Article

Primeness of Relative Annihilators in *BCK*-Algebra

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Abstract: Conditions that are necessary for the relative annihilator in lower *BCK*-semilattices to be a prime ideal are discussed. Given the minimal prime decomposition of an ideal A, a condition for any prime ideal to be one of the minimal prime factors of A is provided. Homomorphic image and pre-image of the minimal prime decomposition of an ideal are considered. Using a semi-prime closure operation "*cl*", we show that every minimal prime factor of a *cl*-closed ideal A is also *cl*-closed.

Keywords: lower *BCK*-semilattice; relative annihilator; semi-prime closure operation; minimal prime decomposition; minimal prime factor

1. Introduction

For the first time, Aslam et al. in [1] discussed the concept of annihilators for a subset in *BCK*-algebras, and after that many researchers generalized it in different research articles (see [2–5]). Except these, the notion related to annihilator in *BCK*-algebras is investigated in the papers [6–8]. In [4], Bordbar et al. introduced the notion of the relative annihilator in a lower *BCK*-semilattice for a subset with respect to another subset as a logical extension of annihilator, and they obtained some properties related to this notion. They provide the conditions that the relative annihilator of an ideal with respect to an ideal needs to be ideal, and discussed conditions for the relative annihilator ideal to be an implicative (resp., positive implicative, commutative) ideal. Moreover, in some articles, different properties of ideals in logical algebras and ordered algebraic structures were concerned (see [9–18]). In order to investigate these kinds of properties for an arbitrary ideal in *BCI/BCK*-algebra, we need to know about the decomposition of an ideal. With this motivation, this article is the first try, as far as we know, to decompose an ideal in a *BCI/BCK*-algebra.

In this paper, we prove that the relative annihilator of a subset with respect to a prime ideal is also a prime ideal. Given the minimal prime decomposition of an ideal A, we provide a condition for any prime ideal to be one of minimal prime factors of A by using the relative annihilator. We consider homomorphic image and preimage of the minimal prime decomposition of an ideal. Using a semi-prime closure operation "*cl*", we show that, if an ideal A is *cl*-closed, then every minimal prime factor of A is also *cl*-closed.

2. Preliminaries

In this section, gather some results related to *BCI/BCK*-algebra and ideals, which will be used in the next section. For more details, the readers are refereed to [19].

The study of *BCI/BCK*-algebras was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi.

Suppose that X is a set and $(X; *, 0)$ of type $(2, 0)$ is an algebra. The set X is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X) \left(((x * y) * (x * z)) * (z * y) = 0 \right),$
(II) $(\forall x, y \in X) \left((x * (x * y)) * y = 0 \right),$
(III) $(\forall x \in X) \left(x * x = 0 \right),$
(IV) $(\forall x, y \in X) \left(x * y = 0, \ y * x = 0 \ \Rightarrow \ x = y \right).$

Every *BCI*-algebra *X* with the following condition

$$(\forall x \in X) \left(0 * x = 0 \right)$$

is called a *BCK-algebra*.

Proposition 1. *Let X be a BCI/BCK-algebra. Then, the following statements are satisfied in every BCI/BCK-algebra:*

(1) $(\forall x \in X) \left(x * 0 = x \right),$
(2) $(\forall x, y, z \in X) \left(x \leq y \ \Rightarrow \ x * z \leq y * z, \ z * y \leq z * x \right),$
(3) $(\forall x, y, z \in X) \left((x * y) * z = (x * z) * y \right),$
(4) $(\forall x, y, z \in X) \left((x * z) * (y * z) \leq x * y \right)$

where $x \leq y$ if and only if $x * y = 0$.

Definition 1. *A BCK-algebra X is called a lower BCK-semilattice (see [19]) if X is a lower semilattice with respect to the BCK-order.*

Definition 2 ([19])**.** *Let X be a a BCI/BCK-algebra. An arbitrary subset A of X is called an ideal of X if it satisfies*

$$0 \in A, \tag{1}$$

$$(\forall x \in X) \, (\forall y \in A) \, (x * y \in A \ \Rightarrow \ x \in A). \tag{2}$$

Remark 1 ([19])**.** *For every ideal A of a BCK-algebra X and for all $x, y \in X$, the following implication is satisfied:*

$$(x \leq y, \ y \in A \ \Rightarrow \ x \in A). \tag{3}$$

Definition 3 ([19])**.** *Let P be a proper ideal of a lower BCK-semilattice X. Then, P is a prime ideal if, for $a, b \in X$ such that $a \wedge b \in P$, we conclude that $a \in P$ or $b \in P$, where $a \wedge b$ is the greatest lower bound of a and b.*

For an ideal *A* of a *BCK*-algebra *X*, the ideal *B* of *X* is called *minimal prime* associated with *A* if *B* is minimal in the set of all prime ideals containing *A*.

Lemma 1 ([20])**.** *If $\varphi : X \to Y$ is an epimorphism of lower BCK-semilattices, then*

$$(\forall x, y \in X) \, (\varphi(x \wedge_X y) = \varphi(x) \wedge_Y \varphi(y)). \tag{4}$$

Lemma 2 ([20])**.** 1. *Let $\varphi : X \to Y$ be an epimorphism of BCK-algebras. If A is an ideal of X, then $\varphi(A)$ is an ideal of Y.*
2. *Let $\varphi : X \to Y$ be an homomorphism of BCK-algebras. If B is an ideal of Y, then $\varphi^{-1}(B)$ is an ideal of X.*

Lemma 3 ([20])**.** *Let $\varphi : X \to Y$ be a homomorphism of BCK-algebras X and Y and let A be an ideal of X such that $Ker(\varphi) \subseteq A$. Then, $\varphi^{-1}(A') = A$ where $A' = \varphi(A)$.*

3. Primeness of Relative Annihilators

In this section, we use the notations X as a lower *BCK*-semilattice, $x \wedge y$ as the g.l.b.(greatest lower bound) of $x, y \in X$ and

$$A \wedge B := \{a \wedge b \mid a \in A, b \in B\}$$

for any two arbitrary subsets A, B of X, unless otherwise.
In a case that, $A = \{a\}$, then we use $a \wedge B$ instead of $\{a\} \wedge B$.

Definition 4 ([4]). *Let A and B be two arbitrary subsets of X. A set $(A :_\wedge B)$ is defined as follows:*

$$(A :_\wedge B) := \{x \in X \mid x \wedge B \subseteq A\} \tag{5}$$

and it is called the relative annihilator of B with respect to A.

Remark 2. *If $A = \{a\}$, then $(\{a\} :_\wedge B)$ is denoted by $(a :_\wedge B)$. Similarly, we use $(A :_\wedge b)$ instead of $(A :_\wedge \{b\})$, when $B = \{b\}$.*

The next two Lemmas are from [4].

Lemma 4. *For any ideal A and a nonempty subset B of X, the following implication*

$$A \subseteq (A :_\wedge B)$$

is satisfied.

Lemma 5. *Let B be an arbitrary nonempty subset of X in which the following statement is valid for all $x, y \in X$*

$$(\forall b \in B)\,((x \wedge b) * (y \wedge b) \leq (x * y) \wedge b). \tag{6}$$

Consider the relative annihilator $(A :_\wedge B)$. If A is an ideal of X, then the the relative annihilator $(A :_\wedge B)$ is an ideal of X.

Theorem 1. *Let B be an arbitrary subset of X such that the condition (6) is satisfied for B. If A is a prime ideal of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is X itself or a prime ideal of X.*

Proof. Suppose that $(A :_\wedge B) \neq X$. Then $(A :_\wedge B)$ is a proper ideal of X by Lemma 5. Now, let $x \wedge y \in (A :_\wedge B)$ and $x \notin (A :_\wedge B)$ for elements $x, y \in X$. Then, $(x \wedge y) \wedge B \subseteq A$ and $x \wedge b \notin A$ for some $b \in B$. Thus,

$$(x \wedge b) \wedge y = (x \wedge y) \wedge b \in A.$$

Since A is a prime ideal of X, it follows from Definition 3 and Lemma 4 that $y \in A \subseteq (A :_\wedge B)$. Therefore, $(A :_\wedge B)$ is a prime ideal of X. □

Corollary 1. *Suppose that X is a commutative BCK-algebra. If A is a prime ideal of X and B is a nonempty subset of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is X itself or a prime ideal of X.*

Lemma 6 ([21]). *If A and B are ideals of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

Theorem 2. *If A is a prime ideal and B is an ideal of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is X itself or a prime ideal of X.*

Proof. Suppose that $(A :_\wedge B) \neq X$. By using Lemma 6, $(A :_\wedge B)$ is a proper ideal of X. The primeness of $(A :_\wedge B)$ can be proved by a similar way as in the proof of Theorem 1.　□

By changing the role of A and B in Theorem 2, the $(A :_\wedge B)$ may not be a prime ideal of X. The following example shows that it is not true in general case.

Example 1. *Let* $X = \{0, 1, 2, 3, 4\}$ *with the following Cayley table.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then, by routine calculation, X is a lower BCK-semilattice. Consider ideals $A = \{0, 1\}$ and $B = \{0, 1, 2, 4\}$ of X. It is easy to show that B is a prime ideal. Then,

$$(A :_\wedge B) = \{x \in X \mid x \wedge B \subseteq A\} = \{0, 1, 3\},$$

and it is not a prime ideal of X because $2 \wedge 4 = 0 \in (A :_\wedge B)$ but $2 \notin (A :_\wedge B)$ and $4 \notin (A :_\wedge B)$.

For any ideal I of X and any $x \in X$, we know that

$$I \subseteq (I :_\wedge x) \subseteq X. \tag{7}$$

Lemma 7. *For any ideal P of X and any $a \in X$, the following statements are satisfied:*

$$(a \in P \implies (P :_\wedge a) = X). \tag{8}$$
$$(a \notin P \text{ and } P \text{ is prime} \implies (P :_\wedge a) = P). \tag{9}$$

Proof. Let $a \in P$. Then, for arbitrary element $x \in X$, $x \wedge a \in P$. Hence, $x \in (P :_\wedge a)$. Therefore, (8) is valid. Let $a \notin P$ and P be a prime ideal of X. Obviously, $P \subseteq (P :_\wedge a)$. If $x \in (P :_\wedge a)$, then $x \wedge a \in P$ and so $x \in P$. Consequently, $(P :_\wedge a) = P$.　□

Theorem 3. *Let A_1 and A_2 be ideals of X. For any prime ideal P of X, the following assertions are equivalent:*

(i)　$A_1 \subseteq P$ or $A_2 \subseteq P$.
(ii)　$A_1 \cap A_2 \subseteq P$.
(iii)　$A_1 \wedge A_2 \subseteq P$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (i) Suppose $A_1 \nsubseteq P$ and $A_2 \nsubseteq P$. Then, there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $a_1, a_2 \notin P$. Since P is a prime ideal, we have $a_1 \wedge a_2 \notin P$. This is a contradiction, and so $A_1 \subseteq P$ or $A_2 \subseteq P$.　□

By using induction on n, the following theorem can be considered as an extension of Theorem 3.

Theorem 4. *Let $A_1, A_2, \cdots, A_n$ be ideals of X. For a prime ideal P of X, the following assertions are equivalent:*

(i)　$A_j \subseteq P$ for some $j \in \{1, 2, \cdots, n\}$.
(ii)　$\bigcap_{i=1}^{n} A_i \subseteq P$.
(iii)　$\bigwedge_{i=1}^{n} A_i \subseteq P$.

Theorem 5. *Let A_1 and A_2 be ideals of X. For any prime ideal P of X, if $P = A_1 \cap A_2$, then $P = A_1$ or $P = A_2$.*

Proof. It is straightforward by Theorem 3. □

Inductively, the following theorem can be proved as an extension of Theorem 5.

Theorem 6. *Let $A_1, A_2, \cdots, A_n$ be ideals of X. For a prime ideal P of X, if $P = \cap_{i=1}^{n} A_i$, then $P = A_j$ for some $j \in \{1, 2, \cdots, n\}$.*

Definition 5. *Letting A be an ideal of a lower BCK-semilattice X, we say that A has a minimal prime decomposition if there exist prime ideals $Q_1, Q_2, \cdots, Q_n$ of X such that*

(1) $A = \bigcap_{i \in \{1, 2, \cdots, n\}} Q_i,$

(2) $\bigcap_{\substack{i \in \{1, 2, \cdots, n\} \\ i \neq j}} Q_i \nsubseteq Q_j.$

The class $\{Q_1, Q_2, \cdots, Q_n\}$ is called a minimal prime decomposition of A, and each Q_i is called a minimal prime factor of A.

Lemma 8 ([22]). *Let A, B, and C be non-empty subsets of X. Then, we have*

$$(A \cap B :_\wedge C) = (A :_\wedge C) \cap (B :_\wedge C).$$

Given the minimal prime decomposition of an ideal A, we provide a condition for any prime ideal to be one of minimal prime factors of A by using the relative annihilator.

Theorem 7. *Let A be an ideal of X and $\{P_1, P_2\}$ be a minimal prime decomposition of A. For a prime ideal P of X, the following statements are equivalent:*

(i) $P = P_1$ or $P = P_2$.
(ii) *There exists $a \in X$ such that $(A :_\wedge a) = P$.*

Proof. (i) $\Rightarrow$ (ii). Since $\{P_1, P_2\}$ is a minimal prime decomposition of A, there exist $a_1 \in P_1 \setminus P_2$ and $a_2 \in P_2 \setminus P_1$. If $P = P_2$, then Lemmas 7 and 8 imply that

$$(A :_\wedge a_1) = (P_1 \cap P_2 :_\wedge a_1) = (P_1 :_\wedge a_1) \cap (P_2 :_\wedge a_1) = X \cap P_2 = P_2 = P.$$

Similarly, if $P = P_1$, then $(A :_\wedge a_2) = P$.

Conversely, suppose that, for an element $a \in X$, we have $(A :_\wedge a) = P$. Then, we have

$$(A :_\wedge a) = (P_1 \cap P_2 :_\wedge a) = (P_1 :_\wedge a) \cap (P_2 :_\wedge a).$$

If $a \in P_1$, then $(P_1 :_\wedge a) = X$, and if $a \notin P_1$, then $(P_1 :_\wedge a) = P_1$ by Lemma 7. Similarly, $(P_2 :_\wedge a) = X$ or $(P_2 :_\wedge a) = P_2$. Thus,

$$P = (A :_\wedge a) = (P_1 \cap P_2 :_\wedge a) = (P_1 :_\wedge a) \cap (P_2 :_\wedge a)$$

is one of $P_1, P_2, P_1 \cap P_2$ and X. We know that $P \neq X$ since P is proper. If $P = P_1 \cap P_2$, then $P = P_1$ or $P = P_2$ by Theorem 5. □

Using an inductive method, the following theorem is satisfied.

Theorem 8. *Let $\{P_1, P_2, \cdots, P_n\}$ be a minimal prime decomposition of an ideal A in X. If P is a prime ideal of X, then the following statements are equivalent:*

(i) $P = P_i$ for some $i \in \{1, 2, \cdots, n\}$.
(ii) *There exists $a \in X$ such that $(A :_\wedge a) = P$.*

Theorem 9. *Suppose that* $\varphi : X \to Y$ *is an epimorphism of lower BCK-semilattices. Then,*

(i) *If P is a prime ideal of X such that* $Ker\varphi \subseteq P$, *then* $\varphi(P)$ *is a prime ideal of Y.*
(ii) *For prime ideals* $P_1, P_2, \cdots, P_n$ *of X, the following equation is satisfied:*

$$\varphi(P_1 \cap P_2 \cap \cdots \cap P_n) = \varphi(P_1) \cap \varphi(P_2) \cap \cdots \cap \varphi(P_n).$$

Proof. (i) Suppose that P is a prime ideal of X and $Ker\varphi \subseteq P$. Then, $\varphi(P)$ is an ideal of Y by using Lemma 2. Now, let $a \wedge_Y b \in \varphi(P)$ for any $a, b \in Y$. Then, there exist x and y in X such that $\varphi(x) = a$ and $\varphi(y) = b$. Using Lemma 1, we have the following:

$$\varphi(x \wedge_X y) = \varphi(x) \wedge_Y \varphi(y) = a \wedge_Y b \in \varphi(P).$$

Hence, there exists $q \in P$ such that $\varphi(x \wedge_X y) = \varphi(q)$. In addition, since φ is a homomorphism, it follows that

$$\varphi((x \wedge_X y) *_X q) = \varphi(x \wedge_X y) *_Y \varphi(q) = 0.$$

Thus, $(x \wedge_X y) *_X q \in Ker\varphi \subseteq P$. Since $q \in P$, we conclude that $x \wedge_X y \in P$. It follows from the primeness of P that

$$a = \varphi(x) \in \varphi(P) \text{ or } b = \varphi(y) \in \varphi(P).$$

Therefore, $\varphi(P)$ is a prime ideal of Y.

(ii) Let $x \in \varphi(P_1 \cap P_2 \cap \cdots \cap P_n)$. Then, there exists $a \in P_1 \cap P_2 \cap \cdots \cap P_n$ such that $x = \varphi(a)$. Since $a \in P_1 \cap P_2 \cap \cdots \cap P_n$, we have $a \in P_i$ and so $\varphi(a) \in \varphi(P_i)$ for all $i \in \{1, 2, \cdots, n\}$. Hence,

$$x = \varphi(a) \in \varphi(P_1) \cap \varphi(P_2) \cap \cdots \cap \varphi(P_n).$$

Therefore, $\varphi(P_1 \cap P_2 \cap \cdots \cap P_n) \subseteq \varphi(P_1) \cap \varphi(P_2) \cap \cdots \cap \varphi(P_n)$.

Assume that $x \in \varphi(P_1) \cap \varphi(P_2) \cap \cdots \cap \varphi(P_n)$. Then, $x \in \varphi(P_i)$, and thus there exists $a_i \in P_i$ such that $x = \varphi(a_i)$ for all $i \in \{1, 2, \cdots, n\}$. Note that $a_1 \wedge_X a_2 \wedge_X \cdots \wedge_X a_n \leq a_i$ for all $i \in \{1, 2, \cdots, n\}$. Since $a_i \in P_i$ and P_i is an ideal, we conclude that $a_1 \wedge_X a_2 \wedge_X \cdots \wedge_X a_n \in P_i$ for all $i \in \{1, 2, \cdots, n\}$. Therefore,

$$a_1 \wedge_X a_2 \wedge_X \cdots \wedge_X a_n \in P_1 \cap P_2 \cap \ldots \cap P_n$$

and so

$$\begin{aligned}
x &= x \wedge_Y x \wedge_Y \cdots \wedge_Y x \\
&= \varphi(a_1) \wedge_Y \varphi(a_2) \wedge_Y \cdots \wedge_Y \varphi(a_n) \\
&= \varphi(a_1 \wedge_X a_2 \wedge_X \cdots \wedge_X a_n) \in \varphi(P_1 \cap P_2 \cap \cdots \cap P_n).
\end{aligned}$$

Hence, $\varphi(P_1) \cap \varphi(P_2) \cap \cdots \cap \varphi(P_n) \subseteq \varphi(P_1 \cap P_2 \cap \cdots \cap P_n)$, and therefore the proof is completed. $\square$

Lemma 9. *Let* $\{P_1, P_2, \cdots, P_n\}$ *be a minimal prime decomposition of an ideal A in X. If P is a prime ideal of X, then* $A \subseteq P$ *if and only if there exists* $i \in \{1, 2, \cdots, n\}$ *such that* $P_i \subseteq P$.

Proof. Straightforward. $\square$

Theorem 10. *Let* $\varphi : X \to Y$ *be an epimorphism of lower BCK-semilattices. Let A be an ideal of X such that* $Ker(\varphi) \subseteq A$. *If* $\{P_1, P_2, \cdots, P_n\}$ *is a minimal prime decomposition of A in X, then* $\{\varphi(P_1), \varphi(P_2), \cdots, \varphi(P_n)\}$ *is a minimal prime decomposition of* $\varphi(A)$ *in Y.*

Proof. Note that $\varphi(A)$ is an ideal of Y (Lemma 1). If $\{P_1, P_2, \cdots, P_n\}$ is a minimal prime decomposition of A in X, then

$$A = \bigcap_{i \in \{1,2,\cdots,n\}} P_i$$

and so $\varphi(A) = \varphi\left(\bigcap_{i \in \{1,2,\cdots,n\}} P_i\right) = \bigcap_{i \in \{1,2,\cdots,n\}} \varphi(P_i)$. Suppose that

$$\bigcap_{\substack{i \in \{1,2,\cdots,n\} \\ i \neq j}} \varphi(P_i) \subseteq \varphi(P_j).$$

Since $Ker(\varphi) \subseteq P_i$, we conclude that $\varphi^{-1}(\varphi(P_i)) = P_i$ for all $i \in \{1,2,\cdots,n\}$ by using Lemma 3. Hence,

$$\bigcap_{\substack{i \in \{1,2,\cdots,n\} \\ i \neq j}} P_i = \bigcap_{\substack{i \in \{1,2,\cdots,n\} \\ i \neq j}} \varphi^{-1}(\varphi(P_i))$$

$$= \varphi^{-1}\left(\bigcap_{\substack{i \in \{1,2,\cdots,n\} \\ i \neq j}} \varphi(P_i)\right)$$

$$\subseteq \varphi^{-1}(\varphi(P_j)) = P_j.$$

This is a contradiction, so $\{\varphi(P_1), \varphi(P_2), \cdots, \varphi(P_n)\}$ is a minimal prime decomposition of $\varphi(A)$ in Y. $\square$

Corollary 2. *Suppose that $\varphi : X \to Y$ is an isomorphism of lower BCK-semilattices. Let A be an ideal of X. If $\{P_1, P_2, \cdots, P_n\}$ is a minimal prime decomposition of A in X, then $\{\varphi(P_1), \varphi(P_2), \cdots, \varphi(P_n)\}$ is a minimal prime decomposition of $\varphi(A)$ in Y.*

Theorem 11. *Suppose that $\varphi : X \to Y$ is an epimorphism of lower BCK-semilattices. Let B be an ideal of Y. If $\{Q_1, Q_2, \cdots, Q_n\}$ is a minimal prime decomposition of B in Y, then $\{\varphi^{-1}(Q_1), \varphi^{-1}(Q_2), \cdots, \varphi^{-1}(Q_n)\}$ is a minimal prime decomposition of $\varphi^{-1}(B)$ in X.*

Proof. Obviously, $\varphi^{-1}(B)$ is an ideal of X. If $\{Q_1, Q_2, \cdots, Q_n\}$ is a minimal prime decomposition of B in Y, then

$$B = \bigcap_{i \in \{1,2,\cdots,n\}} Q_i.$$

Thus,

$$\varphi^{-1}(B) = \varphi^{-1}\left(\bigcap_{i \in \{1,2,\cdots,n\}} Q_i\right) = \bigcap_{i \in \{1,2,\cdots,n\}} \varphi^{-1}(Q_i).$$

Suppose that

$$\bigcap_{\substack{i \in \{1,2,\cdots,n\} \\ i \neq j}} \varphi^{-1}(Q_i) \subseteq \varphi^{-1}(Q_j).$$

Since φ is onto, $\varphi\left(\varphi^{-1}(Q_i)\right) = Q_i$ for all $i \in \{1,2,\cdots,n\}$. Hence,

$$\bigcap_{\substack{i\in\{1,2,\cdots,n\}\\i\neq j}} Q_i = \bigcap_{\substack{i\in\{1,2,\cdots,n\}\\i\neq j}} \varphi\left(\varphi^{-1}(Q_i)\right)$$

$$= \varphi\left(\bigcap_{\substack{i\in\{1,2,\cdots,n\}\\i\neq j}} \varphi^{-1}(Q_i)\right)$$

$$\subseteq \varphi\left(\varphi^{-1}(Q_j)\right) = Q_j.$$

This is a contradiction, and so

$$\bigcap_{\substack{i\in\{1,2,\cdots,n\}\\i\neq j}} \varphi^{-1}(Q_i) \nsubseteq \varphi^{-1}(Q_j).$$

Therefore, $\{\varphi^{-1}(Q_1), \varphi^{-1}(Q_2), \cdots, \varphi^{-1}(Q_n)\}$ is a minimal prime decomposition of $\varphi^{-1}(B)$ in X. $\quad\square$

Lemma 10 ([19])**.** *If X is Noetherian BCK-algebra, then each ideal of X has a unique minimal prime decomposition.*

Lemma 11 ([19])**.** *Every proper ideal of X is equal to the intersection of all minimal prime ideals associated with it.*

For an ideal A of X, consider the set $X \setminus A$. This set is not closed subset under the $\wedge$ operation in X in general. The following example shows it.

Example 2. *Let $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

Then, X is a lower BCK-semilattice. For an ideal $A = \{0, 1, 2\}$ of X, we have $X \setminus A = \{3, 4\}$, which is not a $\wedge$-closed subset of X because $3, 4 \in X \setminus A$, but $3 \wedge 4 = 1 \notin X \setminus A$.

For a subset A of X with $0 \notin A$, we can check that the set $X \setminus A$ may not be an ideal of X. In the following example, we check it.

Example 3. *Suppose that $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then, X is a lower BCK-semilattice. For a subset $A = \{3, 4\}$ of X, we have $X \setminus A = \{0, 1, 2\}$. By routine verification, we can investigate that $X \setminus A$ is not an ideal of X.

The following theorem provided a characterization of a prime ideal.

Theorem 12. *For an arbitrary ideal P of X, the following assertions are equivalent:*

(i) *P is a prime ideal of X.*

(ii) $X \setminus P$ is a closed subset under the $\wedge$ operation in X, that is, $x \wedge y \in X \setminus P$ for all $x, y \in X \setminus P$.

Proof. (i) $\rightarrow$ (ii): Suppose that P is a prime ideal of X and $x, y \in X \setminus P$ are arbitrary elements. If $x \wedge y \notin X \setminus P$, then clearly $x \wedge y \in P$. Since P is a prime ideal, $x \in P$ or $y \in P$, which is contradictory because x and y were chosen from the set $X \setminus P$. Thus, $x \wedge y \in X \setminus P$ and $X \setminus P$ is the closed subset under the $\wedge$ operation.

(ii) $\rightarrow$ (i): Suppose that $x \wedge y \in P$. If $x \notin P$ and $y \notin P$, then clearly $x \in X \setminus P$ and also $y \in X \setminus P$. Using condition (ii), we conclude that $x \wedge y \in X \setminus P$, which is a contradiction from the first assumption $x \wedge y \in P$. Thus, $x \in P$ or $y \in P$ and P is a prime ideal of X. $\square$

Definition 6. *Let X be a BCK-algebra. We defined in [2] the closure operation on $\mathcal{I}_X$, as the following function*

$$cl : \mathcal{I}(X) \rightarrow \mathcal{I}(X), \ A \mapsto A^{cl}$$

such that

$$(\forall A \in \mathcal{I}(X)) \left(A \subseteq A^{cl} \right), \tag{10}$$

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = (A^{cl})^{cl} \right), \tag{11}$$

$$(\forall A, B \in \mathcal{I}(X)) \left(A \subseteq B \ \Rightarrow \ A^{cl} \subseteq B^{cl} \right), \tag{12}$$

where $\mathcal{I}(X)$ is the set of all ideals of X.

An ideal A in a *BCK*-algebra X is said to be *cl-closed* (see [2]) if $A = A^{cl}$.

Definition 7 ([3]). *For a closure operation "cl" on X, we have the following definitions:*

 (i)   *"cl" is a semi-prime closure operation if we have*

$$A \wedge B^{cl} \subseteq (A \wedge B)^{cl} \text{ and } A^{cl} \wedge B \subseteq (A \wedge B)^{cl}$$

  for every $A, B \in \mathcal{I}(X)$.
(ii)   *"cl" is a good semi-prime closure operation, if we have*

$$A \wedge B^{cl} = A^{cl} \wedge B = (A \wedge B)^{cl}$$

  for every $A, B \in \mathcal{I}(X)$.

Theorem 13 ([3]). *Suppose that "cl" is a semi-prime closure operation on X and S is a closed subset of X under the $\wedge$ operation. If X is Noetherian and A is a cl-closed ideal of X, then the set*

$$B := \langle \{x \in X \mid x \wedge s \in A \text{ for some } s \in S\} \rangle$$

is a cl-closed ideal of X.

Lemma 12. *If $\{P_1, P_2, \cdots, P_n\}$ is a minimal prime decomposition of an ideal A of X, then*

$$(\forall i, j \in \{1, 2, \cdots, n\}) \left(i \neq j \ \Rightarrow \ P_i \cap (X \setminus P_j) \neq \varnothing \right). \tag{13}$$

Proof. Suppose that for $i, j \in \{1, 2, \cdots, n\}$ such that $i \neq j$, $P_i \cap (X \setminus P_j) = \varnothing$. Then, it follows that $P_i \subseteq P_j$ and this is a contradiction because $\{P_1, P_2, \cdots, P_n\}$ is a minimal prime decomposition of an ideal A of X. $\square$

Theorem 14. *Suppose that A is an ideal of X with a minimal prime decomposition*

$$\{P_1, P_2, \cdots, P_n\}.$$

Assume that X is Noetherian and "cl" is a semi-prime closure operation on $\mathcal{I}(X)$. If A is cl-closed, then so is P_j for all $j \in \{1, 2, \cdots, n\}$.

Proof. For any $j \in \{1, 2, \cdots, n\}$, let

$$\Omega_j := \{x \in X \mid x \wedge s \in A \text{ for some } s \in X \setminus P_j\}. \tag{14}$$

Then, we will prove that $\Omega_j = P_j$. If $x \in \Omega_j$, then there exists $s \in X \setminus P_j$ such that $x \wedge s \in A$. It follows that $x \wedge s \in P_j$ and so $x \in P_j$. Thus, $\Omega_j \subseteq P_j$ for all $j \in \{1, 2, \cdots, n\}$. Now, assume that $y \in P_j$. Using Lemma 12, we can take an element $a \in P_i \cap (X \setminus P_j)$, and so $a \in P_i$ and $a \in X \setminus P_j$ for all $i \in \{1, 2, \cdots, n\}$ with $i \neq j$. Then, $y \wedge a \in P_j$ and $y \wedge a \in P_i$ for all $i \in \{1, 2, \cdots, n\}$ with $i \neq j$. Thus,

$$y \wedge a \in \bigcap_{i \in \{1, 2, \cdots, n\}} P_i = A,$$

and so $y \in \Omega_j$. Therefore, $\Omega_j = P_j$, which implies that $\langle \Omega_j \rangle = \langle P_j \rangle = P_j$ for all $j \in \{1, 2, \cdots, n\}$. Since $X \setminus P_j$ is a $\wedge$-closed subset of X for all $j \in \{1, 2, \cdots, n\}$ by Theorem 12, we conclude from Theorem 13 that P_j is a *cl*-closed ideal of X for all $j \in \{1, 2, \cdots, n\}$. $\square$

4. Conclusions

Necessary conditions for the relative annihilator in lower *BCK*-semilattices to be a prime ideal are discussed. In addition, we provided conditions for any prime ideal in the minimal prime decomposition of an ideal A, to be one of the minimal prime factors of A. Homomorphic image and pre-image of the minimal prime decomposition of an ideal are considered. Using a semi-prime closure operation "*cl*", we showed that every minimal prime factor of a *cl*-closed ideal A is also *cl*-closed.

These results can be applied to characterize the composable ideals in a *BCK*-algebra with their associated prime ideals. In our future research, we will focus on some properties of decomposable ideal such as intersections, unions, maximality, and height, and try to find the relations between these properties of ideals and the associated prime ideals. For instance, is the height of the arbitrary decomposable ideal, equal to the sum of the height of associated prime ideals? For information about the height of ideals, please refer to [23–25].

In addition, other kinds of closure operations such as meet, tender, nave, finite, prime, etc. can be checked for prime ideals in prime decompositions. For further information about other kinds of (weak) closure operation, please refer to [2,3,21,22].

In addition, for future research, we invite the researchers to join us and apply the results of this paper to new concepts in [26–28].

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