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Reduction of Order: Analytical Solution of Film Formation in the Electrostatic Rotary Bell Sprayer

Mark Doerre * and **Nelson K. Akafuah**

Institute of Research for Technology Development (IR4TD), College of Engineering, University of Kentucky, Lexington, KY 40506, USA

* Correspondence: mark.doerre@uky.edu

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Abstract: This brief paper explains the slight differences in governing equations for a fluid film in a spinning cone, and the mechanism that reduces the order of a solution. Spinning cones with a centrally supplied fluid that spreads over its inner surface as a thin film have been the subject of interest for many years. Though often cast as a mathematical analysis, understanding this process is important, especially in the application of automotive painting. The analysis consists of a system of equations obtained from the Navier–Stokes equations along with simple boundary conditions that describe radial and tangential momentum conservation. Solutions to this system of equations are shown using several techniques. The connection between these techniques is slightly subtle. However, the conditions that enable reduction of order are clear once they are exposed. Directional velocity profiles in the film can be a combination of four roots in the complex plane. This system of roots also contains two diagonal axes of symmetry that are offset by 90 degrees. Alternatively, if the radial and tangential velocity profiles are expressed as a single complex function, a reduced order solution that is a combination of one set of diagonal set of roots can be found.

Keywords: spinning cone; reduction of order; centrifugal force; Coriolis force; Navier-Stokes

1. Introduction

About half of a modern automotive assembly plant is dedicated to producing the finish, or in colloquial terms, the paint job. A luxurious finish is of utmost importance in satisfying the consumer. Paint and other organic coatings are atomized using spinning cones or, as they are known in the industry, Electrostatic Rotary Bell Sprayers (ERBS) [1]. Fluid is centrifugally flung off the edge of the bell and aerodynamically shredded. Atomization transforms bulk fluid into a mist with a drop diameter and diameter range that is appropriate for the material being applied. Advection and electrostatic forces transport the droplets to the target surface [2]. Mathematical models [3] of atomization processes rely on the thickness of the fluid film as an input [4], hence why this subject is of interest.

In the published literature, problems are solved many times over a span of years, and each author adds expertise and techniques. In the context of rotary bells, a solution consists of velocity profiles and film thickness. In this paper, slightly different formulations of the equations of motion are compared. Comparison is also made between two solution methods, one based on a fourth order ODE (ordinary differential equation), and the other on a change of variables. The change of variables leads to a reduction of order and a second order ODE. The fourth order solution is based on substitution methods similar to those found in linear algebra. Whereas the genesis of the second order solution is slightly subtle, it becomes clear once derived.

2. Methods

Prior to formulating the equations of motion, several key assumptions are made. A thin film model is validated because the ratio of film thickness to the radius of the bell is <<1. This renders the analysis as two-dimensional, and the solution consists of velocity profiles in the radial and tangential directions. Film thickness is determined by integrating the radial velocity over a computationally convenient area and equating that to supply flow. The simplest boundary conditions are no slip at the fluid/bell interface, and no shear at the fluid/air interface.

The Navier–Stokes equations (NSEs) in a stationary spherical coordinate system are modified to account for motions seen by an observer in a non-inertial rotating system. In the case of a rotary bell, the angular velocity vector is along the radial direction that has a polar angle of zero. Several methods of formulation are found in the referenced literature. For example, there is a formulation based on the Coriolis Theorem [5] shown by Expression (1). It can be included with the other acceleration terms in the NSEs or it can appear as terms grouped with body forces. Another formulation is based on substitution of the velocity difference between the bell surface velocity and the fluid velocity seen by the stationary observer [6], as shown by Equation (2). This substitution, and its derivatives, replaces v_ϕ and corresponding derivatives in the NSEs. Lastly, another method is based on applying Newton's Law to a differential fluid parcel [7]. This alternative is similar to the NSEs, albeit less formal in the procedure.

$$2\Omega \times v + \Omega \times (\Omega \times r) \tag{1}$$

$$v'_\phi = -v_\phi + \omega\, r \sin\beta \tag{2}$$

Two slightly different forms of the NSEs are obtained after "correcting" for a rotating coordinate system. Based on assumptions of steady state, azimuthal symmetry, thin film and fully developed flow, neglecting gravity, and an order of magnitude analysis, the NSEs are reduced to an equation involving viscous forces and inertial forces. If Expression (1) is used, the resulting equations are (3)a,b.

$$\begin{aligned} -r\omega^2 \sin^2\beta - 2\omega v_\phi \sin\beta &= \frac{\nu}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\ (a) \\ 2v_r\omega\sin\beta &= \frac{\nu}{r^2}\frac{\partial^2 v'_\phi}{\partial\theta^2}\ (b) \end{aligned} \tag{3}$$

If Equation (2) is used, the resulting equations are (4)a,b.

$$\begin{aligned} -r\omega^2 \sin^2\beta + 2\omega v_\phi \sin\beta &= \frac{\nu}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\,(a) \\ -2v_r\omega\sin\beta &= \frac{\nu}{r^2}\frac{\partial^2 v'_\phi}{\partial\theta^2}\,(b) \end{aligned} \tag{4}$$

The slight differences between systems of Equations (3) and (4) are caused by the assumed direction of the tangential velocity. In (3), it is assumed in the same direction as the surface velocity, on the other hand, in (4), it is assumed the opposite of the surface velocity. This difference is evident in the sign change in (4) between the centrifugal term and the Coriolis term. As it turns out, the tangential velocity, in the rotating frame, is opposite of the surface velocity, though surface velocity appears stationary to the rotating observer. For the remainder of this paper, dimensionless forms of Equation (4)a,b will be used since the solution methods apply to both formulations of the NSEs.

Equation (4) are divided by $r\omega^2 \sin^2\beta$ and s is set equal to $r\theta\,\sqrt{\omega\sin\beta/\nu}$ thereby casting them into a dimensionless form [6,8], the results are Equation (5).

$$\begin{aligned} -1 + 2W = U'' \quad (a) \\ -2U = W''\,(b) \end{aligned} \tag{5}$$

Equation (5)a,b, along with the no-slip boundary, expressed as $U(s=0) = W(s=0) = 0$ and the no shear boundary, expressed as $U'(s = s^+) = W'(s = s^+) = 0$, where s^+ is the dimensionless film thickness, lead to a discussion on connections between solutions. The two dimensionless numbers, U

$(v_r/\omega r \sin\beta)$ and W $(v'_\phi/\omega r \sin\beta)$ are in the form of Rossby $(Vel/(\omega\ Len))$ numbers, while U'' and W'' are approximate curvatures of Ekman $\left(Visc/\left(\omega\ Len^2\right)\right)$ numbers over the film thickness.

3. Discussion

One solution to the system of Equation (5)a,b consists of differentiating (5)a twice, then substituting (5)b into (5)a, yielding a fourth order equation for U, shown as Equation (6)a.

$$\begin{gathered} U^{(4)} + 4U = 0\ (a) \\ \alpha^4 + 4 = 0\ (b) \end{gathered} \tag{6}$$

Letting $U = e^{\alpha s}$ leads to the characteristic polynomial (6)b. A linear combination of the roots of the polynomial is the homogenous solution, as shown by Equation (7).

$$U = Ae^{(1+i)S} + Be^{(1-i)S} + Ce^{(-1+i)S} + De^{(-1-i)S} \tag{7}$$

Shifting to the next method, another solution to the system of Equation (5) found in the literature

$$-1 = (U - Wi)'' - 2i\ (U - Wi) \tag{8}$$

consists of multiplying (5)b by i, and subtracting this result from (5)a, yielding Equation (8). The change of variables $\Psi = U - Wi$ simplifies Equation (8), yielding Equation (9). This is a second order equation where the solution Ψ contains U and W.

$$-1 = \Psi'' - 2i\Psi \tag{9}$$

The first key operation that enables reduction of order results from two derivative operations on Equation (7) that rotate and scale diagonal pairs of roots in equal and opposite directions. After these operations, two roots are rotated and scaled by $2i$ and the other two by negative $2i$, yielding Equation (10). Next, W is obtained from U'' by substituting it into Equation (5)a.

$$U'' = 2i\ Ae^{(1+i)S} - 2i\ Be^{(1-i)S} - 2i\ Ce^{(-1+i)S} + 2i\ De^{(-1-i)S} \tag{10}$$

After the multiplication of W by i, two roots have a coefficient of −2, and +2 for the others. The second key operation that enables reduction of order is the factor of 2 from the derivative operations negates the factor of 2 from the Coriolis term. Finally, Wi is subtracted from U as shown by Equation (11), and the result is a solution to Equation (9).

$$U - (Wi) = U - \left(\frac{U''}{2}i + \frac{1}{2}i\right) = \Psi = C_1e^{(1+i)S} + C_2e^{(-1-i)S} - \frac{1}{2}i \tag{11}$$

On the other hand, if W is multiplied by i, then added to U, Equation (12) is established; it is based on the other diagonal roots, and the result is a solution to Equation (13).

$$U + (Wi) = U + \left(\frac{U''}{2}i + \frac{1}{2}i\right) = \Phi = C_3e^{(1-i)S} + C_4e^{(-1+i)S} + \frac{1}{2}i \tag{12}$$

From Equation (12), the change of variables $\Phi = U + Wi$, is associated with a second order differential equation shown by Equation (13). A similar pattern of equations and changes of variables is obtained by multiplying Equation (5)a by $\pm i$, then adding the result to (5)b. Notice that the characteristic polynomials associated with Equations (9) and (13) are roots of Equation (6)b and correspond with roots on two diagonal axes of symmetry in the original four root system.

$$-1 = \Phi'' + 2i\Phi \tag{13}$$

Since complex numbers carry magnitude and phase, summing equal magnitude roots that are 180 degrees out of phase results in their negation. This allows the combination of two velocity-related components to be represented by a single complex function that is a solution to the reduced order differential equation. The velocity components are recoverable from the real and imaginary parts according to Equation (14) [9].

$$\begin{aligned} \mathcal{R}e(z) &= (z+\overline{z})/2(a) \\ \mathcal{J}m(z) &= (z-\overline{z})/2i(b) \end{aligned} \tag{14}$$

4. Conclusions

The mathematical intricacies of establishing two seemingly distinct solutions for fluid atomization from rotary bells used in automobile painting were shown. These intricacies of multiple solutions are either rarely included in the literature, or left out possibly for brevity reasons. Although not critical in calculating these solutions, this elucidation is expected to be of value to other researchers seeking to understand the mathematics and to check on the correctness of the solutions.

For automotive painting, the high angular velocity of rotary bells enables simplifying the radial momentum equation to factors related to only centrifugal and viscous effects, thereby leaving an ordinary differential equation to be solved. Rotary bell atomizers and similar fluid mechanics are also found in other industrial processes, for example, wafer coating processes in the semi-conductor industry. However, for wafer coatings the fluid is usually dosed [10] rather than from a continuous supply, and while being spun, it undergoes significant evaporation resulting in reduced volume and increased viscosity.

We expect that the current trend of using numerical calculations in the area of this research will increase; this rigorous mathematical formulation and analysis can help check accuracy of assumptions made for those numerical calculations and codes in the similar role played by experiments [11].

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Nomenclature

U	dimensionless radial velocity
W	dimensionless tangential velocity
i	$\sqrt{-1}$
r	position vector
s	dimensionless film coordinate
s^+	dimensionless film thickness
v	velocity vector
v_r	radial fluid velocity
v_ϕ	tangential fluid velocity
v'_ϕ	tangential fluid velocity in rotating coordinates
z	complex number
Ψ	Complex solution to reduced order equation
Φ	Complex solution to reduced order equation
β	cone half angle
ω	angular velocity
Ω	angular velocity

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