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One-Dimensional Optimal System for 2D Rotating Ideal Gas

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Abstract: We derive the one-dimensional optimal system for a system of three partial differential equations, which describe the two-dimensional rotating ideal gas with polytropic parameter $\gamma > 2$. The Lie symmetries and the one-dimensional optimal system are determined for the nonrotating and rotating systems. We compare the results, and we find that when there is no Coriolis force, the system admits eight Lie point symmetries, while the rotating system admits seven Lie point symmetries. Consequently, the two systems are not algebraic equivalent as in the case of $\gamma = 2$, which was found by previous studies. For the one-dimensional optimal system, we determine all the Lie invariants, while we demonstrate our results by reducing the system of partial differential equations into a system of first-order ordinary differential equations, which can be solved by quadratures.

Keywords: lie symmetries; invariants; shallow water; similarity solutions; optimal system

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1. Introduction

A powerful mathematical treatment for the determination of exact solutions for nonlinear differential equations is the Lie symmetry analysis [1–3]. Specifically, Lie point symmetries help us in the simplification of differential equations by means of similarity transformations, which reduce the differential equation. The reduction process is based on the existence of functions that are invariant under a specific group of point transformations. When someone uses these invariants as new dependent and independent variables, the differential equation is reduced. The reduction process differs between ordinary differential equations (ODEs) and partial differential equations (PDEs). For ODEs, Lie point symmetries are applied to reduce the order of ODE by one; while on PDEs, Lie point symmetries are applied to reduce by one the number of independent variables, while the order of the PDEs remains the same. The solutions that are found with the application of those invariant functions are called similarity solutions. Some applications on the determination of similarity solutions for nonlinear differential equations can be found in [4–9] and the references therein.

A common characteristic in the reduction process is that the Lie point symmetries are not preserved during the reduction; hence, we can say that the symmetries can be lost. That is not an accurate statement, because symmetries are not “destroyed” or “created” under point transformations, but the “nature” of the symmetry changes. In addition, Lie symmetries can be used to construct new similarity solutions for a given differential equation by applying the adjoint representation of the Lie group [10].

It is possible that a given differential equation admits more than one similarity solution when the given differential equation admits a “large” number of Lie point symmetries. Hence, in order for someone to classify a differential equation according to the admitted similarity solutions, all the inequivalent Lie subalgebras of the admitted Lie symmetries should be determined.

The first group classification problem was carried out by Ovsiannikov [11], who demonstrated the construction of the one-dimensional optimal system for the Lie algebra. Since then, the classification of the one-dimensional optimal system has become a main tool for the study of nonlinear differential equations [12–15].

In this work, we focus on the classification of the one-dimensional optimal system for the two-dimensional rotating ideal gas system described by the following system of PDEs [16–18]:

$$h_t + (hu)_x + (hv)_y = 0, \tag{1}$$

$$u_t + uu_x + vu_y + h^{\gamma-2}h_x - fv = 0, \tag{2}$$

$$v_t + uv_x + vv_y + h^{\gamma-2}h_y + fu = 0. \tag{3}$$

where u and v are the velocity components in the x and y directions, respectively, h is the density of the ideal gas, f is the Coriolis parameter, and γ is the polytropic parameter of the fluid. Usually, γ is assumed to be $\gamma = 2$ where Equations (1)–(3) reduce to the shallow water system. However, in this work, we consider that $\gamma > 2$. In this work, polytropic index γ is defined as $\frac{C_p}{C_v} = \gamma - 1$.

Shallow water equations describe the flow of a fluid under a pressure surface. There are various physical phenomena that are described by the shallow water system with emphasis on atmospheric and oceanic phenomena [19–21]. Hence, the existence of the Coriolis force becomes critical in the description of the physical phenomena.

In the case of $\gamma = 2$, the complete symmetry analysis of the system (1)–(3) is presented in [22]. It was found that for $\gamma = 2$, the given system of PDEs is invariant under a nine-dimensional Lie algebra. The same Lie algebra, but in a different representation, is also admitted by the nonrotating system, i.e., $f = 0$. One of the main results of [22] is that the transformation that relates the two representations of the admitted Lie algebras for the rotating and nonrotating system transforms the rotating system (1)–(3) into the nonrotating one. For other applications of Lie symmetries on shallow water equations, we refer the reader to [23–28].

For the case of an ideal gas [17], i.e., parameter $\gamma > 1$ from our analysis, it follows that this property is lost. The nonrotating system and the rotating one are invariant under a different number of Lie symmetries and consequently under different Lie algebras. For each of the Lie algebras, we have the one-dimensional optimal system and all the Lie invariants. The results are presented in tables. We demonstrate the application of the Lie invariants by determining some similarity solutions for the system (1)–(3) for $\gamma > 2$. The paper is structured as follows.

In Section 2, we briefly discuss the theory of Lie symmetries for differential equations and the adjoint representation. The nonrotating system (1)–(3) is studied in Section 3. Specifically, we determine the Lie points symmetries, which form an eight-dimensional Lie algebra. The commutators and the adjoint representation are presented. We make use of these results, and we perform, a classification of the one-dimensional optimal system. We found that in total, there are twenty-three one-dimensional independent Lie symmetries and possible reductions, and the corresponding invariants are determined and presented in tables. In Section 4, we perform the same analysis for the rotating system. There, we find that the admitted Lie symmetries form a seven-dimensional Lie algebra, while there are twenty independent one-dimensional Lie algebras. We demonstrate the results by reducing the system of PDEs (1)–(3) into an integrable system of three first-order ODEs, the solution of which is given by quadratures. In Section 5, we discuss our results and draw our conclusions. Finally, in Appendix A, we present the tables, which include the results of our analysis.

2. Lie Symmetry Analysis

Let $H^A\left(x^i, \Phi^A, \Phi^A_i, ...\right) = 0$ be a system of partial differential equations (PDEs) where Φ^A denotes the dependent variables and x^i are the independent variables. At this point, it is important to mention that we make use of the Einstein summation convention. By definition, under the action of the infinitesimal one-parameter point transformation (1PPT):

$$\bar{x}^i = x^i\left(x^j, \Phi^B; \varepsilon\right), \quad \bar{\Phi}^A = \Phi^A\left(x^j, \Phi^B; \varepsilon\right), \tag{4}$$

which connects two different points $P\left(x^j,\Phi^B\right) \to Q\left(\bar{x}^j,\bar{\Phi}^B,\varepsilon\right)$, the differential equation $H^A = 0$ remains invariant if and only if $\bar{H}^A = H^A$, that is [2]:

$$\lim_{\varepsilon\to 0}\frac{\bar{H}^A\left(\bar{y}^i,\bar{u}^A,...;\varepsilon\right) - H^A\left(y^i,u^A,...\right)}{\varepsilon} = 0. \tag{5}$$

The latter condition means that the $\Phi^A\left(P\right)$ and $\Phi^A\left(Q\right)$ are connected through the transformation.

The lhs of Expression (5) defines the Lie derivative of H^A along the vector field X of the one-parameter point transformation (4), in which X is defined as:

$$X = \frac{\partial\bar{x}^i}{\partial\varepsilon}\partial_i + \frac{\partial\bar{\Phi}}{\partial\varepsilon}\partial_A.$$

Thus, Condition (5) is equivalent to the following expression: [2]

$$\mathcal{L}_X\left(H^A\right) = 0, \tag{6}$$

where $\mathcal{L}$ denotes the Lie derivative with respect to the vector field $X^{[n]}$, which is the n^{th}-extension of generator X of the transformation (4) in the jet space $\left\{x^i,\Phi^A,\Phi^A_{,i},\Phi^A_{,ij},...\right\}$ given by the expression [2]:

$$X^{[n]} = X + \eta^{[1]}\partial_{\Phi^A_i} + ... + \eta^{[n]}\partial_{\Phi^A_{i_i i_j ... i_n}}, \tag{7}$$

in which:

$$\eta^{[n]} = D_i\eta^{[n-1]} - u_{i_1 i_2 ... i_{n-1}}D_i\left(\frac{\partial\bar{x}^j}{\partial\varepsilon}\right), \; i \succeq 1, \; \eta^{[0]} = \left(\frac{\partial\bar{\Phi}^A}{\partial\varepsilon}\right). \tag{8}$$

Condition (6) provides a system of PDEs whose solution determines the components of the X, consequently the infinitesimal transformation. The vector fields X, which satisfy condition (6), are called Lie symmetries for the differential equation $H^A = 0$. The Lie symmetries for a given differential equation form a Lie algebra.

Lie symmetries can be used in different ways [2] in order to study a differential equation. However, their direct application is on the determination of the so-called similarity solutions. The steps that we follow to determine a similarity solution are based on the determination and application of the Lie invariant functions.

Let X be a Lie symmetry for a given differential equation $H^A = 0$, then the differential equation $X\left(F\right) = 0$, where F is a function, provides the Lie invariants where by replacing in the differential equation $H^A = 0$, we reduce the number of the independent variables (in the case of PDEs) or the order of the differential equation (in the case of ordinary differential equations (ODEs)).

Optimal System

Consider the n-dimensional Lie algebra G_n with elements $X_1,\ X_2,\ ...\ X_n$. Then, we shall say that the two vector fields [2]:

$$Z = \sum_{i=1}^{n} a_i X_i\;,\; W = \sum_{i=1}^{n} b_i X_i\;,\;\; a_i,\ b_i \text{ are constants.} \tag{9}$$

are equivalent iff there:

$$\mathbf{W} = lim^n_{j=i} Ad\left(\exp\left(\varepsilon_i X_i\right)\right)\mathbf{Z} \tag{10}$$

or:

$$W = cZ\;,\; c = const. \tag{11}$$

where the operator [2]:

$$Ad\left(\exp\left(\varepsilon X_i\right)\right) X_j = X_j - \varepsilon\left[X_i, X_j\right] + \frac{1}{2}\varepsilon^2\left[X_i, \left[X_i, X_j\right]\right] + \ldots \tag{12}$$

is called the adjoint representation.

Therefore, in order to perform a complete classification for the similarity solutions of a given differential equation, we should determine all the one-dimensional independent symmetry vectors of the Lie algebra G_n.

We continue our analysis by calculating the Lie point symmetries for the system (1)–(3) for the case where the system is rotating $(f \neq 0)$ and nonrotating $(f = 0)$.

3. Symmetries and the Optimal System for Nonrotating Shallow Water

We start our analysis by applying the symmetry condition (6) for the Coriolis free system (1)–(3) with $f = 0$. We found that the system of PDEs admits eight Lie point symmetries, as are presented in the following [11]:

$$\begin{aligned}
X_1 &= \partial_t\,,\ X_2 = \partial_x\,,\ X_3 = \partial_y\,,\\
X_4 &= t\partial_x + \partial_u\,,\ X_5 = t\partial_y + \partial_v\,,\\
X_6 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v,\\
X_7 &= t\partial_t + x\partial_x + y\partial_y,\\
X_8 &= (\gamma-1)\left(x\partial_x + y\partial_y + u\partial_u + v\partial_v\right) + 2h\partial_h.
\end{aligned}$$

The commutators of the Lie symmetries and the adjoint representation are presented in Table 1 and Table A1, respectively.

Table 1. Commutators of the admitted Lie point symmetries for the nonrotating 2D shallow water.

[,]	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$
$\mathbf{X}_1$	0	0	0	X_2	X_3	0	$-(\gamma-1)X_1$	0
$\mathbf{X}_2$	0	0	0	0	0	$-X_3$	0	$(\gamma-1)X_2$
$\mathbf{X}_3$	0	0	0	0	0	X_2	0	$(\gamma-1)X_3$
$\mathbf{X}_4$	$-X_2$	0	0	0	0	$-X_5$	$(\gamma-1)X_4$	$(\gamma-1)X_4$
$\mathbf{X}_5$	$-X_3$	0	0	0	0	X_4	$(\gamma-1)X_5$	$(\gamma-1)X_5$
$\mathbf{X}_6$	0	X_3	$-X_2$	X_5	$-X_4$	0	0	0
$\mathbf{X}_7$	$(\gamma-1)X_1$	0	0	$-(\gamma-1)X_4$	$-(\gamma-1)X_5$	0	0	0
$\mathbf{X}_8$	0	$-(\gamma-1)X_2$	$-(\gamma-1)X_3$	$-(\gamma-1)X_4$	$-(\gamma-1)X_5$	0	0	0

We continue by determining the one-dimensional optimal system. Let us consider the generic symmetry vector:

$$Z^8 = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7 + a_8X_8$$

From Table A1, we see that by applying the following adjoint representations:

$$Z'^8 = Ad\left(\exp\left(\varepsilon_5X_5\right)\right) Ad\left(\exp\left(\varepsilon_4X_4\right)\right) Ad\left(\exp\left(\varepsilon_3X_3\right)\right) Ad\left(\exp\left(\varepsilon_2X_2\right)\right) Ad\left(\exp\left(\varepsilon_1X_1\right)\right) Z^8$$

parameters $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3, \varepsilon_4$, and ε_5 can be determined such that:

$$Z'^8 = a_6'X_6 + a_7'X_7 + a_8'X_8$$

Parameters $a_6,\ a_7$, and a_8 are the relative invariants of the full adjoint action. Indeed, in order to determine the relative invariants, we solve the following system of partial differential equations [1]:

$$\Delta\left(\phi\left(a_i\right)\right) = C_{ij}^k a^i \frac{\partial}{\partial a_j}$$

where C_{ij}^k are the structure constants of the admitted Lie algebra as presented in Table 1. Consequently, in order to derive all the possible one-dimensional Lie symmetries, we should study various cases were none of the invariants are zero, one of the invariants is zero, two of the invariants are zero, or all the invariants are zero.

Hence, for the first three cases, infer the following one-dimensional independent Lie algebras:

$$X_6\ ,\ X_7\ ,\ X_8\ ,\ \xi_{(67)} = X_6 + \alpha X_7\ ,\ \xi_{(68)} = X_6 + \alpha X_8$$

$$\xi_{(78)} = X_7 + \alpha X_8\ ,\ \xi_{(678)} = X_6 + \alpha X_7 + \beta X_8.$$

We apply the same procedure for the rest of the possible linear combinations of the symmetry vectors, and we find the one-dimensional-dependent Lie algebras:

$$X_1,\ X_2\ ,\ X_3\ ,\ X_4\ ,\ X_5\ ,\ \xi_{(12)} = X_1 + \alpha X_2\ ,\ \xi_{(13)} = X_1 + \alpha X_3\ ,\ \xi_{(23)} = X_2 + \alpha X_3\ ,\ \xi_{(14)} = X_1 + \alpha X_4\ ,$$

$$\xi_{(15)} = X_1 + \alpha X_5\ ,\ \xi_{(16)} = X_1 + \alpha X_6,\ \xi_{(34)} = X_3 + \alpha X_4\ ,\ \xi_{(25)} = X_2 + \alpha X_5\ \ \xi_{(45)} = X_4 + \alpha X_5\ ,$$

$$\xi_{(123)} = X_1 + \alpha X_2 + \beta X_3\ \xi_{(145)} = X_1 + \alpha X_4 + \beta X_5\ ,\ \xi_{(125)} = X_1 + \alpha X_2 + \beta X_5\ ,\ \xi_{(134)} = X_1 + \alpha X_3 + \beta X_4,$$

in which α and β are constants.

Therefore, by applying one of the above Lie symmetry vectors, we find all the possible reductions from a system of $1+2$ PDEs to a system of $1+1$ PDEs. The reduced system will not admit all the remaining Lie symmetries. The Lie symmetries that survive under a reduction process are given as described in the following example.

Let a PDE admit the Lie point symmetries $\Gamma_1,\ \Gamma_2$, which are such that $[\Gamma_1, \Gamma_2] = C_{12}^1 X_1$, with $C_{12}^1 \neq 0$. Reduction with the symmetry vector Γ_1 leads to a reduced differential equation, which admits Γ_2 as the Lie symmetry. On the other hand, reduction of the mother equation with respect to the Lie symmetry Γ_2 leads to a different reduced differential equation, which does not admit as a Lie point symmetry the vector field Γ_1. In case the two Lie symmetries form an Abelian Lie algebra, i.e., $C_{12}^1 = 0$, then under any reduction process, symmetries are preserved by any reduction.

We found that the optimal system admits twenty-three one-dimensional Lie symmetries and possible independent reductions. All the possible twenty-three Lie invariants are presented in Tables A2 and A3.

An application of the Lie invariants is presented below.

Application of ξ_{145}

Let us now demonstrate the results of Tables A2 and A3 by the Lie invariants of the symmetry vector ξ_{145} and construct the similarity solution for the system.

The application of ξ_{145} in the nonrotating system (1)–(3) reduces the PDEs in the following system:

$$(hu)_z + (hv)_w = 0 \tag{13}$$

$$\alpha + uu_z + vu_w + h^{\gamma-2}h_z = 0 \tag{14}$$

$$\beta + uv_z + vv_w + h^{\gamma-2}h_w = 0 \tag{15}$$

where $z = x - \frac{\alpha}{2}t^2$ and $w = y - \frac{\beta}{2}t^2$.

System (13)–(15) admits the Lie point symmetries:

$$\partial_z\ ,\ \partial_w\ ,\ z\partial_z + w\partial_w + \frac{2}{\gamma-1}h\partial_h + u\partial_u + v\partial_v \tag{16}$$

Reduction with the symmetry vector $\partial_z + c\partial_w$ provides the following system of first-order ODEs:

$$Fh_\sigma = (c\alpha-\beta)\,h^2, \tag{17}$$

$$Fv_\sigma = \frac{(\alpha-c\beta)\,ch^\gamma - \alpha h\,(v-cu)^2}{v-cu}, \tag{18}$$

$$Fh_\sigma = \frac{(\alpha-c\beta)\,cu^\gamma - \beta h\,(v-cu)^2}{v-cu}. \tag{19}$$

where $F = \left(1+c^2\right)h^\gamma - h\,(v-cu)^2$ and $\sigma = z + cw$.

By performing the change of variable $d\sigma = f d\tau$, function f can be removed from the above system. For $h\,(\tau) = 0$, the system (17)–(19) admits a solution $u = u_0,\ v = v_0$, which is a critical point. The latter special solutions are always unstable when $\alpha c > \beta$.

We proceed with our analysis by considering the rotating system.

4. Symmetries and Optimal System for Rotating Shallow Water

For the rotating system $(f \neq 0)$, the Lie symmetries are:

$$\begin{aligned}
Y_1 &= \partial_t\ ,\ Y_2 = \partial_x\ ,\ Y_3 = \partial_y\ ,\\
Y_4 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v\ ,\\
Y_5 &= \sin(ft)\,\partial_x + \cos(ft)\,\partial_y + f\left(\cos(ft)\,\partial_u - \sin(ft)\,\partial_v\right)\\
Y_6 &= \cos(ft)\,\partial_x - \sin(ft)\,\partial_y - f\left(\sin(ft)\,\partial_u + \cos(ft)\,\partial_v\right)\\
Y_7 &= (\gamma-1)\left(x\partial_x + y\partial_y + u\partial_u + v\partial_v\right) + 2h\partial_h
\end{aligned}$$

The commutators and the adjoint representation are given in Table 2 and Table A4. The Lie symmetries for the rotating system form a smaller dimension Lie algebra than the non-rotating system. That is not the case when $\gamma = 2$, where the two Lie algebras have the same dimension and are equivalent under point transformation [22]. Therefore, for $\gamma > 2$, the Coriolis force cannot be eliminated by a point transformation as in the $\gamma = 2$ case.

Table 2. Commutators of the admitted Lie point symmetries for the rotating 2D shallow water.

$[\,,\,]$	$\mathbf{Y}_1$	$\mathbf{Y}_2$	$\mathbf{Y}_3$	$\mathbf{Y}_4$	$\mathbf{Y}_5$	$\mathbf{Y}_6$	$\mathbf{Y}_7$
$\mathbf{Y}_1$	0	0	0	0	fY_6	$-fY_5$	0
$\mathbf{Y}_2$	0	0	0	$-Y_3$	0	0	$(\gamma-1)\,Y_2$
$\mathbf{Y}_3$	0	0	0	Y_2	0	0	$(\gamma-1)\,Y_3$
$\mathbf{Y}_4$	0	Y_3	$-Y_2$	0	$-Y_6$	Y_5	0
$\mathbf{Y}_5$	$-fY_6$	0	0	Y_6	0	0	$(\gamma-1)\,Y_5$
$\mathbf{Y}_6$	fY_5	0	0	$-Y_5$	0	0	$(\gamma-1)\,Y_6$
$\mathbf{Y}_7$	0	$-(\gamma-1)\,Y_2$	$-(\gamma-1)\,Y_3$	0	$-(\gamma-1)\,Y_5$	$-(\gamma-1)\,Y_6$	0

As for the admitted Lie symmetries admitted by the given system of PDEs with or without the Coriolis terms for $\gamma > 2$, we remark that the rotating and the nonrotating systems have a common Lie subalgebra of one-parameter point transformations consisting of the symmetry vectors $Y_1,\ Y_2,\ Y_3,\ Y_4,$ and Y_7 or for the nonrotating system $X_1,\ X_2,\ X_3,\ X_6,$ and X_8.

We proceed with the determination of the one-dimensional optimal system and the invariant functions. Specifically, the relative invariants for the adjoint representation are calculated to be a_1 , a_7 and a_8. From Table 2 and Table A4, we can find the one-dimensional optimal system, which is:

$$Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Y_6,\ Y_7,\ \chi_{12}=Y_1+\alpha Y_2,\ \chi_{13}=Y_1+\alpha Y_3,$$

$$\chi_{14}=Y_1+\alpha Y_4\ ,\ \chi_{15}=Y_1+\alpha Y_5,\ \ \chi_{16}=Y_1+\alpha Y_6,\ \chi_{17}=Y_1+\alpha Y_7,$$

$$\chi_{23}=Y_2+\alpha Y_3\ ,\ \ \chi_{45}=Y_4+\alpha Y_5,\ \chi_{46}=Y_4+\alpha Y_6,\ \chi_{56}=Y_5+\alpha Y_6$$

$$\chi_{47}=Y_4+\alpha Y_6\ ,\ \chi_{123}=Y_1+\alpha Y_2+\beta Y_3,\ \chi_{147}=Y_1+\alpha Y_4+\beta Y_7.$$

The Lie invariants, which correspond to all the above one-dimensional Lie algebras, are presented in Tables A5 and A6.

Let us demonstrate the application of the Lie invariants by the following, from which we can see that the Lie invariants reduce the nonlinear field equations into a system of integrable first-order ODEs, which can be solved with quadratures.

4.1. Application of χ_{12}

We consider the travel-wave similarity solution in the x-plane provided by the symmetry vector χ_{12} and the vector field Y_3. The resulting equations are described by the following system of first order ODEs:

$$v_z = f\frac{u}{\alpha-u} \tag{20}$$
$$\bar{F}u_z = f(\alpha-u)\,vh \tag{21}$$
$$\bar{F}h_z = fvh^2 \tag{22}$$

where $\bar{F}=h^{\gamma}-(a-u)^2h$ and $z=t-\alpha x$. Because we performed reduction with a subalgebra admitted by the nonrotating system, by setting $f=0$ in (20)–(22), we get the similarity solution for the nonrotating system, where in this case, it is found to be $h(z)=h_0,\ u(z)=u_0$ and $v(z)=v_0$.

We perform the substitution $dz=\frac{\bar{F}}{fv}d\tau$, and the latter system is simplified as follows:

$$\frac{v}{\bar{F}}v_\tau = \frac{u}{\alpha-u} \tag{23}$$
$$u_\tau = (\alpha-u)\,h \tag{24}$$
$$h_\tau = h^2 \tag{25}$$

from which we get the solution:

$$h(\tau)=(h_0-\tau)^{-1}\ ,\ u(\tau)=\alpha+u_0-\frac{u_0}{h_0}\tau \tag{26}$$

and:

$$v(t)^2=2\int\frac{\left(a+u_0-\frac{u_0}{h_0}\tau\right)}{\frac{u_0}{h_0}(h_0-\tau)}\left((h_0-\tau)^{-\gamma}+\left(\frac{u_0}{h_0}\right)^2\tau-\frac{(u_0)^2}{h_0}\right)d\tau. \tag{27}$$

4.2. Application of χ_{23}

Consider now the reduction with the symmetry vector fields χ_{23}. The resulting system of 1 + 1 differential equations admits five Lie point symmetries, and they are:

$$\partial_t,\ \partial_w\ ,\ (\sin(ft)+\alpha\cos(ft))\,\partial_w+f(\sin(ft)\,\partial_u+\cos(ft)\,\partial_v)$$
$$(\alpha\sin(ft)-\cos(ft))\,\partial_w-f(\cos(ft)\,\partial_u-\sin(ft)\,\partial_v)\ ,\ (\gamma-1)(\partial_w+u\partial_u+v\partial_v)+2h\partial_h.$$

where $w = y - \alpha x$. For simplicity of our calculations, let us assume $\gamma = 3$.

Reduction with the scaling symmetry provides the following system of first order ODEs:

$$H_t = 2H(\alpha U - V), \tag{28}$$

$$U_t = \alpha H^2 + u(\alpha U - V) + fV, \tag{29}$$

$$V_t = -H^2 - v(\alpha U - V) - fU, \tag{30}$$

where $h = wH$, $u = wU$, and $v = wU$. The latter system is integrable and can be solved with quadratures.

Reducing with respect to the symmetry vector $(\alpha \sin(ft) - \cos(ft))\partial_w - f(\cos(ft)\partial_u - \sin(ft)\partial_v)$, we find the reduced system:

$$\frac{H_t}{H} = -\frac{\alpha\cos(ft) + \sin(ft)}{\cos(ft) - \alpha\sin(ft)}, \tag{31}$$

$$U_t = -\alpha f\frac{\sin(ft)V - \cos(ft)U}{\cos(ft) - \alpha\sin(ft)}, \tag{32}$$

$$V_t = -f\frac{\sin(ft)V - \cos(ft)U}{\cos(ft) - \alpha\sin(ft)}, \tag{33}$$

where now:

$$h = H(t), \tag{34}$$

$$u = \frac{\cos(ft)}{\cos(ft) - \alpha\sin(ft)}fw + U(t), \tag{35}$$

$$v = -\frac{\sin(ft)}{\cos(ft) - \alpha\sin(ft)}fw + V(t). \tag{36}$$

System (31)–(33) is integrable, and the solution is expressed in terms of quadratures.

5. Conclusions

In this work, we determined the one-dimensional optimal system for the two-dimensional ideal gas equations. The nonrotating system was found to be invariant under an eight-dimensional group of one-parameter point transformations. and there were twenty-three independent one-dimensional Lie algebras. One the other hand, when the Coriolis force was introduced, the dynamical admitted seven Lie point symmetries and twenty one-dimensional Lie algebras.

For all the independent Lie algebras, we determined all the invariant functions, which corresponded to all the independent similarity solutions.

In a future work, we plan to classify all the independent one-dimensional Lie algebras, which lead to analytic forms for the similarity solutions.

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Appendix A

In this Appendix, we present the Tables A1–A6, which are referenced in the main article.

Table A1. Adjoint representation of the admitted Lie point symmetries for the nonrotating 2D shallow water.

$Ad\left(e^{(\varepsilon X_i)}\right) X_j$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$
$\mathbf{X}_1$	X_1	X_2	X_3	$X_4-\varepsilon X_2$	$X_5-\varepsilon X_3$	X_6	$X_7+\varepsilon(\gamma-1)X_1$	X_8
$\mathbf{X}_2$	X_1	X_2	X_3	X_4	X_5	$X_6+\varepsilon X_3$	X_7	$X_8-\varepsilon(\gamma-1)X_2$
$\mathbf{X}_3$	X_1	X_2	X_3	X_4	X_5	$X_6-\varepsilon X_2$	X_7	$X_8-\varepsilon(\gamma-1)X_3$
$\mathbf{X}_4$	$X_1+\varepsilon X_2$	X_2	X_3	X_4	X_5	$X_6+\varepsilon X_5$	$X_7-\varepsilon(\gamma-1)X_4$	$X_8-\varepsilon(\gamma-1)X_4$
$\mathbf{X}_5$	$X_1+\varepsilon X_3$	X_2	X_3	X_4	X_5	$X_6-\varepsilon X_4$	$X_7-\varepsilon(\gamma-1)X_5$	$X_8-\varepsilon(\gamma-1)X_5$
$\mathbf{X}_6$	X_1	$X_2\cos\varepsilon-X_3\sin\varepsilon$	$X_2\sin\varepsilon+X_3\cos\varepsilon$	$X_4\cos\varepsilon-X_5\sin\varepsilon$	$X_4\sin\varepsilon+X_5\cos\varepsilon$	X_6	X_7	X_8
$\mathbf{X}_7$	$e^{-(\gamma-1)\varepsilon}X_1$	X_2	X_3	$e^{-(\gamma-1)\varepsilon}X_4$	$e^{-(\gamma-1)\varepsilon}X_5$	X_6	X_7	X_8
$\mathbf{X}_8$	X_1	$e^{(\gamma-1)\varepsilon}X_3$	$e^{(\gamma-1)\varepsilon}X_4$	$e^{-(\gamma-1)\varepsilon}X_4$	$e^{-(\gamma-1)\varepsilon}X_5$	X_6	X_7	X_8

Table A2. Lie invariants for the optimal system of the nonrotating system.

Symmetry	Invariants
$\mathbf{X}_1$	$x,\ y,\ h(x,y),\ u(x,y),\ v(x,y)$
$\mathbf{X}_2$	$t,\ y,\ h(t,y),\ u(t,y),\ v(t,y)$
$\mathbf{X}_3$	$t,\ x,\ h(t,x),\ u(t,x),\ v(t,x)$
$\mathbf{X}_4$	$t,\ y,\ h(t,y),\ \frac{x}{t}+U(t,y),\ v(t,y)$
$\mathbf{X}_5$	$t,\ x,\ h(t,x),\ u(t,x),\ \frac{y}{t}+V(t,x)$
$\mathbf{X}_6$	$t,\ x^2+y^2,\ h\left(t,x^2+y^2\right),\ \frac{xU\left(t,x^2+y^2\right)+yV\left(t,x^2+y^2\right)}{\sqrt{x^2+y^2}},\ \frac{yU\left(t,x^2+y^2\right)-xV\left(t,x^2+y^2\right)}{\sqrt{x^2+y^2}}$
$\mathbf{X}_7$	$\frac{x}{t},\ \frac{y}{t},\ h\left(\frac{x}{t},\frac{y}{t}\right),\ u\left(\frac{x}{t},\frac{y}{t}\right),\ v\left(\frac{x}{t},\frac{y}{t}\right)$
$\mathbf{X}_8$	$H(x,y)\,t^{\frac{2}{1-\gamma}},\ U(x,y)\,t^{-1},\ V(x,y)\,t^{-1}$
$\xi_{(12)}$	$x-\alpha t,\ y,\ h(x-\alpha t,y),\ u(x-\alpha t,y),\ v(x-\alpha t,y)$
$\xi_{(13)}$	$x,\ y-\alpha t,\ h(x,y-\alpha t),\ u(x,y-\alpha t),\ v(x,y-\alpha t)$
$\xi_{(14)}$	$x-\frac{\alpha}{2}t^2,\ y,\ h\left(x-\frac{\alpha}{2}t^2,y\right),\ u\left(x-\frac{\alpha}{2}t^2,y\right),\ v\left(x-\frac{\alpha}{2}t^2,y\right)$
$\xi_{(15)}$	$x,\ y-\frac{\alpha}{2}t^2,\ h\left(x,y-\frac{\alpha}{2}t^2\right),\ u\left(x,y-\frac{\alpha}{2}t^2\right),\ v\left(x,y-\frac{\alpha}{2}t^2\right)$

Table A3. Lie invariants for the optimal system of the nonrotating system.

Symmetry	Invariants
$\xi_{(16)}$	$t,\ e^{-\alpha t}\left(x^2+y^2\right),\ u\left(t,e^{-\alpha t}x^2+y^2\right)\cos(\alpha t)+v\left(t,e^{-\alpha t}x^2+y^2\right)\sin(\alpha t)$ $\frac{y}{x},\ h\left(e^{-\alpha t}x^2+y^2,\frac{y}{x}\right),\ u\left(t,e^{-\alpha t}x^2+y^2\right)\sin(\alpha t)-v\left(t,e^{-\alpha t}x^2+y^2\right)\cos(\alpha t)$
$\xi_{(23)}$	$t,\ x-\alpha y,\ h\left(t,x-\alpha y\right),\ u\left(t,x-\alpha y\right),v\left(t,x-\alpha y\right)$
$\xi_{(34)}$	$t,\ y-\frac{x}{\alpha t},\ h\left(t,y-\frac{x}{\alpha t}\right),\ u\left(t,y-\frac{x}{\alpha t}\right),\ v\left(t,y-\frac{x}{\alpha t}\right)$
$\xi_{(25)}$	$t,\ y-\alpha tx,\ h\left(t,y-\alpha tx\right),\ u\left(t,y-\alpha tx\right),\ v\left(t,y-\alpha tx\right)$
$\xi_{(45)}$	$t,\ y-\alpha x,\ h\left(t,y-\alpha x\right),\ \alpha\frac{x}{t}+U\left(t,y-\alpha x\right),\ \alpha\frac{x}{t}+V\left(t,y-\alpha x\right)$
$\xi_{(123)}$	$t-\alpha x,\ t-\beta y,\ h\left(t-\alpha x,t-\beta y\right),\ u\left(t-\alpha x,t-\beta y\right),v\left(t-\alpha x,t-\beta y\right)$
$\xi_{(145)}$	$x-\frac{\alpha}{2}t^2,\ y-\frac{\beta}{2}t^2,\ h\left(x-\frac{\alpha}{2}t^2,y-\frac{\beta}{2}t^2\right),\ \alpha t+U\left(x-\frac{\alpha}{2}t^2,\ y-\frac{\beta}{2}t^2\right),\ \beta t+V\left(x-\frac{\alpha}{2}t^2,\ y-\frac{\beta}{2}t^2\right)$
$\xi_{(125)}$	$x-\alpha t,\ y-\frac{\beta}{2}t^2,\ h\left(x-\alpha t,y-\frac{\beta}{2}t^2\right),\ u\left(x-\alpha t,y-\frac{\beta}{2}t^2\right),\ \beta t+V\left(x-\alpha t,y-\frac{\beta}{2}t^2\right)$
$\xi_{(134)}$	$x-\frac{\beta}{2}t^2,\ y-\alpha t,\ h\left(x-\frac{\beta}{2}t^2,y-\alpha t\right),\ \beta t+U\left(x-\frac{\beta}{2}t^2,y-\alpha t\right),\ V\left(x-\frac{\beta}{2}t^2,y-\alpha t\right)$
$\xi_{(67)}$	$\frac{\ln t}{\alpha},w=\ \frac{t^{-\frac{\alpha+\sqrt{\alpha(\alpha-4)-4}}{2\alpha}}}{2\sqrt{\alpha(\alpha-4)-4}}\left(x-\left(\alpha+\sqrt{\alpha\left(\alpha-4\right)-4}\right)y\right),\ z=\ \frac{t^{-\frac{\alpha+\sqrt{\alpha(\alpha-4)-4}}{2\alpha}}}{2\sqrt{\alpha(\alpha-4)-4}}\left(x+\left(\alpha+\sqrt{\alpha\left(\alpha-4\right)-4}\right)y\right)$ $h\left(w,z\right),\ U\left(w,z\right)\sin\left(\frac{\ln t}{\alpha}\right)+V\left(w,z\right)\sin\left(\frac{\ln t}{\alpha}\right),\ U\left(w,z\right)\cos\left(\frac{\ln t}{\alpha}\right)-\ V\left(w,z\right)\sin\left(\frac{\ln t}{\alpha}\right)$
$\xi_{(68)}$	$t,\ x^2+y^2,\ x^{-\frac{2}{\gamma-1}}h\left(t,x^2+y^2\right),\ \frac{U\left(t,x^2+y^2\right)\cos\left(\frac{\ln x}{\alpha}\right)+V\left(t,x^2+y^2\right)\sin\left(\frac{\ln x}{\alpha}\right)}{x},\ \frac{U\left(t,x^2+y^2\right)\sin\left(\frac{\ln x}{\alpha}\right)-V\left(t,x^2+y^2\right)\cos\left(\frac{\ln x}{\alpha}\right)}{x}$
$\xi_{(78)}$	$w=xt^{-\frac{(\gamma-1)}{\alpha(\gamma-1)-2}},\ z=yt^{-\frac{(\gamma-1)}{\alpha(\gamma-1)-2}},\ t^{-\frac{2\alpha}{\alpha(\gamma-1)-2}}h\left(w,z\right),t^{-\frac{(\gamma-1)\alpha}{\alpha(\gamma-1)-2}}u\left(w,z\right),\ t^{-\frac{(\gamma-1)\alpha}{\alpha(\gamma-1)-2}}v\left(w,z\right)$
$\xi_{(678)}$	$t,\ t^{-1-\beta}\left(x^2+y^2\right),t^{-\beta}\left(U\left(t,x^2+y^2\right)\sin(\alpha t)+V\left(t,x^2+y^2\right)\cos(\alpha t)\right)$ $t^{-\frac{2\beta}{\gamma-1}}H\left(t,x^2+y^2\right),\ t^{-\beta}\left(U\left(t,x^2+y^2\right)\cos(\alpha t)-V\left(t,x^2+y^2\right)\sin(\alpha t)\right)$

Table A4. Adjoint representation of the admitted Lie point symmetries for the rotating 2D shallow water.

$Ad\left(e^{(\varepsilon Y_i)}\right)\mathbf{Y}_j$	$\mathbf{Y}_1$	$\mathbf{Y}_2$	$\mathbf{Y}_3$	$\mathbf{Y}_4$	$\mathbf{Y}_5$	$\mathbf{Y}_6$	$\mathbf{Y}_7$
$\mathbf{Y}_1$	Y_1	Y_2	Y_3	Y_4	$Y_5\cos(f\varepsilon)-Y_6\sin(f\varepsilon)$	$Y_5\sin(f\varepsilon)+Y_6\cos(f\varepsilon)$	Y_7
$\mathbf{Y}_2$	Y_1	Y_2	Y_3	$Y_4+\varepsilon Y_3$	Y_5	Y_6	$Y_7-\varepsilon(\gamma-1)Y_2$
$\mathbf{Y}_3$	Y_1	Y_2	Y_3	$Y_4-\varepsilon Y_2$	Y_5	Y_6	$Y_7-\varepsilon(\gamma-1)Y_3$
$\mathbf{Y}_4$	Y_1	$Y_2\cos\varepsilon-Y_3\sin\varepsilon$	$Y_2\sin\varepsilon+Y_3\cos\varepsilon$	Y_4	$Y_5\cos\varepsilon+Y_6\sin\varepsilon$	$Y_6\cos\varepsilon-Y_5\sin\varepsilon$	Y_7
$\mathbf{Y}_5$	$Y_1+f\varepsilon Y_6$	Y_2	Y_3	$Y_4-\varepsilon Y_6$	Y_5	Y_6	$Y_7-\varepsilon(\gamma-1)Y_5$
$\mathbf{Y}_6$	$Y_1-f\varepsilon Y_5$	Y_2	Y_3	$Y_4+\varepsilon Y_5$	Y_5	Y_6	$Y_7-\varepsilon(\gamma-1)Y_6$
$\mathbf{Y}_7$	Y_1	$e^{(\gamma-1)\varepsilon}Y_2$	$e^{(\gamma-1)\varepsilon}Y_3$	Y_4	$e^{(\gamma-1)\varepsilon}Y_5$	$e^{(\gamma-1)\varepsilon}Y_6$	Y_7

Table A5. Lie invariants for the optimal system of the rotating system.

Symmetry	Invariants
$\mathbf{Y}_1$	$x,\ y,\ h\left(x,y\right),\ u\left(x,y\right),\ v\left(x,y\right)$
$\mathbf{Y}_2$	$t,\ y,\ h\left(t,y\right),\ u\left(t,y\right),\ v\left(t,y\right)$
$\mathbf{Y}_3$	$t,\ x,\ h\left(t,x\right),\ u\left(t,x\right),\ v\left(t,x\right)$
$\mathbf{Y}_4$	$t,\ x^2+y^2,\ h\left(t,x^2+y^2\right),\ \frac{xU\left(t,x^2+y^2\right)+yV\left(t,x^2+y^2\right)}{\sqrt{x^2+y^2}},\ \frac{yU\left(t,x^2+y^2\right)-xV\left(t,x^2+y^2\right)}{\sqrt{x^2+y^2}}$
$\mathbf{Y}_5$	$t,\ x\cot\left(ft\right)-y,\ h\left(t,x\cot\left(ft\right)-y\right),\ fx\cot\left(ft\right)+U\left(t,x\cot\left(ft\right)-y\right),\ -fx+V\left(t,x\cot\left(ft\right)-y\right)$
$\mathbf{Y}_6$	$t,\ x\tan\left(ft\right)+y,\ h\left(t,x\tan\left(ft\right)+y\right),\ -fx\tan\left(ft\right)+U\left(t,x\tan\left(ft\right)+y\right),\ -fx+V\left(t,x\tan\left(ft\right)+y\right)$
$\mathbf{Y}_7$	$\frac{x}{t},\ \frac{y}{t},\ h\left(\frac{x}{t},\frac{y}{t}\right),\ u\left(\frac{x}{t},\frac{y}{t}\right),\ v\left(\frac{x}{t},\frac{y}{t}\right)$
$\mathcal{X}_{(12)}$	$x-\alpha t,\ y,\ h\left(x-\alpha t,y\right),\ u\left(x-\alpha t,y\right),\ v\left(x-\alpha t,y\right)$
$\mathcal{X}_{(13)}$	$x,\ y-\alpha t,\ h\left(x,y-\alpha t\right),\ u\left(x,y-\alpha t\right),\ v\left(x,y-\alpha t\right)$
$\mathcal{X}_{(14)}$	$t,\ e^{-\alpha t}\left(x^2+y^2\right),\ u\left(t,e^{-\alpha t}x^2+y^2\right)\cos\left(\alpha t\right)+v\left(t,e^{-\alpha t}x^2+y^2\right)\sin\left(\alpha t\right)$ $\frac{y}{x},\ h\left(e^{-\alpha t}x^2+y^2,\frac{y}{x}\right),\ u\left(t,e^{-\alpha t}x^2+y^2\right)\sin\left(\alpha t\right)-v\left(t,e^{-\alpha t}x^2+y^2\right)\cos\left(\alpha t\right)$

Table A6. Lie invariants for the optimal system of the rotating system.

Symmetry	Invariants
$\mathcal{X}_{(15)}$	$x+\frac{\alpha}{f}\cos\left(ft\right),\ y-\frac{\alpha}{f}\sin\left(ft\right)\ ,\ h\left(x+\frac{\alpha}{f}\cos\left(ft\right),y-\frac{\alpha}{f}\sin\left(ft\right)\right),$ $\alpha\sin\left(ft\right)+U\left(x+\frac{\alpha}{f}\cos\left(ft\right),y-\frac{\alpha}{f}\sin\left(ft\right)\right),\ \alpha\cos\left(ft\right)+V\left(x+\frac{\alpha}{f}\cos\left(ft\right),y-\frac{\alpha}{f}\sin\left(ft\right)\right)$
$\mathcal{X}_{(16)}$	$x-\frac{\alpha}{f}\sin\left(ft\right),\ y-\frac{\alpha}{f}\cos\left(ft\right)\ ,\ h\left(x-\frac{\alpha}{f}\sin\left(ft\right),y-\frac{\alpha}{f}\cos\left(ft\right)\right),$ $\alpha\cos\left(ft\right)+U\left(x-\frac{\alpha}{f}\sin\left(ft\right),y-\frac{\alpha}{f}\cos\left(ft\right)\right),\ -\alpha\sin\left(ft\right)+V\left(x-\frac{\alpha}{f}\sin\left(ft\right),y-\frac{\alpha}{f}\cos\left(ft\right)\right)$
$\mathcal{X}_{(17)}$	$xe^{-\alpha t},\ ye^{-\alpha t},\ e^{\frac{2\alpha}{\gamma-1}t}h\left(xe^{-\alpha t},ye^{-\alpha t}\right),\ e^{\alpha t}u\left(xe^{-\alpha t},ye^{-\alpha t}\right),\ e^{\alpha t}v\left(xe^{-\alpha t},ye^{-\alpha t}\right)$
$\mathcal{X}_{(23)}$	$t,\ x-\alpha y,\ h\left(t,x-\alpha y\right),\ u\left(t,x-\alpha y\right),\ v\left(t,x-\alpha y\right)$
$\mathcal{X}_{(45)}$	$t,\ w=\left(x^2+y^2-2x\ \cos\left(ft\right)+2y\sin\left(ft\right)\right),\ \frac{U(t,w)+f\sin(ft)}{V(t,w)+f\cos(ft)},\ \frac{U(t,w)^2+V(t,w)^2}{2}+f\left(U\left(t,w\right)\sin\left(ft\right)+V\left(t,w\right)\cos\left(ft\right)\right)$
$\mathcal{X}_{(46)}$	$t,\ w=\left(x^2+y^2-2x\ \sin\left(ft\right)-2y\cos\left(ft\right)\right),\ \frac{U(t,w)-f\cos(ft)}{V(t,w)+f\sin(ft)},\ \frac{U(t,w)^2+V(t,w)^2}{2}+f\left(V\left(t,w\right)\sin\left(ft\right)-U\left(t,w\right)\cos\left(ft\right)\right)$
$\mathcal{X}_{(56)}$	$t,\ z=y-\frac{x(\cos(ft)-\alpha\sin(ft))}{\sin(ft)+\alpha\cos(ft)},\ h\left(t,z\right),\ f\frac{x(\cos(ft)-\alpha\sin(ft))}{\sin(ft)+\alpha\cos(ft)}+U\left(t,z\right),\ -x+V\left(t,z\right)$
$\mathcal{X}_{(47)}$	$t,\ x^2+y^2\ ,\ x^{-\frac{2}{\gamma-1}}h\left(t,x^2+y^2\right)\ ,\ \frac{U\left(t,x^2+y^2\right)\cos\left(\frac{\ln x}{\alpha}\right)+V\left(t,x^2+y^2\right)\sin\left(\frac{\ln x}{\alpha}\right)}{x}\ ,\ \frac{U\left(t,x^2+y^2\right)\sin\left(\frac{\ln x}{\alpha}\right)-V\left(t,x^2+y^2\right)\cos\left(\frac{\ln x}{\alpha}\right)}{x}$
$\mathcal{X}_{(123)}$	$t-\alpha x,\ t-\beta y,\ h\left(t-\alpha x,t-\beta y\right),\ u\left(t-\alpha x,t-\beta y\right),v\left(t-\alpha x,t-\beta y\right)$
$\mathcal{X}_{(147)}$	$z=e^{-t(\gamma-1)}\left(x\cos t-y\sin t\right)\ ,\ w=e^{-t(\gamma-1)}\left(y\cos t+x\sin t\right)\ ,$ $e^{-t(\gamma-1)}h\left(z,w\right),\ e^{-t(\gamma-1)}\left(U\left(z,w\right)\cos t-V\left(z,w\right)\sin t\right)\ ,\ e^{-t(\gamma-1)}\left(U\left(z,w\right)\sin t+V\left(z,w\right)\cos t\right)$

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