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Representations of a Comparison Measure between Two Fuzzy Sets

Juin-Han Chen [1] and Hui-Chin Tang [2,*]

[1] Department of Industrial Engineering and Management, Cheng Shiu University, Kaohsiung 83347, Taiwan; K0411@gcloud.csu.edu.tw

[2] Department of Industrial Engineering and Management, National Kaohsiung University of Science and Technology, Kaohsiung 80778, Taiwan

* Correspondence: tang@nkust.edu.tw

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Abstract: This paper analyzes the representation behaviors of a comparison measure between two compared fuzzy sets. Three types of restrictions on two fuzzy sets are considered in this paper: two disjoint union fuzzy sets, two disjoint fuzzy sets and two general fuzzy sets. Differences exist among the numbers of possible representations of a comparison measure for the three types of fuzzy sets restrictions. The value of comparison measure is constant for two disjoint union fuzzy sets. There are 49 candidate representations of a comparison measure for two disjoint fuzzy sets, of which 13 candidate representations with one or two terms are obtained. For each candidate representation, a variant of the general axiomatic definition for a comparison measure is presented. Choosing the right candidate representation for a given application, we can easily and efficiently calculate and compare a comparison measure.

Keywords: fuzzy set; comparison measure; representation; disjoint

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1. Introduction

The concept of fuzzy sets (FSs), proposed by Zadeh [1], is characterized by a membership function and has successfully been applied in various fields. This paper deals with the well-known notions of comparison measures between two compared FSs. A comparison measure calculates the degree of equality or inequality between two compared FSs. Some related definitions such as similarity, similitude, proximity or resemblance were proposed for the equality measures [2–18], as well as some other dual definitions such as dissimilarity, dissimilitude, divergence or distance for the inequality measures [6,9,13,18–23]. The inequality measures have received much less attention in the literature. The degree of comparison measure is an important tool for cluster analysis [8], decision-making [7,14,21,22], e-waste [2,17,20], image processing [10], medical diagnosis [13], pattern recognition [3,4,9,12] and service quality [11,18]. Recently, many papers [5,9,10,12,15,16,18,20–22] have been dedicated to the comparison measures, and research on this area is still carried out in the literature.

Couso et al. [6] surveyed a large collection of axiomatic definitions from the literature regarding the notions of comparison measures between two compared FSs. Three separate lists of properties are provided: general axioms, axioms for the equality measures and axioms for the inequality measures. One of the general axioms of a comparison measure is as follows. For two disjoint FSs, A and B, if both comparison measures between A and empty set and that of B and empty set are less, then the degree of a comparison measure $m(A, B)$ between A and B is less. More precisely, if $A \cap B = \varnothing$, $A' \cap B' = \varnothing$, $m(A, \varnothing) \le m(A', \varnothing)$ and $m(B, \varnothing) \le m(B', \varnothing)$, then $m(A, B) \le m(A', B')$, for all FSs A, B, A', B'. From this axiomatic definition, we analyze the comparison measure behaviors of two FSs, A and B, in terms of the other simple comparison measures, especially for the intersection and the union of A and B, the empty set and the universal set. This general axiomatic definition has received much less attention in the literature. Consider a local divergence $d(A,\ B)$ for two disjoint FSs A and B, proposed by Montes et al. [19]. Couso et al. [6] showed that $d(A,\ B) = d(A,\ \varnothing) + d(B,\ \varnothing)$, which satisfies this general axiomatic definition. This paper adopts the representations of a comparison measure to generalize this axiomatic definition and to efficiently compare the comparison measure between two FSs. The representations of a comparison measure between two FSs can not only present the important components of a comparison measure but also analyze the comparison measure behaviors of two FSs in terms of other simple comparison measures. The representative equivalence between two representations indicates that these representations fulfill symmetric property.

To analyze the representation behaviors of a comparison measure between two FSs, three kinds of two FSs are considered in this paper: two disjoint union FSs, two disjoint FSs and two general FSs. The two disjoint union FSs is a special case of two disjoint FSs, and the latter is a special case of two general FSs. Both special cases derive some interesting results, especially for the case of two disjoint FSs. For two FSs A and B, this paper deals with the representations of a comparison measure for the case that $A \cap B = \varnothing$ and $A \cup B = U$, the case that $A \cap B = \varnothing$ and the general FSs A and B.

The organization of this paper is as follows. Section 2 briefly reviews the FSs and the comparison measures between two compared FSs. We present representations of a comparison measure for two disjoint union FSs in Section 3, two disjoint FSs in Section 4 and two general FSs in Section 5. Finally, some concluding remarks and future research are presented.

2. Fuzzy Sets and Comparison Measures

We firstly review the basic notations of FSs. Let U be a non-empty universal set or referential set.

Definition 1. *A FS A over U is defined as*

$$A = \{(x,\ \mu_A(x)) \big| x \in U\}$$

where the membership function $\mu_A(x) : U \to [0,1]$. We denote by $\mathcal{F}(U)$ the set of all FSs over U.

Definition 2. *For two FSs $A, B \in \mathcal{F}(U)$, define the membership functions of $A \cap B$, $A \cup B$ and $A \backslash B$ as follows:*

1. $\mu_{A\cap B}(x) = min\{\mu_A(x), \mu_B(x)\}$.
2. $\mu_{A\cup B}(x) = max\{\mu_A(x), \mu_B(x)\}$.
3. $\mu_{A\backslash B}(x) = min\{\mu_A(x), 1 - \mu_B(x)\}$.

We now recall the definition of comparison measures between two FSs. The following properties are general axioms that may be required in equality measure and inequality measure between two FSs [6].

Definition 3. *A comparison measure $m : \mathcal{F}(U)^2 \to \mathcal{R}$ should satisfy the following properties:*

- *G1:* $0 \le m(A, B) \le 1, \forall\, A, B \in \mathcal{F}(U)$.
- *G1*:* $0 \le m(A, B) \le 1, \forall\, A, B \in \mathcal{F}(U)$ *and there exists two FSs $C, D \in \mathcal{F}(U)$ such that $m(C, D) = 1$.*
- *G2:* $m(A, B) = m(B, A), \forall\, A, B \in \mathcal{F}(U)$.
- *G3: Let $\rho : U \to U$ be a permutation for finite U. Define $A^\rho \in \mathcal{F}(U)$ with membership function $\mu_{A^\rho}(x) = \mu_A(\rho(x))$ for $A \in \mathcal{F}(U)$. Then $m(A, B) = m(A^\rho, B^\rho)$.*
- *G3*: For finite set U, there exists a function $h : [0,1] \times [0,1] \to \mathcal{R}$ such that $m(A, B) = \sum_{x \in U} h(\mu_A(x), \mu_B(x)), \forall\, A, B \in \mathcal{F}(U)$.*
- *G4: There exists a function $f : \mathcal{F}(\mathrm{U})^3 \to \mathcal{R}$ such that $m(A, B) = f(A \cap B, A\backslash B, B\backslash A), \forall\, A, B \in \mathcal{F}(U)$.*
- *G4*: There exists a function $F : \mathcal{R}^3 \to \mathcal{R}$ and a fuzzy measure $M : \mathcal{F}(U) \to \mathcal{R}$ such that $m(A, B) = F(M(A \cap B), M(A\backslash B), M(B\backslash A)), \forall\, A, B \in \mathcal{F}(U)$.*

- *G5: If $A \cap B = \varnothing$, $A' \cap B' = \varnothing$, $m(A, \varnothing) \le m(A', \varnothing)$ and $m(B, \varnothing) \le m(B', \varnothing)$, then $m(A, B) \le m(A', B')$, $\forall\, A, B, A', B' \in \mathcal{F}(U)$.*

Couso et al. [6] showed that the asterisk will be understood as stronger than. More precisely, if a comparison measure satisfies Gi*, then it fulfills Gi, for $i = 1,\ 3,\ 4$.

Consider a fuzzy measure $M : \mathcal{F}(U) \to \mathcal{R}$ with $M(A \cap B) = c$, $M(A \backslash B) = a$ and $M(B \backslash A) = b$, $a,\ b,\ c \in [0,\ 1]$. Define a comparison measure as follows:

$$m(A, B) = F(M(A \cap B), M(A \backslash B), M(B \backslash A)) = F(c, a, b) = \frac{c + 1 - a + 1 - b}{3}.$$

For two disjoint FSs A and B, we have

$$m(A, B) = F(0, a, b) = \frac{2 - a - b}{3},$$

$$m(A, \varnothing) = F(0,\ a,\ 0) = \frac{2 - a}{3}$$

and

$$m(B, \varnothing) = F(0,\ b,\ 0) = \frac{2 - b}{3},$$

it implies that

$$m(A,\ B) = -\frac{2}{3} + m(A,\ \varnothing) + m(B, \varnothing).$$

If $A \cap B = \varnothing$, $A' \cap B' = \varnothing$, $m(A, \varnothing) \le m(A', \varnothing)$ and $m(B, \varnothing) \le m(B', \varnothing)$, we obtain

$$m(A, B) = -\frac{2}{3} + m(A,\ \varnothing) + m(B, \varnothing) \le -\frac{2}{3} + m(A',\ \varnothing) + m(B', \varnothing) = m(A', B')$$

which coincides with the result of G5. The representation of $m(A,\ B)$ can not only present its important ingredients but also compare $m(A,\ B)$ in terms of other measures $m(A,\ \varnothing)$ and $m(B, \varnothing)$. On the other hand, we have

$$m(A, U) = F(a,\ 0,\ 1 - a) = \frac{1 + 2a}{3},$$

$$m(B, U) = F(b,\ 0,\ 1 - b) = \frac{1 + 2b}{3}$$

and

$$m(A,\ B) = 1 - \frac{1}{2} m(A, U) - \frac{1}{2} m(B, U).$$

The general axiomatic definition G5 can be written as follows.

If $A \cap B = \varnothing$, $A' \cap B' = \varnothing$, $m(A, U) \ge m(A', U)$ and $m(B, U) \ge m(B', U)$, then $m(A, B) \le m(A', B')$, $\forall\, A, B, A', B' \in \mathcal{F}(U)$.

Applying different representations of $m(A, B)$, the alternative expressions of the general axiom G5 are presented. For two FSs, A and B, the adopted components of a comparison measure are A, B, the intersection and the union of A and B, the empty set and the universal set. To represent a comparison measure $m(A,\ B)$, the adopted comparison measures other than $m(A,\ B)$ are $m(X, Y)$ for different FSs, X and Y, $(X, Y) \ne (A,\ B)$, $X, Y \in \{\varnothing,\ A,\ B,\ A \cap B,\ A \cup B,\ U\}$.

The following sections list the representations of a comparison measure $m(A,\ B)$ for two disjoint union FSs A and B, two disjoint FSs A and B and two general FSs A and B. More precisely, we consider the case that $A \cap B = \varnothing$ and $A \cup B = U$ for Section 3, $A \cap B = \varnothing$ for Section 4 and the general FSs A and B for Section 5. Sections 3 and 4 are special cases of Section 5. Some interesting conclusions can be drawn from these special cases.

3. Representations of a Comparison Measure for Two Disjoint Union Fuzzy Sets

This section will present the representations of a comparison measure $m(A,\ B)$ for two disjoint union FSs A and B. For $A \cap B = \varnothing$ and $A \cup B = U$, we have that $M(A \cap B) = 0$, $M(A \cup B) = 1$, $M(A \backslash B) = a$, $M(B \backslash A) = b$, $a + b = 1$, $a,\ b \in [0,\ 1]$ and

$$m(A,B) = F(M(A \cap B), M(A \backslash B), M(B \backslash A)) = F(0,a,b) = \frac{2-a-b}{3} = \frac{1}{3}.$$

Since $A \cap B = \varnothing$ and $A \cup B = U$, the adopted components of a comparison measure are $\{\varnothing,\ A,\ B,\ U\}$. For two different FSs, X and Y, $X, Y \in \{\varnothing,\ A,\ B,\ U\}$, the number of the possible forms of $m(X,Y)$ is six described as follows:

$$m(A,B) = F(0,a,b) = \frac{1}{3},$$

$$m(A,\varnothing) = F(0,\ a,\ 0) = \frac{2-a}{3},$$

$$m(B,\varnothing) = F(0,\ b,\ 0) = \frac{2-b}{3},$$

$$m(A,U) = F(a,\ 0,\ 1-a) = \frac{1+2a}{3},$$

$$m(B,U) = F(b,\ 0,\ 1-b) = \frac{1+2b}{3}$$

and

$$m(\varnothing,U) = F(0,0,1) = \frac{1}{3}.$$

From these six measures $m(X,Y)$, we obtain six equations for the representations of $\frac{a}{3}$ and six equations for those of $\frac{b}{3}$ presented as follows:

$$\begin{gathered}\tfrac{a}{3} = \tfrac{1}{2}(m(A,\ U) - m(A,\ B)) = m(B,\ \varnothing) - m(A,\ B) \\ = \tfrac{1}{2}\left(\tfrac{4}{3} - m(A,\ B) - m(B,\ U)\right) = \tfrac{4}{3} - m(B,\ U) - m(B,\ \varnothing) \\ = m(A,\ U) - m(B,\varnothing) = \tfrac{1}{2}(1 - m(B,\ U))\end{gathered}$$

and

$$\begin{gathered}\tfrac{b}{3} = \tfrac{1}{2}(m(B,\ U) - m(A,\ B)) = m(A,\ \varnothing) - m(A,\ B) \\ = \tfrac{1}{2}\left(\tfrac{4}{3} - m(A,\ B) - m(A,U)\right) = \tfrac{4}{3} - m(A,\ U) - m(A,\ \varnothing) \\ = m(B,U) - m(A,\varnothing) = \tfrac{1}{2}(1 - m(A,\ U)).\end{gathered}$$

Since the constant value of

$$m(A,B) = \frac{1}{3}$$

for $A \cap B = \varnothing$ and $A \cup B = U$, we cannot compare the degree of $m(A,B)$ for two disjoint union FSs A and B.

4. Representations of a Comparison Measure for Two Disjoint Fuzzy Sets

For two disjoint FSs A and B, $A \cap B = \varnothing$, we denote $M(A \cap B) = 0$, $M(A \backslash B) = a$, $M(B \backslash A) = b$, $a,\ b \in [0,\ 1]$ and

$$m(A,B) = F(M(A \cap B), M(A \backslash B), M(B \backslash A)) = F(0,a,b) = \frac{2-a-b}{3}.$$

The number of total combinations $m(X,Y)$ for two different FSs, X and Y, $X,Y \in \{\varnothing,\ A,\ B,\ A\cup B,\ U\}$, is 10 presented as follows:

$$m(A,B) = F(0,a,b) = \frac{2-a-b}{3},$$

$$m(A,\varnothing) = F(0,\ a,\ 0) = \frac{2-a}{3},$$

$$m(B,\varnothing) = F(0,\ b,\ 0) = \frac{2-b}{3},$$

$$m(A,A\cup B) = F(a,\ 0,\ b) = \frac{2+a-b}{3},$$

$$m(B,A\cup B) = F(b,\ 0,\ a) = \frac{2-a+b}{3},$$

$$m(A,U) = F(a,\ 0,\ 1-a) = \frac{1+2a}{3},$$

$$m(B,U) = F(b,\ 0,\ 1-b) = \frac{1+2b}{3},$$

$$m(A\cup B,\varnothing) = F(0,\ a+b,\ 0) = \frac{2-a-b}{3},$$

$$m(A\cup B,U) = F(a+b,0,1-a-b) = \frac{1+2a+2b}{3}$$

and

$$m(\varnothing,U) = F(0,0,1) = \frac{1}{3}.$$

To represent $m(A,B) = \frac{2-a-b}{3}$, from above ten measures $m(X,Y)$, we obtain nine equations for the representations of $\frac{a}{3}$ and nine equations for those of $\frac{b}{3}$ described as follows:

$$\frac{a}{3}$$

$$\begin{aligned}
&[1] = \tfrac{1}{2}(m(A,\ A\cup B) - m(A,\ B))\\
&[2] = m(B,\ \varnothing) - m(A,\ B)\\
&[3] = \tfrac{1}{2}\left(\tfrac{4}{3} - m(A,\ B) - m(B,\ A\cup B)\right)\\
&[4] = \tfrac{1}{2}\left(\tfrac{5}{3} - 2m(A,\ B) - m(B,\ U)\right)\\
&[5] = \tfrac{4}{3} - m(B,\ A\cup B) - m(B,\ \varnothing)\\
&[6] = m(A,\ A\cup B) - m(B,\varnothing)\\
&[7] = \tfrac{1}{2}\left(-\tfrac{5}{3} + m(A\cup B,U) + 2m(B,\ \varnothing)\right)\\
&[8] = \tfrac{1}{4}(1 + m(A\cup B,U) - 2m(B,\ A\cup B))\\
&[9] = \tfrac{1}{2}(m(A\cup B,U) - m(B,\ U))
\end{aligned}$$

and

$$\frac{b}{3}$$

$$\begin{aligned}
&[1]= \tfrac{1}{2}(m(B,\ A\cup B)-m(A,\ B))\\
&[2]= m(A,\ \varnothing)-m(A,\ B)\\
&[3]= \tfrac{1}{2}\Big(\tfrac{4}{3}-m(A,\ B)-m(A,\ A\cup B)\Big)\\
&[4]= \tfrac{1}{2}\Big(\tfrac{5}{3}-2m(A,\ B)-m(A,\ U)\Big)\\
&[5]= \tfrac{4}{3}-m(A,\ A\cup B)-m(A,\ \varnothing)\\
&[6]= m(B,\ A\cup B)-m(A,\varnothing)\\
&[7]= \tfrac{1}{2}\Big(-\tfrac{5}{3}+m(A\cup B,U)+2m(A,\ \varnothing)\Big)\\
&[8]= \tfrac{1}{4}(1+m(A\cup B,U)-2m(A,\ A\cup B))\\
&[9]= \tfrac{1}{2}(m(A\cup B,U)-m(A,\ U)).
\end{aligned}$$

The number of the total combinations of forms of $\frac{a}{3}$ and $\frac{b}{3}$ to represent a comparison measure $m(A,\ B)$ is $9\times 9=81$. We will denote by [i]-[j], the combination of ith form of $\frac{a}{3}$ and jth form of $\frac{b}{3}$ to represent $m(A,\ B)$. We classify these 81 combinations into four types (I, II, III, IV). The first type I is the candidate representation of a comparison measure $m(A,\ B)$. For example, the combination [1]-[2], the 1st form of $\frac{a}{3}$ and the 2nd form of $\frac{b}{3}$ are adopted. Applying $\frac{a}{3}=\frac{1}{2}(m(A,\ A\cup B)-m(A,\ B))$ and $\frac{b}{3}=m(A,\ \varnothing)-m(A,\ B)$ to $m(A,B)=\frac{2-a-b}{3}$, we obtain

$$m(A,B)=\frac{2}{3}-\frac{1}{2}(m(A,\ A\cup B)-m(A,\ B))-(m(A,\ \varnothing)-m(A,\ B))=-\frac{4}{3}+m(A,\ A\cup B)+2m(A,\ \varnothing).$$

Among these 81 combinations, there are 49 candidate representations of $m(A,\ B)$ for type I. The number of terms $m(X,Y)$, $X,Y\in\{\varnothing,\ A,\ B,\ A\cup B,\ U\}$ of a candidate representation of $m(A,\ B)$ is 1, 2, 3 and 4, except for the constant term. There are 1, 12, 27 and 9 candidate representations of $m(A,\ B)$ for the number of terms being 1, 2, 3 and 4, respectively. The combination [1]-[8] is the one term $m(X,Y)$ of a candidate representation of $m(A,\ B)$ as follows.

$$[1]\text{-}[8]:\ m(A,B)=\frac{5}{6}-\ \frac{1}{2}m(A\cup B,U).$$

Using this candidate representation, a variant of general axiom G5 is described as follows. If $A\cap B=\varnothing$, $A'\cap B'=\varnothing$ and $m(A\cup B,U)\geq m(A'\cup B',U)$, then $m(A,B)\leq m(A',B')$. The two terms $m(X,Y)$, $X,Y\in\{\varnothing,\ A,\ B,\ A\cup B,\ U\}$ of candidate representations of $m(A,\ B)$ are as follows.

$$[1]\text{-}[2]:\ m(A,\ B)=-\frac{4}{3}+m(A,\ A\cup B)+2m(A,\ \varnothing)$$

$$[2]\text{-}[1]:\ m(A,B)=-\frac{4}{3}+m(B,\ A\cup B)+2m(B,\ \varnothing)$$

$$[2]\text{-}[2]:\ m(A,\ B)=-\frac{2}{3}+m(A,\ \varnothing)+m(B,\varnothing)$$

$$[2]\text{-}[3]:\ m(A,B)=-\ m(A,\ A\cup B)+2m(B,\ \varnothing)$$

$$[3]\text{-}[2]:\ m(A,\ B)=-m(B,\ A\cup B)+2m(A,\ \varnothing)$$

$$[2]\text{-}[4]:\ m(A,\ B)=\frac{1}{6}-\frac{1}{2}m(A,U)+m(B,\ \varnothing)$$

$$[4]\text{-}[2]:\ m(A,\ B)=\frac{1}{6}-\frac{1}{2}m(B,U)+m(A,\ \varnothing)$$

$$[1]\text{-}[4]:\ m(A,\ B)=\frac{1}{3}+m(A,\ A\cup B)-m(A,U)$$

$$[4]\text{-}[1]:\ m(A,B)=\frac{1}{3}+m(B,\ A\cup B)-m(B,U)$$

$$[4]\text{-}[4]:\ m(A,\ B) = 1 - \frac{1}{2}m(A,U) - \frac{1}{2}m(B,U)$$

$$[3]\text{-}[4]:\ m(A,\ B) = \frac{5}{3} - m(B,\ A \cup B) - m(A,U)$$

and

$$[4]\text{-}[3]:\ m(A,B) = \frac{5}{3} - m(A,\ A \cup B) - m(B,\ U).$$

If FSs A and B are interchanged, the representation of combination [1]-[2] becomes

$$m(B,A) = -\frac{4}{3} + m(B,\ A \cup B) + 2m(B,\ \varnothing)$$

which is equal to the representation of combination [2]-[1]. This symmetric property is also satisfied for combinations [2]-[3] and [3]-[2], combinations [2]-[4] and [4]-[2], combinations [1]-[4] and [4]-[1], combinations [3]-[4] and [4]-[3]. While combinations [2]-[2] and [4]-[4] derive the same representation when FSs A and B are interchanged. Therefore, if combination [i]-[j]: $m(A,\ B)$ is a candidate representation, then both combination [i]-[j]: $m(B,A)$ and combination [j]-[i]: $m(A,\ B)$ are equal and are also candidate representations.

The second type II is the relationship between different terms of $m(X,Y)$, $X,Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$ other than $m(A,\ B)$. For example, the combination [1]-[1], $\frac{a}{3} = \frac{1}{2}(m(A,\ A \cup B) - m(A,\ B))$ and $\frac{b}{3} = \frac{1}{2}(m(B,\ A \cup B) - m(A,\ B))$, we get that

$$m(A,B) = \frac{2}{3} - \frac{1}{2}(m(A,\ A \cup B) - m(A,\ B)) - \frac{1}{2}(m(B,\ A \cup B) - m(A,\ B))$$

so

$$m(A,\ A \cup B) + m(B,\ A \cup B) = \frac{4}{3}.$$

The combinations [1]-[1], [2]-[5], [2]-[6], [2]-[7], [2]-[8], [2]-[9], [3]-[3], [4]-[5], [4]-[6], [4]-[7], [4]-[8], [4]-[9], [5]-[2], [5]-[4], [6]-[2], [6]-[4], [7]-[2], [7]-[4], [8]-[2], [8]-[4] and [9]-[2] are included in type II. Among these 21 combinations, the number of different relationships between different terms of $m(X,Y)$, $X,Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$ other than $m(A,\ B)$ is 16.

The third type III is the identical equation 0=0. For example, for the combination [1]-[3], we obtain

$$m(A,B) = \frac{2}{3} - \frac{1}{2}(m(A,\ A \cup B) - m(A,\ B)) - \frac{1}{2}\Big(\frac{4}{3} - m(A,\ B) - m(A,\ A \cup B)\Big)$$

so

$$0 = 0.$$

The combinations [1]-[3], [3]-[1] and [9]-[4] are listed in the type III.

The fourth type IV is the duplicate representations of a comparison measure $m(A,\ B)$ which appear in type I. For example, for the combination [1]-[5], we obtain that

$$m(A,B) = \frac{2}{3} - \frac{1}{2}(m(A,\ A \cup B) - m(A,\ B)) - \Big(\frac{4}{3} - m(A,\ A \cup B) - m(A,\ \varnothing)\Big)$$

$$m(A,B) = -\frac{4}{3} + m(A,\ A \cup B) + 2m(A,\ \varnothing)$$

which is the same as that of combination [1]-[2]. For simplicity, we adopt the notation [1]-[2] ≡ [1]-[5] to denote the representative equivalence between [1]-[2] and [1]-[5]. There are eight combinations in type IV described as follows:

[2]-[2] ≡[5]-[6] ≡[6]-[5] ≡ [7]-[6],

[1]-[2] ≡[[8]-[1], [2]-[1] ≡[[6]-[3] and [3]-[2] ≡ [3]-[6].

The largest number of duplicate representations is three with the associated combination [2]-[2]

$$m(A,\ B) = -\frac{2}{3} + m(A,\ \varnothing) + m(B, \varnothing).$$

Therefore, there are 81 combinations of a comparison measure $m(A,\ B)$ for $A \cap B = \varnothing$. Among these 81 combinations, we obtain 49 candidate representations of $m(A,\ B)$, 8 duplicate representations of $m(A,\ B)$, 21 relationships between different terms of $m(X, Y)$, $X, Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$ other than $m(A,\ B)$ and 3 identical equations. There are one and 12 candidate representations of $m(A,\ B)$ for one and two terms $m(X, Y)$, $X, Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$, respectively. These 13 candidate representations can be used to easily compare $m(A,\ B)$ with $A \cap B = \varnothing$.

5. Representations of a Comparison Measure for Two General Fuzzy Sets

This section lists the representations of a comparison measure $m(A,\ B)$ for two general FSs, A and B. Let $M(A \cap B) = c$, $M(A \backslash B) = a$, $M(B \backslash A) = b$, $a,\ b,\ c \in [0,\ 1]$ and

$$m(A, B) = F(M(A \cap B), M(A \backslash B), M(B \backslash A)) = F(c, a, b) = \frac{c + 1 - a + 1 - b}{3}.$$

The adopted components of a comparison measure are $\{\varnothing,\ A,\ B,\ A \cap B,\ A \cup B,\ U\}$. There are 15 combinations $m(X, Y)$ of different FSs, X and Y, $X, Y \in \{\varnothing,\ A,\ B,\ A \cap B,\ A \cup B,\ U\}$ as follows.

$$m(A, B) = F(c, a, b) = \frac{2 - a - b + c}{3},$$

$$m(A, \varnothing) = F(0,\ a + c,\ 0) = \frac{2 - a - c}{3},$$

$$m(B, \varnothing) = F(0,\ b + c,\ 0) = \frac{2 - b - c}{3},$$

$$m(A, A \cap B) = F(c,\ a,\ 0) = \frac{2 - a + c}{3},$$

$$m(B, A \cap B) = F(c,\ b,\ 0) = \frac{2 - b + c}{3},$$

$$m(A, A \cup B) = F(a + c,\ 0,\ b) = \frac{2 + a - b + c}{3},$$

$$m(B, A \cup B) = F(b + c,\ 0,\ a) = \frac{2 - a + b + c}{3},$$

$$m(A, U) = F(a + c,\ 0,\ 1 - a - c) = \frac{1 + 2a + 2c}{3},$$

$$m(B, U) = F(b + c,\ 0,\ 1 - b - c) = \frac{1 + 2b + 2c}{3},$$

$$m(A \cap B, \varnothing) = F(0,\ c,\ 0) = \frac{2 - c}{3},$$

$$m(A \cup B, \varnothing) = F(0,\ a + b + c,\ 0) = \frac{2 - a - b - c}{3},$$

$$m(A \cap B, \mathrm{A} \cup B) = F(c, a + b, 0) = \frac{2 - a - b + c}{3},$$

$$m(A \cap B, U) = F(c, 0, 1 - c) = \frac{1 + 2c}{3},$$

$$m(A \cup B, U) = F(a + b + c, 0, 1 - a - b - c) = \frac{1 + 2a + 2b + 2c}{3}$$

and

$$m(\varnothing, U) = F(0,0,1) = \frac{1}{3}.$$

One can make several notable observations. Firstly, we have that

$$m(A \cap B, A \cup B) = \frac{2-a-b+c}{3} = m(A,B).$$

So, to calculate the degree of a comparison measure $m(A,B)$ is equivalent to calculate that of $m(A \cap B, A \cup B)$.

Secondly, we have 33 different relationships between different terms of $m(X,Y)$, $X,Y \in \{\varnothing,\ A,\ B,\ A \cap B,\ A \cup B,\ U\}$. Since $m(A,B) = \frac{2-a-b+c}{3}$, from these 33 relationships, we obtain 12 equations for the representations of $\frac{a}{3}$, 12 equations for those of $\frac{b}{3}$ and 3 equations for those of $\frac{c}{3}$ presented as follows:

$$\begin{aligned}
\tfrac{a}{3} &= \tfrac{1}{2}(m(A,\ A \cup B) - m(A,\ B)) = m(B,\ A \cap B) - m(A,\ B) = m(A \cap B, \varnothing) - m(A, \varnothing) \\
&= \tfrac{1}{2}\Big(\tfrac{4}{3} - m(A,\ A \cap B) - m(A,\ \varnothing)\Big) = \tfrac{4}{3} - m(B,\ A \cup B) - m(B,\ \varnothing) = m(A,\ A \cup B) - m(B,\ A \cap B) \\
&= m(B,\ \varnothing) - m(A \cup B,\ \varnothing) = \tfrac{1}{2}\Big(\tfrac{4}{3} - m(A \cup B,\ \varnothing) - m(B,\ A \cup B)\Big) = \tfrac{1}{2}\Big(\tfrac{5}{3} - 2m(A \cup B,\ \varnothing) - m(B,\ U)\Big) \\
&= \tfrac{1}{2}\Big(-\tfrac{5}{3} + m(A \cup B, U) + 2m(B,\ \varnothing)\Big) = \tfrac{1}{4}(1 + m(A \cup B, U) - 2m(B,\ A \cup B)) = \tfrac{1}{2}(m(A \cup B, U) - m(B,\ U)),
\end{aligned}$$

$$\begin{aligned}
\tfrac{b}{3} &= \tfrac{1}{2}(m(B,\ A \cup B) - m(A,\ B)) = m(A,\ A \cap B) - m(A,\ B) = m(A \cap B, \varnothing) - m(B, \varnothing) \\
&= \tfrac{1}{2}\Big(\tfrac{4}{3} - m(B,\ A \cap B) - m(B,\ \varnothing)\Big) = \tfrac{4}{3} - m(A,\ A \cup B) - m(A,\ \varnothing) = m(B,\ A \cup B) - m(A,\ A \cap B) \\
&= m(A,\ \varnothing) - m(A \cup B,\ \varnothing) = \tfrac{1}{2}\Big(\tfrac{4}{3} - m(A \cup B,\ \varnothing) - m(A,\ A \cup B)\Big) = \tfrac{1}{2}\Big(\tfrac{5}{3} - 2m(A \cup B,\ \varnothing) - m(A,\ U)\Big) \\
&= \tfrac{1}{2}\Big(-\tfrac{5}{3} + m(A \cup B, U) + 2m(A,\ \varnothing)\Big) = \tfrac{1}{4}(1 + m(A \cup B, U) - 2m(A,\ A \cup B)) = \tfrac{1}{2}(m(A \cup B, U) - m(A,\ U))
\end{aligned}$$

and

$$\frac{c}{3} = \frac{1}{2}\Big[m(A,\ A \cap B) - m(A,\ \varnothing)\Big] = \frac{1}{2}\Big[m(B,\ A \cap B) - m(B,\ \varnothing)\Big] = \frac{1}{2}\Big(-\frac{4}{3} + m(A,\ A \cup B) + m(B,\ A \cup B)\Big).$$

The number of total combinations $m(A,B)$ of forms of $\frac{a}{3}$, $\frac{b}{3}$ and $\frac{c}{3}$ is $12 \times 12 \times 3 = 432$. The number of combinations is large. Detailed representations of a comparison measure $m(A,\ B)$ are available from authors.

From $\frac{a}{3} = m(B,\ A \cap B) - m(A,\ B)$, $\frac{b}{3} = m(A,\ A \cap B) - m(A,\ B)$ and $\frac{c}{3} = \frac{1}{2}[m(A,\ A \cap B) - m(A,\ \varnothing)]$, it implies that

$$m(A,B) = \frac{2}{3} - m(B,\ A \cap B) + m(A,\ B) - m(A,\ A \cap B) + m(A,\ B) + \frac{1}{2}[m(A,\ A \cap B) - m(A,\ \varnothing)]$$

$$m(A,B) = -\frac{2}{3} + \frac{1}{2}m(A,\ A \cap B) + m(B,\ A \cap B) + \frac{1}{2}m(A,\ \varnothing).$$

Similarly, we have that

$$m(A,B) = -\frac{2}{3} + m(A,\ A \cap B) + \frac{1}{2}m(B,\ A \cap B) + \frac{1}{2}m(B,\ \varnothing),$$

$$m(A,B) = \frac{2}{3} - 2m(A \cap B,\ \varnothing) + \frac{1}{2}m(A,\ A \cap B) + \frac{1}{2}m(A,\ \varnothing) + m(B,\ \varnothing)$$

and

$$m(A,B) = \frac{2}{3} - 2m(A \cap B,\ \varnothing) + \frac{1}{2}m(B,\ A \cap B) + m(A,\ \varnothing) + \frac{1}{2}m(B,\ \varnothing).$$

If $A \cap B = \varnothing$, the above four representations of a comparison measure $m(A,\ B)$ reduce to

$$m(A,B) = -\frac{2}{3} + m(A,\ \varnothing) + m(B,\ \varnothing)$$

which is the representation [2]-[2] of a comparison measure $m(A,\ B)$ with $A \cap B = \varnothing$ appearing in Section 4. Therefore, for two disjoint FSs, A and B, the representation of a comparison measure $m(A, B)$ with general FSs can be reduced to that of $m(A, B)$ with $A \cap B = \varnothing$.

6. Conclusions and Future Research

For two FSs, A and B, this paper presents the representations of a comparison measure $m(A, B)$ for two disjoint union FSs, two disjoint FSs and two general FSs. The numbers of total combinations $m(A, B)$ are 36, 81 and 432 for two disjoint union FSs, two disjoint FSs and two general FSs, respectively. The smaller the number of restrictions placed on two FSs, the greater the number of possible representations of a comparison measure. For two disjoint union FSs, the constant value of $m(A, B) = \frac{1}{3}$ implies that we cannot compare the comparison behaviors of two disjoint union FSs A and B. Among the 81 combinations of two disjoint FSs A and B, there are 49 candidate representations of $m(A,\ B)$, 8 duplicate representations of $m(A,\ B)$, 21 relationships between different $m(X, Y)$, $X, Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$ other than $m(A,\ B)$ and 3 identical equations. There are one and 12 candidate representations of $m(A,\ B)$ for one and two terms $m(X, Y)$, $X, Y \in \{\varnothing,\ A,\ B,\ A \cup B,\ U\}$, respectively. For each candidate representation, if combination [i]-[j]: $m(A,\ B)$ is a candidate representation, then both combination [i]-[j]: $m(B, A)$ and combination [j]-[i]: $m(A,\ B)$ are also candidate representations. The representative equivalence between combination [i]-[j]: $m(B, A)$ and combination [j]-[i]: $m(A,\ B)$ indicates that the candidate representation fulfills symmetric property. Applying these 13 candidate representations, the alternative expressions of the general axiom G5 are presented. Choosing the right general axiom G5 for a given application, we can easily and efficiently calculate and compare the degree of a comparison measure $m(A,\ B)$ with $A \cap B = \varnothing$.

In the future, we will analyze the representation behaviors of comparison measures for the generalization of FSs and the general forms of a comparison measure. In particular, the analysis can be extended to the intuitionistic fuzzy sets, hesitant fuzzy sets and neutrosophic sets. Thus, the representation analysis of comparison measures for the intuitionistic fuzzy sets is a subject of considerable ongoing research.

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