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Topological Symmetry Groups of the Heawood Graph

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Abstract: We classify all groups which can occur as the topological symmetry group of some embedding of the Heawood graph in S^3.

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MSC: 57M25; 57M15; 57M27; 92E10; 05C10

1. Introduction

Topological symmetry groups were originally introduced to classify the symmetries of non-rigid molecules. In particular, the symmetries of rigid molecules are represented by the *point group*, which is the group of rigid motions of the molecule in space. However, non-rigid molecules can have symmetries which are not included in the point group. The symmetries of such molecules can instead be represented by the subgroup of the automorphism group of the molecular graph which are induced by homeomorphisms of the ambient space. In this way, the molecular graph is treated as a topological object, and hence this group is referred to as the *topological symmetry group* of the graph in space.

Although initially motivated by chemistry, the study of topological symmetry groups of graphs embedded in S^3 can be thought of as a generalization of the study of symmetries of knots and links. Various results have been obtained about topological symmetry groups in general ([1–4]) as well as topological symmetry groups of embeddings of particular graphs or families of graphs in S^3 ([5–10]).

In this paper, we classify the topological symmetry groups of embeddings of the Heawood graph in S^3, whose (combinatorial) automorphism group is $\mathrm{PGL}(2,7)$. This graph, denoted by C_{14}, is illustrated in Figure 1. The Heawood graph is of interest to topologists because it is obtained from the intrinsically knotted graph K_7 by what are known as "$\Delta - Y$" moves. Such moves alter the graph by replacing three edges that form a triangle by three edges in the form of the letter Y with a new 3-valent vertex in the center. Since $\Delta - Y$ moves preserve intrinsic knotting [11], the Heawood graph is intrinsically knotted. This means that every embedding of C_{14} in S^3 contains a non-trivial knot. It also follows from [12] that C_{14} is *intrinsically chiral*, that is, no embedding of C_{14} in S^3 has an orientation reversing homeomorphism.

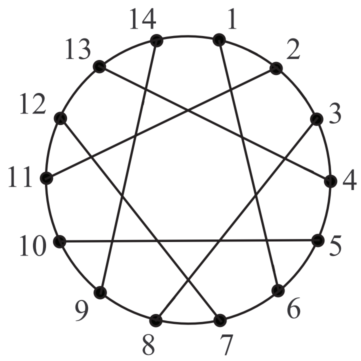

Figure 1. The Heawood graph, which we denote by C_{14}.

We begin with some terminology.

Definition 1. *Let Γ be a graph embedded in S^3. We define the* **topological symmetry group** $\mathrm{TSG}(\Gamma)$ *as the subgroup of the automorphism group* $\mathrm{Aut}(\Gamma)$ *induced by homeomorphisms of* (S^3, Γ)*. We define the* **orientation preserving topological symmetry group** $\mathrm{TSG}_+(\Gamma)$ *as the subgroup of* $\mathrm{Aut}(\Gamma)$ *induced by orientation preserving homeomorphisms of* (S^3, Γ)*.*

Definition 2. *Let G be a group and let γ denote an abstract graph. If there is some embedding Γ of γ in S^3 such that $\mathrm{TSG}(\Gamma) = G$, then we say that G is* **realizable** *for γ. If there is some embedding Γ of γ in S^3 such that $\mathrm{TSG}_+(\Gamma) = G$, then we say that the group G is* **positively realizable** *for γ.*

Definition 3. *Let φ be an automorphism of an abstract graph γ. We say φ is* **realizable** *if for some embedding Γ of γ in S^3, the automorphism φ is induced by a homeomorphism of (S^3, Γ). If such a homeomorphism exists which is orientation preserving, then we say φ is* **positively realizable***.*

Since the Heawood graph is intrinsically chiral, a group is realizable if and only if it is positively realizable. Our main result is the following classification theorem.

Theorem 1. *A group G is realizable as the topological symmetry group of an embedding of C_{14} if and only if G is the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, D_3, or D_7.*

In Section 2, we present some background material about C_{14}. In Section 3, we determine which of the automorphisms of C_{14} are realizable. We then use the results of Section 3 to prove our main result in Section 4.

2. Background About the Heawood Graph

We will be interested in the action of automorphisms of C_{14} on cycles of particular lengths. The graph C_{14} has 28 6-cycles, its shortest cycles, and 24 14-cycles [13,14]. The following results about the 12-cycles and 14-cycles of C_{14} are proved in the paper [15]. While some of these results may be well known, the authors could not find proofs in the graph theory literature.

Lemma 1. *([15])*

1. *C_{14} has 56 12-cycles.*
2. *$\mathrm{Aut}(C_{14})$ acts transitively on the set of 14-cycles and the set of 12-cycles.*
3. *The graph obtained from C_{14} by removing any pair of vertices which are a distance 3 apart has exactly two 12-cycles.*

By part (2) of Lemma 1, we can assume that any 14-cycle in C_{14} looks like the outer circle in Figure 1 and any 12-cycle looks like the round circle in Figure 2. We will always label the vertices of C_{14} either as in Figure 1 or as in Figure 2.

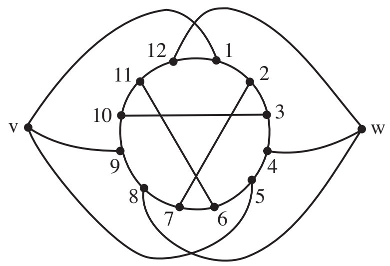

Figure 2. Any 12-cycle looks like the round circle in this illustration.

The automorphism group of C_{14} is isomorphic to the projective linear group $\mathrm{PGL}(2,7)$ whose order is $336 = 2^4 \times 3 \times 7$ ([13]). The program Magma was used to determine that all of the non-trivial elements of $\mathrm{PGL}(2,7)$ have order 2, 3, 4, 6, 7, and 8. The following lemma gives us information about the action of automorphisms with order 3 and 7 on the 12-cycles and 14-cycles of C_{14}.

Lemma 2. *Let α be an automorphism of C_{14}. Then the following hold.*

1. *If α has order 7, then α setwise fixes precisely three 14-cycles, rotating each by $\frac{2\pi n}{7}$ for some $n < 7$, when considered as a round circle (see Figure 1).*
2. *If α has order 3, then α fixes precisely two vertices and setwise fixes precisely two 12-cycles in their complement, rotating each by $\pm\frac{2\pi}{3}$, when considered as a round circle (see Figure 2).*

Proof. (1) Suppose that the order of α is 7. Since C_{14} has 24 14-cycles, α must setwise fix at least three of them. Observe that any 14-cycle which is setwise fixed by α must be rotated by $\frac{2\pi n}{7}$ for some $n < 7$. Thus, every edge must be in an orbit of size 7. Since there are 21 edges, there are precisely three such edge orbits. Now any 14-cycle which is setwise fixed must be made up of two of these three edge orbits, and hence there are at most three 14-cycles which are invariant under α. It follows that there are precisely three invariant 14-cycles.
(2) Suppose that the order of α is 3. Since there are 14 vertices, α must fix at least two vertices v and w. Furthermore, since C_{14} has 56 12-cycles (by part (1) of Lemma 1), α must setwise fix at least two 12-cycles. If some vertex on an invariant 12-cycle were fixed, the entire 12-cycle would be fixed and hence α could not have order 3. Thus, neither v nor w can be on an invariant 12-cycle. By part (2) of Lemma 1, we can assume that one of the invariant 12-cycles is the round circle in Figure 2, and hence v and w are as in Figure 2. Since v and w are a distance 3 apart, it follows from part (3) of Lemma 1 that there are precisely two 12-cycles in the complement of $\{v, w\}$. Therefore, α must rotate each of the two 12-cycles in the complement of $\{v, w\}$ by $\pm\frac{2\pi}{3}$. $\square$

Lemma 3. *Let α be an order 2 automorphism of C_{14} which setwise fixes a 12-cycle or a 14-cycle. Then no vertex is fixed by α.*

Proof. First suppose α setwise fixes a 14-cycle and fixes at least one vertex. Then without loss of generality, α setwise fixes the round circle C in Figure 1 and fixes vertex 1. It follows that either α interchanges vertices 2 and 14 or fixes both. In the latter case α would be the identity. Thus, we can assume that α interchanges vertices 2 and 14. However, since vertex 6 is also adjacent to vertex 1, it must also be fixed by α. This implies that α interchanges the two components of $C - \{1, 6\}$. However, this is impossible because one component of $C - \{1, 6\}$ has four vertices while the other has eight vertices.

Next suppose that α setwise fixes a 12-cycle. Then without loss of generality, α setwise fixes the round circle D in Figure 2. Then $\alpha(\{v, w\}) = \{v, w\}$. However, every vertex on D has precisely one neighbor on D which is adjacent to $\{v, w\}$. Thus, if α fixed any vertex on D, it would have to fix every vertex on D, and hence would be the identity. Now suppose α fixes v. Since α has order 2 and v has three neighbors on D, one of these neighbors would have to be fixed by α. As we have already ruled out the possibility that α fixes a vertex on D, this again gives us a contradiction. $\square$

3. Realizable Automorphisms of C_{14}

Lemma 4. *Let α be a realizable automorphism of C_{14}. Then the following hold.*

1. *For some embedding Γ of C_{14} in S^3, α is induced by an orientation preserving homeomorphism h: $(S^3, \Gamma) \to (S^3, \Gamma)$ with $\mathrm{order}(h) = \mathrm{order}(\alpha)$.*
2. *If $\mathrm{order}(\alpha)$ is a power of 2, then α leaves at least two 14-cycles or at least two 12-cycles setwise invariant, and if $\mathrm{order}(\alpha) = 2$, then α fixes no vertices.*
3. *If $\mathrm{order}(\alpha)$ is even, then $\mathrm{order}(\alpha) = 2$ or 6.*

Proof. (1) Since α is realizable, there is some embedding Λ of C_{14} in S^3 such that α is induced by a homeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$. Now by Theorem 1 of [16], since C_{14} is 3-connected, there is an embedding Γ of C_{14} in S^3 such that α is induced by a finite order homeomorphism $h : (S^3, \Gamma) \to (S^3, \Gamma)$. Furthermore, it follows from [12] that no embedding of C_{14} in S^3 has an orientation reversing homeomorphism. Thus, h is orientation preserving.

Let $\mathrm{order}(\alpha) = p$ and $\mathrm{order}(h) = q$. Since h^q is the identity, $p \leq q$. If $p < q$, then h^p pointwise fixes Γ, yet h^p is not the identity. However, by Smith Theory [17], the fixed-point set of h^p is either the empty set or S^1. But, this is impossible since Γ is contained in the fixed-point set of h^p. Thus, $\mathrm{order}(h) = \mathrm{order}(\alpha)$.

(2) Suppose that $\mathrm{order}(\alpha)$ is a power of 2. Let h be given by part (1). Then $\mathrm{order}(h)$ is the same power of 2. Let S_1 and S_2 denote the sets of 12-cycles and 14-cycles, respectively. By [18], for any embedding of C_{14} in S^3, the mod 2 sum of the arf invariants of all 12-cycles and 14-cycles is 1. Thus, an odd number of cycles in $S_1 \cup S_2$ have arf invariant 1. Hence for precisely one i, the set S_i has an odd number of cycles with arf invariant 1. Since $|S_1| = 56$ and $|S_2| = 24$ are each even, S_i must have an odd number of cycles with arf invariant 0 and an odd number of cycles with arf invariant 1.

We know that $h(S_i) = S_i$ and h preserves arf invariants. Hence h setwise fixes T_0 the set of cycles in S_i with arf invariant 0 and T_1 the set of cycles in S_i with arf invariant 1. Since $\mathrm{order}(h)$ is a power of 2, and $|T_0|$ and $|T_1|$ are each odd, h setwise fixes at least one cycle in T_0 and at least one cycle in T_1. Hence at least two 12-cycles or at least two 14-cycles are setwise fixed by h, and hence by α. It now follows from Lemma 3 that if $\mathrm{order}(\alpha) = 2$, then α fixes no vertices.

(3) Suppose that $\mathrm{order}(\alpha)$ is even and $\mathrm{order}(\alpha) \neq 2, 6$. Recall that every even order automorphism of C_{14} has order 2, 4, 6 or 8. Then by part (2), α setwise fixes a 12-cycle or 14-cycle. If α setwise fixes a 14-cycle, then $\mathrm{order}(\alpha) = 2$ since $\mathrm{order}(\alpha)$ is even and cannot be 14. Thus, we suppose that α setwise fixes a 12-cycle Q, and hence $\mathrm{order}(\alpha) \neq 8$

Since $\mathrm{order}(\alpha) \neq 2, 6$, we must have $\mathrm{order}(\alpha) = 4$. Without loss of generality we can assume that Q is the round 12-cycle in Figure 2 and $\alpha|Q = (1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)$. However, this is impossible because $\alpha(\{v, w\}) = \{v, w\}$, and hence α cannot take vertex 4 (which is adjacent to w) to vertex 7 (which is adjacent to neither v nor w). Thus, $\mathrm{order}(\alpha) \neq 4$. $\square$

Theorem 2. *A non-trivial automorphism of C_{14} is realizable if and only if it has order 2, 3, 6 or 7.*

Proof. Figure 3 illustrates an embedding of C_{14} with vertices labeled as in Figure 2 where vertex w is at ∞ and the grey arrows are the edges incident to w. This embedding has a glide rotation h obtained by rotating the picture by $\frac{2\pi}{3}$ around a vertical axis going through vertices v and w while rotating by π around the circular waist of the picture. Then h induces the order 6 automorphism $(v, w)(10, 11, 6, 7, 2, 3)(1, 4, 9, 12, 5, 8)$. Now h^3 and h^2 induce automorphisms of order 2 and 3 respectively. Thus, automorphisms of orders 2, 3, and 6 are all realizable.

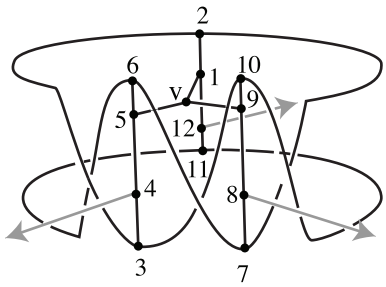

Figure 3. This embedding has a glide rotation inducing $(v,w)(10,11,6,7,2,3)(1,4,9,12,5,8)$.

Figure 4 shows an embedding of C_{14} with a rotation of order 7 about the center of the picture. Thus, C_{14} has realizable automorphisms of order 2, 3, 6, and 7, as required.

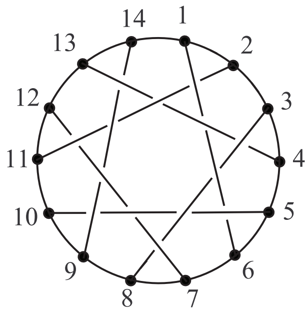

Figure 4. This embedding has a rotation of order 7.

For the converse, we know that that the only odd order automorphisms of C_{14} have order 3 or 7, and part (3) of Lemma 4 shows that the only realizable even order automorphisms of C_{14} have order 2 or 6. □

4. Topological Symmetry Groups of Embeddings of C_{14}

Since C_{14} is intrinsically chiral, for any embedding Γ of C_{14} in S^3, $\mathrm{TSG}(\Gamma) = \mathrm{TSG}_+(\Gamma)$. Thus, a finite group G is realizable for C_{14} if and only if G is positively realizable. Let Γ be an embedding of C_{14} in S^3. We know that $\mathrm{TSG}(\Gamma)$ is a subgroup of $\mathrm{Aut}(C_{14}) \cong \mathrm{PGL}(2,7)$. According to [19], the non-trivial proper subgroups of $\mathrm{PGL}(2,7)$ are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, D_2, D_3, D_4, D_6, D_7, D_8, A_4, S_4, $\mathrm{PSL}(2,7)$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$. We can eliminate the groups $\mathbb{Z}_4$, $\mathbb{Z}_8$, D_4, D_8, S_4, $\mathrm{PSL}(2,7)$, and $\mathrm{PGL}(2,7)$ as possibilities for $\mathrm{TSG}(\Gamma)$ because we know from Theorem 2 that no realizable automorphism of C_{14} has order 4. Thus, the only groups that are possibilities for $\mathrm{TSG}(\Gamma)$ for some embedding Γ of C_{14} are the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, D_2, D_3, D_6, D_7, A_4, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$.

Theorem 3. *The trivial group and the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, D_3, and D_7 are realizable for C_{14}.*

To prove Theorem 3, we will use the following prior result.

Theorem 4. *([20]) Let γ be a 3-connected graph embedded in S^3 as a graph Γ which has an edge e that is not pointwise fixed by any non-trivial element of $G = \mathrm{TSG}_+(\Gamma)$. Then every subgroup of G is positively realizable for γ.*

Proof of Theorem 3. We begin with the embedding Γ of C_{14} illustrated in Figure 5 where the grey squares represent the same trefoil knot.

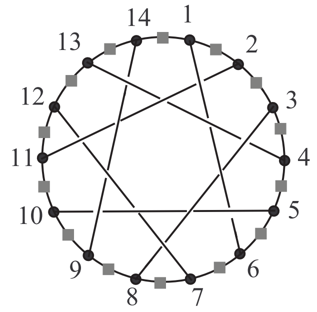

Figure 5. $\mathrm{TSG}(\Gamma) = D_7$.

The outer circle C is setwise invariant under any homeomorphism of (S^3, Γ) because C is the only 14-cycle with 14 trefoil knots, and by [20] any such homeomorphism must preserve the set of knotted edges. It follows that $\mathrm{TSG}(\Gamma) \leq D_{14}$. Also, Γ is invariant under a rotation by $\frac{2\pi}{7}$ inducing the automorphism $(1,3,5,7,9,11,13)(2,4,6,8,10,12)$ and a homeomorphism turning C over inducing $(1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)$. Thus, $D_7 \leq \mathrm{TSG}(\Gamma)$. However, D_7 is the only subgroup of D_{14} containing D_7 which has no element of order 14. Thus, $\mathrm{TSG}(\Gamma) = D_7$.

Observe that no edge of Γ is pointwise fixed by any non-trivial element of $\mathrm{TSG}(\Gamma) = \mathrm{TSG}_+(\Gamma)$. Hence by Theorem 4, every subgroup of $\mathrm{TSG}(\Gamma)$ is realizable. In particular, the groups D_7, $\mathbb{Z}_7$, $\mathbb{Z}_2$, and the trivial group are each realizable for C_{14}.

In the embedding Γ' illustrated in Figure 6, v is above the plane of projection, w is below the plane, and the three grey squares represent the same trefoil knot. Now $C = \overline{1,12,5,4,9,8}$ is the only 6-cycle containing three trefoil knots. It follows that any homeomorphism of (S^3, Γ') must take C to itself taking the set of three trefoils to itself. Thus, $\mathrm{TSG}(\Gamma') \leq D_3$. Since Γ' is invariant under a $\frac{2\pi}{3}$ rotation as well as under turning the picture over, $\mathrm{TSG}(\Gamma') = D_3$. Now if we replace the three trefoils on C by three identical non-invertible knots, we will get an embedding Γ'' such that $\mathrm{TSG}(\Gamma'') = \mathbb{Z}_3$.

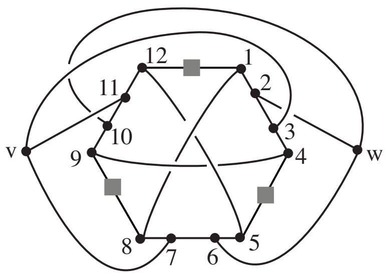

Figure 6. $\mathrm{TSG}(\Gamma') = D_3$ and $\mathrm{TSG}(\Gamma'') = \mathbb{Z}_3$.

Finally, let Λ be the embedding in Figure 3. Then the 6-cycle $C = \overline{10,11,6,7,2,3}$ is the only 6-cycle which contains a trefoil knot. Thus, C is setwise invariant under any homeomorphism of (S^3, Λ). Hence $\mathrm{TSG}(\Lambda) \leq D_6$. We also saw in Figure 3 that a glide rotation of S^3 induces the order 6 automorphism $(v,w)(10,11,6,7,2,3)(1,4,9,12,5,8)$. Thus, $\mathbb{Z}_6 \leq \mathrm{TSG}(\Lambda)$. Since we know from Theorem 5 that D_6 is not realizable for C_{14}, it follows that $\mathrm{TSG}(\Lambda) = \mathbb{Z}_6$. □

In what follows, we prove that no other groups are realizable for C_{14}.

Theorem 5. *The groups D_2 and D_6 are not realizable for C_{14}.*

Proof. Suppose that there exist realizable order 2 automorphisms α and β of C_{14} such that $\langle \alpha, \beta \rangle = D_2$. Since C_{14} has 21 edges, α and β each setwise fix an odd number of edges. Let E_α denote the set of edges which are invariant under α. Let $\varepsilon \in E_\alpha$. Then $\alpha(\beta(\varepsilon)) = \beta(\alpha(\varepsilon)) = \beta(\varepsilon)$. Thus, $\beta(\varepsilon) \in E_\alpha$. It follows that $\beta(E_\alpha) = E_\alpha$. However, since E_α has an odd number of elements and β has order 2, there is some edge $e \in E_\alpha$ such that $\beta(e) = e$. Thus, α and β both setwise fix the edge e, and hence at least one of the involutions α, β, or $\alpha\beta$ must pointwise fix e.

Now by Lemma 3, none of α, β, or $\alpha\beta$ can fix any vertex. Thus, D_2 is not realizable. However, since D_6 contains involutions α and β such that $\langle \alpha, \beta \rangle = D_2$, it follows that D_6 also cannot be realizable for C_{14}. □

Theorem 6. *The group A_4 is not realizable for C_{14}.*

Proof. Suppose that Γ is an embedding of C_{14} such that $\mathrm{TSG}(\Gamma) = A_4$. According to Burnside's Lemma [21], the number of vertex orbits of Γ under $\mathrm{TSG}(\Gamma)$ is:

$$\frac{1}{|A_4|} \sum_{\alpha \in A_4} |\mathrm{fix}(\alpha)|$$

where $|\mathrm{fix}(\alpha)|$ denotes the number of vertices fixed by an automorphism $\alpha \in \mathrm{TSG}(\Gamma)$. Observe that A_4 contains eight elements of order 3, three elements of order 2, and no other non-trivial elements. Now by part (2) of Lemma 2, each order 3 automorphism fixes precisely two vertices, and by Lemma 4 part (2), no realizable order 2 automorphism fixes any vertex. Thus, the number of vertex orbits of Γ under $\mathrm{TSG}(\Gamma)$ is:

$$\frac{1}{|A_4|} \sum_{\alpha \in A_4} |\mathrm{fix}(\alpha)| = \frac{1}{12}((8 \cdot 2) + (3 \cdot 0) + (1 \cdot 14)) = \frac{30}{12}.$$

As this is not an integer, A_4 cannot be realizable for C_{14}. □

To show that the groups $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ are not realizable for C_{14}, we will make use of the definition and results below.

Definition 4. *A finite group G of orientation preserving diffeomorphisms of S^3 is said to satisfy the* involution condition *if for every involution $g \in G$, we have $\mathrm{fix}(g) \cong S^1$ and no $h \in G$ with $h \neq g$ has $\mathrm{fix}(h) = \mathrm{fix}(g)$.*

Theorem 7 ([2])**.** *Let Γ be a 3-connected graph embedded in S^3 with $H = \mathrm{TSG}_+(\Gamma)$. Then Γ can be re-embedded in S^3 as Δ such that $H \leq \mathrm{TSG}_+(\Delta)$ and H is induced by an isomorphic finite group of orientation preserving diffeomorphisms of S^3.*

Theorem 8 ([22])**.** *Let G be a finite group of orientation preserving isometries of S^3 which satisfies the involution condition. Then the following hold.*

1. *If G preserves a standard Hopf fibration of S^3, then G is cyclic, dihedral, or a subgroup of $D_m \times D_m$ for some odd m.*
2. *If G does not preserve a standard Hopf fibration of S^3, then G is S_4, A_4, or A_5.*

Theorem 9. *The groups $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ are not realizable for C_{14}.*

Proof. Suppose that for some embedding Γ of C_{14} in S^3, $\mathrm{TSG}_+(\Gamma)$ is $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$. In either case, $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3 \leq \mathrm{TSG}_+(\Gamma)$. Now since C_{14} is 3-connected, we can apply Theorem 7, to re-embed C_{14} in S^3 as Δ such that $G \leq \mathrm{TSG}_+(\Delta)$ and G is induced by an isomorphic finite group of orientation preserving diffeomorphisms of S^3. However, by the proof of the Geometrization Conjecture, every finite group of orientation preserving diffeomorphisms of S^3 is conjugate to a group of orientation preserving isometries of S^3 [23]. Thus, we abuse notation and treat G as a group of orientation preserving isometries of S^3.

Since G has no elements of order 2, it vacuously satisfies the involution condition, and hence by Theorem 8, G is cyclic, dihedral, a subgroup of $D_m \times D_m$ for some odd m, S_4, A_4, or A_5. However, since $|G| = 21$, it cannot be dihedral, S_4, A_4, or A_5. Also, since $G \leq \mathrm{Aut}(C_{14})$ has no element of order 21, the elements of G of order 3 and 7 cannot commute. Thus, G cannot be cyclic; and since all elements of odd order in $D_m \times D_m$ commute, G cannot be a subgroup of any $D_m \times D_m$. By this contradiction, we conclude that neither $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ nor $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ is realizable for C_{14}. $\square$

The following corollary summarizes our classification of which groups can occur as topological symmetry groups of some embedding of the Heawood graph in S^3.

Corollary 1. *A group G is realizable as a topological symmetry group of C_{14} if and only if G is the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, D_3, or D_7.*

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