period $2\pi/\sqrt{s}$.

Example 3. *Equation*

$$u_t = \left[\exp(u) u_x\right]_x + s \exp(u) + q \exp(-u)$$

admits the CLBS

$$\sigma = \left[\exp(u)\right]_{xx} + \sqrt{s} \tan\left(\sqrt{s} x\right) \left[\exp(u)\right]_x$$

for the case of $s > 0$ *and*

$$\sigma = \left[\exp(u)\right]_{xx} - \sqrt{-s} \tanh\left(\sqrt{-s} x\right) \left[\exp(u)\right]_x$$

for the case of $s < 0$.

The corresponding separable solutions are respectively

$$u(x,t) = \ln\left[\alpha(t) + \beta(t) \sin\left(\sqrt{s} x\right)\right]$$

and

$$u(x,t) = \ln\left[\alpha(t) + \beta(t)\sinh\left(\sqrt{-s}x\right)\right],$$

where $\alpha(t)$ and $\beta(t)$ satisfy two-dimensional dynamical system

$$\alpha' = s\alpha^2 + q,\ \beta' = s\alpha\beta.$$

$\alpha(t)$ and $\beta(t)$ are listed as below.
(i) For $qs > 0$,

$$\alpha(t) = \frac{\sqrt{qs}}{s}\tan\left[\sqrt{qs}(t+c_1)\right],\ \beta(t) = c_2\sec\left[\sqrt{qs}(t+c_1)\right].$$

(ii) For $qs < 0$,

$$\alpha(t) = -\frac{\sqrt{-qs}}{s}\tanh\left[\sqrt{-qs}(t+c_1)\right],\ \beta(t) = c_2\mathrm{sech}\left[\sqrt{-qs}(t+c_1)\right].$$

(iii) For $q = 0$,

$$\alpha(t) = \frac{1}{c_1 - st},\ \beta(t) = \frac{c_2}{c_1 - st}.$$

Example 4. *Equation*

$$u_t = \left(u^{\frac{s}{2-s}} u_x\right)_x$$

admits the CLBS

$$\sigma = \left(u^{\frac{s}{2-s}}\right)_{xx} - \frac{s+2}{2x}\left(u^{\frac{s}{2-s}}\right)_x + \frac{s}{x^2}u^{\frac{s}{2-s}}.$$

The corresponding separable solutions are given as follows.
(i) For $s = 4$,

$$u(x,t) = \left[\frac{c_2 + \frac{1}{2}\ln(c_1 - t)}{c_1 - t}x^2 + \frac{1}{c_1 - t}x^2\ln x\right]^{-\frac{1}{2}}.$$

(ii) For $s \neq 4$,

$$u(x,t) = \left\{\frac{1}{\frac{2(s-4)}{s}t + c_1}x^2 + c_2\left[\frac{2(s-4)}{s}t + c_1\right]^{\frac{s(s-6)}{8}} x^{\frac{s}{2}}\right\}^{\frac{2-s}{s}}.$$

Example 5. *Equation*

$$u_t = \left(u^{-\frac{4}{3}} u_x\right)_x$$

admits the CLBS

$$\sigma = \left(u^{-\frac{4}{3}}\right)_{xxx} - \frac{6}{x}\left(u^{-\frac{4}{3}}\right)_{xx} + \frac{18}{x^2}\left(u^{-\frac{4}{3}}\right)_x - \frac{24}{x^3}u^{-\frac{4}{3}}.$$

The corresponding solutions are given by

$$u(x,t) = \left[\frac{x^2 + c_2 x^3}{c_1 + t} + \left(\frac{c_2^2}{4(c_1+t)} + c_3(c_1+t)^2\right) x^4\right]^{-\frac{3}{4}}.$$

It is easy to see that the solutions have the asymptotical behavior $u \longrightarrow 0$ as $x \longrightarrow \infty$ and $u \longrightarrow 0$ as $t \longrightarrow +\infty$.

Example 6. *Equation*

$$u_t = \left(u^{-\frac{4}{3}} u_x\right)_x - \frac{3}{4} s u^{-\frac{1}{3}} - \frac{3}{4} q u^{\frac{7}{3}}$$

admits the CLBS

$$\sigma = \left(u^{-\frac{4}{3}}\right)_{xxx} - 3\sqrt{s}\cot\left(\sqrt{s}x\right)\left(u^{-\frac{4}{3}}\right)_{xx} + s\left[1 + 3\cot^2\left(\sqrt{s}x\right)\right]\left(u^{-\frac{4}{3}}\right)_x$$

for $s > 0$ and the CLBS

$$\sigma = \left(u^{-\frac{4}{3}}\right)_{xxx} - 3\sqrt{-s}\coth\left(\sqrt{-s}x\right)\left(u^{-\frac{4}{3}}\right)_{xx} + s\left[1 - 3\coth^2\left(\sqrt{-s}x\right)\right]\left(u^{-\frac{4}{3}}\right)_x$$

The corresponding solutions are given respectively as follows.
(i) For $s > 0$,

$$u(x,t) = \left[\alpha(t) + \beta(t)\cos\left(\sqrt{s}x\right) + \gamma(t)\cos\left(2\sqrt{s}x\right)\right]^{-\frac{3}{4}}.$$

(ii) For $s < 0$,

$$u(x,t) = \left[\alpha(t) + \beta(t)\cosh\left(\sqrt{-s}x\right) + \gamma(t)\cosh\left(2\sqrt{-s}x\right)\right]^{-\frac{3}{4}}.$$

The unknown functions in the solutions satisfy three-dimensional dynamical system

$$\begin{aligned}
\alpha' &= s\alpha^2 - \frac{3}{8}s\beta^2 - 3s\gamma^2 + q,\\
\beta' &= s\alpha\beta - 3s\beta\gamma,\\
\gamma' &= \frac{3}{8}s\beta^2 - 2s\alpha\gamma.
\end{aligned}$$

It is noted that the constant c_1 can be removed from all the resulting solutions because Equation (7) is invariant with respect to time translation. The exact solutions of Example 1 were found in [35] for the first time and later rediscovered in [11]. The exact solutions presented in Example 3 are also constructed due to conditional symmetry [36]. In addition, the exact solutions of Examples 2 and 3 are both particular cases of those derived in [11,31]. The exact solutions listed in Example 5 are obtainable from the dynamical system presented in [13].

4. Conclusions

We obtained the classification of the quasi-linear diffusion Equation (7) due to the CLBS method, which provided a symmetry interpretation to the FGSV. The variant forms of (7) admitting the CLBS (6) were presented. As a consequence, we constructed exact solutions in separable form (4) to the resulting equations due to the corresponding symmetry reductions. Most of these solutions could not be obtained by the other symmetry reduction methods.

This method could also be used to consider other types of PDEs, including KdV-type equations, fourth-order equations, etc. Moreover, it was also effective to deal with PDE systems, such as all kinds of diffusion systems. In addition, the discussion of multi-dimensional evolution equations, including all kinds of multi-dimensional diffusion equations in 2D, will be involved in our further study.

Author Contributions: All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Chinese National Natural Science Foundation (Grant No. 11501175) and Key project of Colleges and Universities in Henan Province (Grant No. 20A110017).

Acknowledgments: Our deepest gratitude goes to the anonymous reviewers for their careful work and thoughtful suggestions that have helped improve this paper substantially.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Tikhonov, A.N.; Samarskii, A.A. *Equations of Mathematical Physics*; Nauka: Moscow, Russian, 1972.
2. Vladimirov, V.S.; Zharinov, V.V. *Equations of Mathematical Physics*; Fizmatlit: Moscow, Russian, 2000.
3. Zwillinger, D. *Handbook of Differential Equations*; Academic: San Diego, CA, USA,1998.
4. Polyanin, A.D.; Zaitsev, V.F.; Moussiaux, A. *Handbook of First Order Partial Differential Equations*; Taylor and Fransis: London, UK, 2002.
5. Ibragimov, N.H. *CRC Handbook of Lie Group to Differential Equations*; CRC Press: Boca Raton, FL, USA, 1994.
6. Polyanin, A.D.; Vyaz'min, A.V.; Zhurov, A.I.; Kazemin, D.A. *Handbook of Exact solutions to Heat- and Mass-Transfer Equations*; Faktorial: Moscow, Russian, 1998.
7. Doyle, P.W.; Vassiliou, P.J. Separation of variables for the 1-dimensional nonlinear diffusion equation. *Int. J. Nonl. Mech.* **1998**, *33*, 315–326. [CrossRef]
8. Galaktionov, V.A. On new exact blow-up solutions for nonlinear heat equations with source and applications. *Diff. Int. Eq.* **1990**, *3*, 863–874.
9. Galaktionov, V.A.; Posashkov, S.A. New exact solutions to parabolic equations with quadratic nonlinearities. *USSR Comput. Math. Math. Phys.* **1989**, *29*, 112–119. [CrossRef]
10. Galaktionov, V.A.; Svirshchevski, S. *Exact solutions and Invariant subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics*; Chapman and Hall: London, UK, 2007.
11. Cherniha, R. New non-Lie ansätze and exact solutions of nonlinear reaction-diffusion-convection equations. *J. Phys. A Math. Gen.* **1998**, *31*, 8179–8198. [CrossRef]
12. Samarskii, A.A.; Galaktionov, V.A.; Kurdyumov, S.P.; Mikhailov, A.P. *Blow-Up in Quasilinear Parabolic Equations*; Walter de Gruyter Co.: Berlin, Germany, 1995.
13. King, J.R. Exact polynomial solutions to some nonliner diffusion equations. *Phys. D* **1993**, *64*, 35–65. [CrossRef]
14. Qu, C.Z.; Zhang, S.L.; Liu, R.C. Separation of variables and exaxt solutions to quasilinear diffusion equations with nonlinear source. *Phys. D* **2000**, *144*, 97–123. [CrossRef]
15. Hill, J.M.; Avagliano, A.J.; Edwards, M.P. Some exact results for nonlinear diffusion with absorption. *IMA J. Appl. Math.* **1992**, *48*, 283–304. [CrossRef]
16. Clarkson, P.A.; Mansfield, E.L. symmetry reductions and exact solutions of a class of nonlinear heat equations. *Phys. D* **1993**, *70*, 250–288. [CrossRef]
17. Arrigo, D.J.; Hill, J.M. Nonclassical symmetries for nonlinear diffusion and absorption. *Stud. Appl. Math.* **1995**, *94*, 21–39. [CrossRef]
18. Zhdanov, R.Z. Conditional Lie–Bäcklund symmetry and reduction of evolution equation. *J. Phys. A* **1995**, *28*, 3841–3850. [CrossRef]
19. Fokas, A.S.; Liu, Q.M. Nonlinear interaction of traveling waves of nonintegrable equations. *Phys. Rev. Lett.* **1994**, *72*, 3293–3296. [CrossRef]
20. Qu, C.Z. Group classification and generalized conditional symmetry reduction of the nonlinear diffusion-convection equation with a nonlinear source. *Stud. Appl. Math.* **1997**, *99*, 107–136. [CrossRef]
21. Qu, C.Z. Exact solutions to nonlinear diffusion equations obtained by generalized conditional symmetry. *IMA J. Appl. Math.* **1999**, *62*, 283–302.

22. Bluman, G.W.; Reid, J.D.; Kumei, S. New classes of symmetries for partial differential equations. *J. Math. Phys.* **1998**, *29*, 806–812. [CrossRef]
23. Newell, A.C.; Tabor, M.; Zeng, Y.B. A unified approach to Painlevé expansions. *Phys. D* **1987**, *29*, 1–68. [CrossRef]
24. Kaptsov, O.V.; Verevkin, I.V. Differential constraints and exact solutions of nonlinear diffusion equations. *J. Phys. A Math. Gen.* **2003**, *36*, 1401–1414. [CrossRef]
25. Galaktionov, V.A.; Posashkov, S.A. New explicit solutions of quasi-linear heat equations with general first-order sign-invariants. *Phys. D* **1996**, *99*, 217–236. [CrossRef]
26. Galaktionov, V.A. Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities, *Proc. R. Soc. Edinb.* **1995**, *125*, 225–246. [CrossRef]
27. King, J.R. Exact similarity solutions to some nonlinear diffusion equations. *J. Phys. A Math. Gen.* **1990**, *23*, 3681–3697. [CrossRef]
28. Burgan, J.R.; Munier, A.; Feix, M.R.; Fijalkow, E. Homology and the nonlinear heat diffusion equation. *SIAM J. Appl. Math.* **1984**, *44*, 11–18. [CrossRef]
29. Hood, S. On direct, implicit reductions of a nonlinear diffusion equation with an arbitrary function-generalizations of Clarkson's and Kruskal's method. *IMA J. Appl. Math.* **2000**, *64*, 223–244. [CrossRef]
30. Kannan, R.; Zhang, Z.J. A study of viscous flux formulations for a p-multigrid spectral volume Navier Stokes solver. *J. Sci. Comp.* **2009**, *41*, 165–199. [CrossRef]
31. Cherniha, R.; Serov, M.; Pliukhin, O. *Nonlinear Reaction-Diffusion-Convection Equations: Lie and Conditional Symmetry, Exact Solutions and Their Applications*; Chapman and Hall/CRC Press: New York, NY, USA, 2018.
32. Bertsch, M.; Kersner, R.; Peletier, L.A. Positivity versus localization in degenerate diffusion equations. *Nonl. Anal. TMA* **1985**, *9*, 987–1008. [CrossRef]
33. Svirshchevskii, S.R. Nonlinear differential operator of first and second order possessing invarinant linear spaces of maximal dimension. *Theor. Math. Phys.* **1995**, *105*, 1346–1353. [CrossRef]
34. Ji, L.N.; Qu, C.Z. Conditional Lie–Bäcklund symmetries and invariant subspaces to nonlinear diffusion equations with convection and source. *Stud. Appl. Math.* **2013**, *131*, 266–301. [CrossRef]
35. Kersner, R. Some properties of generalized solutions of quasilinear degenerate parabolic equations. *Acta Math. Acad. Sci. Hungar.* **1978**, *32*, 301–330.
36. Cherniha, R.; Pliukhin, O. New conditional symmetries and exact solutions of reaction-diffusion-convection equations with exponential nonlinearities. *J. Math. Anal. Appl.* **2013**, *403*, 23–37. [CrossRef]