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Nonlinear Rayleigh Quotients and Nonlinear Spectral Theory

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Abstract: We give a new and simplified definition of *spectrum* for a nonlinear operator F acting in a real Banach space X, and study some of its features in terms of (qualitative and) quantitative properties of F such as the measure of noncompactness, $\alpha(F)$, of F. Then, using as a main tool the Ekeland Variational Principle, we focus our attention on the spectral properties of F when F is a gradient operator in a real Hilbert space, and in particular on the role played by its Rayleigh quotient $R(F)$ and by the best lower and upper bounds, $m(F)$ and $M(F)$, of $R(F)$.

Keywords: surjective operators; measure of noncompactness; Darbo's fixed point theorem; gradient operators; Ekeland variational principle

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1. Introduction

Let H be a real Hilbert space with scalar product denoted $\langle .,. \rangle$ and corresponding norm $\|x\| = \sqrt{\langle x, x \rangle}$. If F is any map of H into itself, it makes sense to define its *Rayleigh quotient* by the formula

$$\frac{\langle F(x), x \rangle}{\|x\|^2} \qquad (x \in H, x \neq 0). \tag{1}$$

More generally, given two operators $F, G : H \to H$, with $\langle G(x), x \rangle \neq 0$ for $x \neq 0$, the *Rayleigh quotient of the pair* F, G is defined by the ratio

$$R(x) = \frac{\langle F(x), x \rangle}{\langle G(x), x \rangle} \qquad (x \in H, x \neq 0). \tag{2}$$

The importance of this real-valued function defined on $H \setminus \{0\}$ is evident on observing that if λ is an *eigenvalue* of the pair (F, G), that is, if

$$F(x) = \lambda G(x)$$

for some *eigenvector* $x \neq 0$, then $\lambda = R(x)$. That is to say, eigenvalues of (F, G) are values of the corresponding Rayleigh quotient, and this is in fact the way that they have been systematically studied in the spectral theory of linear differential operators, see in particular Chapter 3 of Weinberger's Lectures on eigenvalue approximation [1]. An interesting question is: can the Rayleigh quotient be usefully employed also for *nonlinear* operators? (by "nonlinear" we mean, as usual, *not necessarily* linear). The answer is an easy "yes" if we look at the many concrete eigenvalue problems driven by nonlinear differential equations that can be found in the literature, a most famous instance being the Dirichlet problem for the $p-$Laplacian, that is,

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda |u|^{p-2}u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{cases} \tag{3}$$

where $p > 1$ and Ω is a bounded domain in $\mathbb{R}^n$. Indeed, it is well known (see, e.g., [2]) that, as for the linear case $p = 2$, that is for the ordinary Laplacian $\Delta u = \operatorname{div} \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, the problem (3) has an infinite sequence of eigenvalues

$$\lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots, \qquad \lambda_k \to +\infty \tag{4}$$

that are obtained by a minimax procedure, over suitable families of subsets of the Sobolev space $W_0^{1,p}(\Omega)$, of the *nonlinear Rayleigh quotient* [3]

$$\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} \qquad (v \in W_0^{1,p}(\Omega), v \neq 0)$$

which is just the explicit form of the ratio in (2) (in fact, of the generalized form of it suited for maps from a Banach space to its dual) when F, G are the operators associated with the weak form of (3).

The specific purpose of this paper is to show that the properties of R can be, more generally, employed in the *spectral theory* of nonlinear operators [4]. Indeed it is natural—reminding of Linear Algebra and Linear Functional Analysis—that we think of the eigenvalues as being a relevant part of the spectrum (or even the whole of it), and is by now acknowledged that the same can be thought of nonlinear operators, see [4] Chapter 7 and in particular Feng's modification [5] of the original construction of Furi, Martelli and Vignoli [6], motivated among others by the remarks of Edmunds and Webb [7]. However, for a nonlinear operator F as well, the *spectrum* of F is a wider concept, based on the property of $F - \lambda I$ (I the identity map) being a *regular* map ([4–6]), that replaces and generalizes that of being a homeomorphism, required in the linear case.

To prove our claim that the Rayleigh quotient is significant in the larger context of nonlinear spectral theory, and not solely for nonlinear eigenvalue problems, we propose in our turn to modify the definition of spectrum of a nonlinear operator given in [5] in the following single point: we replace, in the three requirements for regularity listed in [5] Definition 3.1, that of being

$$(0, \epsilon) - epi\, on \;\; B_r = \{x \in E : ||x|| < r\} \; for\ some\ \epsilon > 0\ and\ every\ r > 0$$

with the weaker one ([5], Proposition 3.2) of being merely *surjective*. Of course, this replacement modifies the spectrum restricting it somewhere, and it may seem perverse to insist giving one more definition of spectrum for a nonlinear map besides the many already existing [4]; however, the simplicity and universality of the concept of surjectivity—together with the fact that for linear maps the newly defined spectrum still coincides with the ordinary one, see Remark 3—hopefully justifies this choice. As a matter of fact, the use of the *simplified spectrum* (see Definition 2) allows us to give a new, improved and clearer presentation of the results on nonlinear spectral theory appeared in [8,9].

This paper is organized as follows. In Section 2 we first recall the definition and properties of some fundamental constants, such as the norm $||F||$ and the measure of noncompactness $\alpha(F)$, of a nonlinear operator F acting in a general (real) Banach space X; on the basis of these constants, and of the idea of surjectivity as indicated above, we then give our new definitions of regularity and of simplified spectrum $\sigma_S(F)$ of F, and establish a first result (Theorem 1) on the location of $\sigma_S(F)$ with respect to the constants. Namely, we prove that

$$\sigma_S(F) \subset [-\|F\|, \|F\|] \cup [-\alpha(F), \alpha(F)]$$

which is the same result stated in ([5], Theorem 3.6) for the spectrum as there defined, save that the proof here is much simpler because of a more direct use of Darbo's Fixed Point Theorem.

Section 3 is devoted to the spectral properties of gradient operators in a real Hilbert space: these are the nonlinear counterpart of self-adjoint operators, and share some of their special properties. In the linear case, such properties are consequential to the special symmetry of these operators, actually defined via the corresponding bilinear form: indeed, in the present context they are also known as *symmetric* operators, see e.g., [10,11]. In general, our attention will be focused on the best lower and upper bounds for the Rayleigh quotient of F, namely the constants defined by the formulae

$$m(F) = \inf_{x \neq 0} \frac{\langle F(x), x \rangle}{\|x\|^2}, \qquad M(F) = \sup_{x \neq 0} \frac{\langle F(x), x \rangle}{\|x\|^2}. \tag{5}$$

We study their role in the spectrum $\sigma_S(F)$ of F, and in Theorems 5 and 6 we establish some generalization of well known properties enjoyed in this sense by linear self-adjoint operators, see for instance ([12], Proposition 6.9) or [11] Theorem 6.2-B.

Gradient operators are by definition the derivatives of a functional, and this is the new and more general symmetry property that must be considered. It is therefore clear that a central role in their study is played by the use of variational methods, and in the first instance of those methods regarding the *minimization* (or *maximization*) of the functional itself. In fact our main results, Theorem 5 and Theorem 6, are conceptually connected by the use in their proof of one fundamental principle in the Calculus of Variations, namely the Ekeland Variational Principle [13], for a nice discussion of which we refer the reader and ourselves to De Figueiredo's book [14].

Though some parts of our results have already appeared elsewhere (see especially [8] about Theorem 6), one of the scopes of the present work is precisely to reorganize and unify them in the light of the new definition of spectrum and of the above mentioned Ekeland principle, and also to simplify as much as possible the technical side of the matter, also in the spirit of possibly stimulating new research on the subject.

For a recent review of some features of nonlinear operators and their eigenvalues, with applications to ordinary and partial differential equations, we refer the interested reader to [15].

2. A Simplified Spectrum

Let X be a real Banach space. If $F : X \to X$, we put

$$\|F\| = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|}, \qquad b(F) = \inf_{x \neq 0} \frac{\|F(x)\|}{\|x\|}. \tag{6}$$

Note that $\|F\|$ can be ∞ unless we assume that F is *sublinear* ("linearly bounded" in the terminology of [4]), that is, satisfies an inequality of the form

$$\|F(x)\| \leq A\|x\| \tag{7}$$

for some $A \geq 0$ and all $x \in X$ with $x \neq 0$. This implies in particular that F is bounded on bounded subsets of X; when a map $F : X \to X$ satisfies this condition, we merely say that F is *bounded*. From now on we shall mostly consider maps $F : X \to X$ that are sublinear and *continuous* on X; these two conditions also imply that $F(0) = 0$, as follows at once from (7). Clearly, this class of maps constitutes a real vector space containing (properly) the vector subspace $L(X)$ of the bounded linear operators acting in X. Moreover, it is readily checked that

- the number $||F||$ given by the first equality in (6) defines a norm in the vector space just described, and $||F||$ coincides with the usual linear operator norm when restricted to $L(X)$;
- the definition of $b(F)$ given by the second equality in (6) implies that

$$||F(x)|| \geq b(F)||x|| \qquad (x \in X)$$

so that the condition $b(F) > 0$ implies a coercivity property for a general F, in the sense that necessarily $||F(x)|| \to \infty$ as $||x|| \to \infty$; while in particular, for $F \in L(X)$—in which case $b(F)$ is sometimes called the *minimum modulus* of F, see e.g., page 231 of Kato's book [10]—the same condition characterizes the property of F of being *boundedly invertible*, that is, injective with bounded inverse F^{-1} (defined on the range of F).

We now come to recall some definitions related to compactness. If A is a bounded subset of X, let $\alpha(A)$ denote the (Kuratowski) *measure of noncompactness* of A defined by

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many subsets of diameter} \leq \epsilon\}.$$

For the elementary properties of α we refer, for instance, to the books [4,16] and to the papers [6,17]. In particular we recall that $\alpha(A) = 0$ if and only if A is *relatively compact*, meaning that the closure $\overline{A}$ of A is compact.

A bounded map $F : X \to X$ is said to be $\alpha-$*Lipschitz* if $\alpha(F(A)) \leq k\alpha(A)$ for some $k \geq 0$ and all bounded subsets A of X; in this case we put

$$\alpha(F) = \inf\{k \geq 0 : \alpha(F(A)) \leq k\alpha(A) \text{ for all bounded } A \subset X\}, \tag{8}$$

that is,

$$\alpha(F) = \sup\{\frac{\alpha(F(A))}{\alpha(A)} : A \subset X, A \text{ bounded}, \alpha(A) > 0\}. \tag{9}$$

(We assume that $dim X = \infty$, so that there exist bounded sets $A \subset X$ with $\alpha(A) > 0$). Note that $\alpha(F) = 0$ if and only if F is *compact*, i.e., such that $F(A)$ is relatively compact whenever $A \subset X$ is bounded. The importance and usefulness of α can be appreciated thinking for instance to Darbo's generalization of the Schauder Fixed Point Theorem, that we shall recall and employ below.
Next, let $\omega(F)$ be defined as follows:

$$\omega(F) = \inf\{\frac{\alpha(F(A))}{\alpha(A)} : A \subset X, A \text{ bounded}, \alpha(A) > 0\}. \tag{10}$$

Though quite obvious, we remark here for completeness that $\omega(F)$ is well defined for any bounded F; while $\alpha(F)$ is defined as a real number only if, in addition, the ratio $\alpha(F(A))/\alpha(A)$ appearing in (9) is bounded from above.

There are useful relations between the various constants introduced so far, that can be easily obtained by the definitions and are shown for instance in [4,6,17]. We report here, for further use in the present paper, only the following:

$$\alpha(\lambda F) = |\lambda|\alpha(F), \qquad \omega(\lambda F) = |\lambda|\omega(F) \tag{11}$$

$$b(F+G) \geq b(F) - ||G||, \qquad \omega(F+G) \geq \omega(F) - \alpha(G) \tag{12}$$

that hold for any $\lambda \in \mathbb{R}$ and any bounded maps F, G of X into itself. Also note that if I denotes the identity map in X, then evidently

$$||I|| = b(I) = \alpha(I) = \omega(I) = 1. \tag{13}$$

We remark that in general, the study of the measure of noncompactness in Banach spaces forms a vast and active research field in Functional Analysis, that has received further interest and expansion from the axiomatic approach presented in [16]; for an updated overview of this, see for instance [18] and the references therein. While in particular, the importance of ω for the study of nonlinear operators, originally shown in [6], has been further demonstrated especially in works by M. Furi and his school, see for instance their recent paper [17]. One basic property of $\omega(F)$ that we shall use repeatedly in Section 3 is expressed by the following statement (see [6], Proposition 3.1.3), the proof of which will be given there for the reader's convenience: if $\omega(F) > 0$, then F is *proper on closed bounded sets*; that is, given any compact $K \subset X$ and any closed bounded $M \subset X$, it follows that $M \cap F^{-1}(K) = \{x \in M : F(x) \in K\}$ is compact.

Remark 1. *If $T : X \to X$ is a bounded linear operator, then it is $\alpha-$Lipschitz and the following inequalities hold true (see, for instance [6], Proposition 3.2.1):*

$$\alpha(T) \leq ||T||, \qquad \omega(T) \geq b(T). \tag{14}$$

After these preliminaries, we are now in a position to give our new definition of regularity, and consequently of spectrum, for a nonlinear operator acting in a general Banach space X.

Definition 1. *A bounded continuous map $F : X \to X$ is said to be simply regular if $\omega(F) > 0$, $b(F) > 0$ and F is surjective.*

Remark 2. *Suppose that F is linear. It is clear by Remark 1, and by the comments made at the beginning of this Section about the condition $b(F) > 0$ for linear F, that F is simply regular if and only if it is a linear homeomorphism of X onto iself.*

Definition 2. *Let $F : X \to X$ be bounded and continuous. The simplified spectrum of F, denoted $\sigma_S(F)$, is defined as*

$$\sigma_S(F) = \{\lambda \in \mathbb{R} : F - \lambda I \quad \textit{is not simply regular}\}.$$

Remark 3. *It follows by Remark 2 that for a linear F, $\sigma_S(F)$ is nothing but the usual spectrum $\sigma(F)$ of F.*

As for a linear operator, a distinguished part of the spectrum of F is the set of its eigenvalues, namely the *point spectrum* of F.

Definition 3. *A point $\lambda \in \mathbb{R}$ is said to be an eigenvalue of F if $F(x) = \lambda x$ for some $x \in X$ with $x \neq 0$; in this case, x is said to be an eigenvector corresponding to λ.*

Eigenvalues of F do belong to $\sigma_S(F)$, for if $F(x) - \lambda x = 0$ for some $x \neq 0$, then necessarily

$$b(F - \lambda I) = \inf_{x \neq 0} \frac{\|F(x) - \lambda x\|}{\|x\|} = 0,$$

so that $F - \lambda I$ is not simply regular. Here and henceforth we put

$$\sigma_p(F) = \{\lambda \in \mathbb{R} : \lambda \;\text{ is an eigenvalue of } F\}$$

and call $\sigma_p(F)$ the *point spectrum* of F. Of course, these definitions are more significant when F satisfies the condition $F(0) = 0$ (as is necessarily the case if F is sublinear—and not merely bounded—as already indicated before): indeed in this case $x = 0$ solves trivially the equation $F(x) = \lambda x$ for every $\lambda \in \mathbb{R}$, and so the solutions $x \neq 0$ of this equation are appropriately called "nontrivial".

Let us now turn to the simplified spectrum $\sigma_S(F)$ of F in its entirety. Our first result is as follows:

Theorem 1. *Let $F : X \to X$ be sublinear, continuous and $\alpha-$Lipschitz. Then*

$$\sigma_S(F) \subset [-\|F\|, \|F\|] \cup [-\alpha(F), \alpha(F)]. \tag{15}$$

We remark that the above statement is essentially the same as that of Theorem 3.6 of [5] for the spectrum as there defined. Also the proof is similar, but is simplified by the fact that here the required surjectivity of $F - \lambda I$ follows by a direct use of Darbo's Fixed Point Theorem (see, e.g., [4], Theorem 2.1). We first recall the latter for the reader's convenience and then—before proving Theorem 1—we state and prove as intermediate step a simple Corollary to Darbo's Theorem that is particularly convenient for our purposes.

Theorem 2. (Darbo's Fixed Point Theorem) *Let C be a closed, bounded, convex subset of the Banach space X, and let $F : C \to C$ be continuous and $\alpha-$Lipschitz with $\alpha(F) < 1$. Then there exists $x \in C : F(x) = x$.*

Corollary 1. *Let $F : X \to X$ be sublinear, continuous and $\alpha-$Lipschitz. Suppose that $||F|| < 1$ and that $\alpha(F) < 1$. Then $I - F$ is surjective.*

Proof. Let $y \in X$ and consider the equation $x - F(x) = y$, that is equivalent to the fixed point problem

$$x = F(x) + y \equiv G(x) \tag{16}$$

for the map G, that is (continuous and) such that $\alpha(G) = \alpha(F) < 1$. We claim that G maps a closed ball into itself, so that the existence of a solution to (16) follows from Darbo's Theorem. Now, given any $R > 0$ we have, if $||x|| \leq R$,

$$||G(x)|| \leq ||F(x)|| + ||y|| \leq ||F||||x|| + ||y|| \leq ||F||R + ||y||$$

so that we will also have $||G(x)|| \leq R$ as soon as $||F||R + ||y|| \leq R$, that is, as soon as R is taken so large that

$$\frac{||y||}{1 - ||F||} \leq R.$$

□

We can now prove Theorem 1. To this aim we show that, if

$$|\lambda| > \max\{||F||, \alpha(F)\}$$

then $F - \lambda I$ is simply regular; however, this will follow using the properties (11) to (13) of the relevant constants $\alpha(F), \omega(F)$ etc. together with Corollary 1. Indeed, we have

$$b(F - \lambda I) = b(\lambda I - F) \geq |\lambda| - ||F|| > 0 \tag{17}$$

and similarly

$$\omega(F - \lambda I) = \omega(\lambda I - F) \geq |\lambda| - \alpha(F) > 0. \tag{18}$$

Moreover, writing (for $\lambda \neq 0$)

$$F(x) - \lambda x = -\lambda(I - \frac{F}{\lambda})(x) \qquad (x \in X)$$

and observing that

$$\left\|\frac{F}{\lambda}\right\| = \frac{||F||}{|\lambda|} < 1, \qquad \alpha\left(\frac{F}{\lambda}\right) = \frac{\alpha(F)}{|\lambda|} < 1$$

we conclude by virtue of Corollary 1 that $I - F/\lambda$, and thus also $F - \lambda I$, is surjective.

Remark 4. *In the special case that $F = T$, a bounded linear operator, Theorem 1 reduces—on the basis of the Remarks 1 and 3—to the familiar result (see, for instance, [12], Proposition 6.7) $\sigma(T) \subset [-\|T\|, \|T\|]$.*

Remark 5. *The spectrum as defined in [5] is not only bounded but also closed ([5], Theorem 3.5). Up to now, we were unable to prove or disprove this same property for the newly defined spectrum σ_S, so this remains at the* *moment an interesting open problem. We remark that the proof of [5], Theorem 3.5 is based on the homotopy property of the topological degree, and does not seem to be ready for an adaptation to* σ_S.

To relate—as indicated by the title—the newly defined spectrum of a nonlinear operator with its Rayleigh quotient, we suppose now that H is a real Hilbert space and let $F : H \to H$. If F is sublinear, then its Rayleigh quotient defined in (1) is *bounded*, for by Cauchy-Schwarz' inequality we have

$$|\langle F(x), x\rangle| \leq \|F(x)\|\|x\| \leq \|F\|\|x\|^2 \tag{19}$$

for every $x \in H$, so that the numbers $m(F), M(F)$ introduced in (5) are well defined, and moreover by (19) we have

$$-\|F\| \leq m(F) \leq M(F) \leq \|F\|. \tag{20}$$

In the special case that $F = T$, a bounded linear operator, these numbers are quite meaningful from the viewpoint not only of the eigenvalues, for evidently we have

$$\sigma_p(T) \subset [m(T), M(T)] \tag{21}$$

but of the entire spectrum itself: indeed, it follows by the Lax-Milgram Lemma that the inclusion $\sigma(T) \subset [-\|T\|, \|T\|]$ can be improved to

$$\sigma(T) \subset [m(T), M(T)] \tag{22}$$

as shown, for instance, in Proposition 2.1 of [9]. In the special case that T is *self-adjoint* (that is, such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$) one also has

$$m(T), M(T) \in \sigma(T) \tag{23}$$

(see, e.g., [12], Proposition 6.9 or [11], Theorem 6.2-B); finally, if $M(T)$ is larger than the measure of noncompactness $\alpha(T)$ of T, then $M(T)$ is *attained* (thus, it is the *maximum* of the Rayleigh quotient of T, and so the largest possible eigenvalue) and is indeed an eigenvalue of T *of finite multiplicity*, that is, the nullspace

$$\mathrm{Ker}(T - M(T)I)$$

has finite dimension. A similar conclusion holds for $m(T)$ in case $m(T) < -\alpha(T)$.

While it is clear that—as indicated in the Introduction—(21) also holds for nonlinear operators, it is less immediate but equally interesting to see that both formula (15) for the "localization" of the entire spectrum σ_S can be improved, and the property (23) together with the comments following it can be partly extended, for the nonlinear version of self-adjoint operators, namely the gradient operators: this will be shown in the next Section.

Remark 6. *For want of a better place, we state here formally the inequality*

$$b(F) \geq \max\{m(F), -M(F)\} \tag{24}$$

that holds for any sublinear operator $F : H \to H$. To see this, just use the definition (5) of, for instance, $m(F)$ and the Cauchy-Schwarz inequality to write

$$m(F)||x||^2 \leq \langle F(x), x\rangle \leq ||F(x)||||x|| \qquad (x \in H).$$

Dividing by $||x||^2$ ($x \neq 0$) and using the definition (6) of $b(F)$, we thus obtain that $b(F) \geq m(F)$. A similar remark about $M(F)$ leads to (24). Note that (24) is trivial in the case that $m(F) \leq 0 \leq M(F)$; on the *other hand, it turns out to be quite useful if $m(F) > 0$ or $M(F) < 0$—that is, if F is "positive* (resp. negative) *definite*" on H—as will be shown in the proof of Theorem 5.

3. Gradient Operators

An operator $F : H \to H$ is said to be a *gradient operator* if there exists a differentiable functional $f : H \to \mathbb{R}$ such that

$$\langle F(x), y \rangle = f'(x)y \quad \text{for all} \quad x, y \in H, \tag{25}$$

where $f'(x)$ denotes the (Fréchet) derivative of f at the point $x \in H$. When it is so, and when in addition F is continuous, the functional f—the *potential* of F—is uniquely determined by the requirement that $f(0) = 0$, and is explicitly given by the formula

$$f(x) = \int_0^1 \langle F(tx), x \rangle \, dt. \tag{26}$$

For these definitions and statements see, for instance [19]. We also recall that a bounded linear operator is a gradient if and only if it is self-adjoint. For concrete examples of gradient operators that one faces when dealing with boundary value problems for nonlinear differential equations, see for instance [14,19,20].

The results on the nonlinear spectrum contained in this Section, Theorem 5 and Theorem 6, both refer to gradient operators and both are based on the Ekeland Variational Principle [13], used in conjunction with the compactness properties steming from the use of the constants α and ω previously defined, respectively by (9) and (10). To explain this strategy, we need discuss some relevant points concerning each of these two tools. As to the former, the following "weak form" (see, e.g., [14], Theorem 4.1) will be sufficient for our purposes:

Theorem 3. (Ekeland Variational principle) *Let (X, d) be a complete metric space. Let $f : X \to \mathbb{R}$ be lower semicontinuous and bounded below. Put $c = \inf_{x \in X} f(x)$; then given any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that*

$$\begin{cases} f(x_\epsilon) < c + \epsilon \\ f(x_\epsilon) < f(x) + \epsilon d(x, x_\epsilon), \quad \forall x \in X,\, x \neq x_\epsilon. \end{cases} \tag{27}$$

For our use, a first standard way of using Ekeland's principle is provided by the following statement, which is in fact a special form of Theorem 4.4 of [14].

Corollary 2. *Let f be a C^1 functional defined on the Banach space X and suppose that f is bounded below on X. Let $c = \inf_{x \in X} f(x)$. Then given any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that*

$$\begin{cases} f(x_\epsilon) < c + \epsilon \\ \|f'(x_\epsilon)\| \leq \epsilon. \end{cases} \tag{28}$$

For the sake of clarity, let us see how Corollary 2 follows from Theorem 3. First consider that the derivative $f'(x_0)$ of f at a given point $x_0 \in X$ is a bounded linear form on X, that satisfies the equality

$$f'(x_0)y = \lim_{t \to 0} \frac{f(x_0 + ty) - f(x_0)}{t} \tag{29}$$

for every $y \in X$, and whose norm in the dual X^* of X is by definition

$$\|f'(x_0)\| = \sup \left\{ |f'(x_0)v| : v \in X, ||v|| = 1 \right\}. \tag{30}$$

Now given $v \in X$ with $||v|| = 1$, take $x = x_\epsilon + tv, t \neq 0$ in the second inequality in (27): this yields

$$f(x_\epsilon) < f(x_\epsilon + tv) + \epsilon|t|. \tag{31}$$

Thus, taking $t > 0$, we obtain

$$\frac{f(x_\epsilon + tv) - f(x_\epsilon)}{t} > -\epsilon$$

and therefore, letting $t \to 0^+$,

$$f'(x_\epsilon)v \geq -\epsilon. \tag{32}$$

Similarly, taking $t < 0$ in (31) yields

$$\frac{f(x_\epsilon + tv) - f(x_\epsilon)}{t} < \epsilon$$

whence, letting $t \to 0^-$,

$$f'(x_\epsilon)v \leq \epsilon. \tag{33}$$

Using (30), (32) and (33) then yields the second inequality in (28).

We come now to compactness, and in particular to the important property, indicated in Section 2, that is owned by the numerical constant $\omega(F)$ defined via (10) for any bounded operator F acting in a Banach space X.

Proposition 1. *Let $F : X \to X$ be continuous and bounded. If $\omega(F) > 0$, then F is proper on closed bounded sets: that is, given any compact $K \subset X$ and any closed bounded $M \subset X$, it follows that $M \cap F^{-1}(K) = \{x \in M : F(x) \in K\}$ is compact. It follows in particular that given any bounded sequence $(x_n) \subset X$ such that $F(x_n)$ converges, (x_n) contains a convergent subsequence.*

Proof. Let M and K be as in the statement. We have

$$K \supset F(M) \cap K \supset F(M) \cap F(F^{-1}(K)) \supset F(M \cap F^{-1}(K))$$

and therefore

$$\alpha(K) \geq \alpha(F(M \cap F^{-1}(K)))$$

whence, using the definition (10) of $\omega(F)$, it follows that

$$\alpha(K) \geq \omega(F)\alpha(M \cap F^{-1}(K)). \tag{34}$$

As K is compact by assumption, the left-hand side of (34) is zero. As $\omega(F) > 0$, (34) thus implies that $M \cap F^{-1}(K)$ is relatively compact, whence the result follows since $M \cap F^{-1}(K)$ is also closed by the continuity of F. The last statement of Proposition 1 follows on considering—for a given bounded sequence $(x_n) \subset X$ such that $F(x_n) \to y_0 \in X$, say, the compact set

$$K = \{F(x_n) : n \in \mathbb{N}\} \cup \{y_0\}$$

and a bounded set M containing (x_n). By what has been just proved, it follows that the set

$$\{x_n : n \in \mathbb{N}\} \subset M \cap F^{-1}(K)$$

is relatively compact, and therefore (x_n) contains a convergent subsequence. □

On the basis of Corollary 2 and of Proposition 1, we have recently obtained the following simple surjectivity result, first proved in [9] and further generalized in [20].

Theorem 4. *Let $F : H \to H$ be a sublinear continuous gradient operator. Suppose that*

$$m(F) > 0 \qquad \textit{and} \qquad \omega(F) > 0 \tag{35}$$

where $m(F)$ and $\omega(F)$ are as in (5) and in (10) respectively. Then F is surjective.

We give here for completeness a sketch of the proof; for more details, see [9] or [20]. Put $k = m(F)$; thus by definition F satisfies the inequality

$$\langle F(x), x\rangle \geq k||x||^2 \qquad (x \in H). \tag{36}$$

Using (26), it follows that a similar inequality is satisfied by the potential f of F. In turn this easily implies that, given any fixed $y \in H$, the functional f_y defined putting

$$f_y(x) = f(x) - \langle y, x\rangle \qquad (x \in H) \tag{37}$$

is *coercive* (i.e., $f_y(x) \to \infty$ as $||x|| \to \infty$) and bounded below on H. Moreover, f_y is of class C^1 by virtue of the continuity assumption on F; so that using Corollary 2, it follows that there is a sequence $(x_n) \subset H$ such that

$$f_y(x_n) \to c_y \equiv \inf f_y \qquad \text{and} \qquad f'_y(x_n) \to 0. \tag{38}$$

However, since (by (25)) we have

$$f'_y(x)v = \langle F(x) - y, v\rangle$$

for every $x, v \in H$, we see that the second relation in (38) is equivalent to

$$F(x_n) \to y. \tag{39}$$

We also have that (x_n) is bounded: for otherwise, extracting a subsequence (x_{n_k}) with $||x_{n_k}|| \to \infty$ and using the coercivity of f_y, we would contradict the first relation in (38). Therefore, since $\omega(F) > 0$ by assumption, it follows by Proposition 1 that (x_n) contains a subsequence, still denoted (x_{n_k}) for convenience, such that $x_{n_k} \to x_0$, say; by (39) and the continuity of F we then have that $F(x_0) = y$. Thus the equation $F(x) = y$ has a solution, and as y is arbitrary this proves the surjectivity of F, as desired.

Remark 7. *In the language of Critical Point Theory, we would say that under the stated assumptions on F, the functional f_y defined in (37) satisfies the Palais-Smale condition*, see for instance page 37 of [14]. The relation between the Palais-Smale condition and the spectrum of a linear self-adjoint operator has been thoroughly investigated by C.A. Stuart in his paper [21].

Remark 8. *The conclusion of Theorem 4 holds unaltered if the assumption $m(F) > 0$ is replaced with $M(F) < 0$. Indeed in this case we have $m(-F) = -M(F) > 0$, and since $\omega(-F) = \omega(F)$, the statement above guarantees that $-F$ is surjective, whence the conclusion follows.*

Remark 9. *In the statement of Theorem 4 it is assumed that F is sublinear, meaning that it satisfies the growth restriction (7). This condition is unnecessarily strong, for in Theorem 1.5 of [9] it is proved that the surjectivity of F still holds if we merely assume that F is bounded (as required "apriori" and once for all on p. 172 of [9]) and that $m(F)$ is defined and > 0, which amounts to the requirement that F satisfies (36) for some $k > 0$. Moreover in [20], it has been shown that the exponent 2 in (36) can be replaced by any $p > 1$ and that F, rather than acting in a Hilbert space, can be assumed to operate from any Banach space X to its dual X^*; in this more general situation, the scalar product appearing on the l.h.s. of (25) must be evidently thought as the pairing between X^* and X.*

We are now ready to establish the improvement of Theorem 1 about the spectrum of gradient operators that was announced by the end of Section 2.

Theorem 5. *Let $F : H \to H$ be sublinear, continuous and $\alpha-$Lipschitz. Suppose moreover that F is a gradient. Then*

$$\sigma_S(F) \subset [m(F), M(F)] \cup [-\alpha(F), \alpha(F)]. \tag{40}$$

Proof. To prove the inclusion (40), we consider a $\lambda \in \mathbb{R}$ such that $|\lambda| > \alpha(F)$ and $\lambda < m(F)$ (or $\lambda > M(F)$), and show that the bounded continuous operator

$$F_\lambda \equiv F - \lambda I$$

is simply regular. As in the proof of Theorem 1, (18) shows at once that $\omega(F_\lambda) > 0$. Furthermore, suppose for instance that $\lambda < m(F)$; then as we have

$$\langle F_\lambda(x), x\rangle = \langle F(x), x\rangle - \lambda\langle x, x\rangle \geq (m(F) - \lambda)||x||^2 \tag{41}$$

for every $x \in H$, it follows that

$$m(F_\lambda) = \inf_{x \neq 0} \frac{\langle F_\lambda(x), x\rangle}{\|x\|^2} \geq m(F) - \lambda > 0.$$

Since of course F_λ is a gradient operator as well as F, Theorem 4 then guarantees that F_λ is surjective. Finally, to achieve the proof that F_λ is simply regular, simply use the inequality (24) to obtain

$$b(F_\lambda) \geq m(F_\lambda) > 0.$$

□

Remark 10. *In the special case that $F = T$, a self-adjoint bounded linear operator, Theorem 5 as it stands does not reproduce the inclusion (22). The reason is that the coercivity conditions $m(F) > 0$ and $\omega(F) > 0$, that by virtue of Theorem 4 guarantee the surjectivity of a gradient operator F, in the case that $F = T$ is linear simplify to $m(T) > 0$: indeed the latter condition implies via (24) that $b(T) > 0$ as well, and in turn this yields $\omega(T) > 0$ by virtue of the second of the two inequalities in (14), that hold for bounded linear operators. More remarkably, as already pointed out in [9], the implication "$m(T) > 0 \Rightarrow T$ surjective" holds for any bounded linear operator T acting in a real Hilbert space, without reference to self-adjointness.*

Our last result on the nonlinear spectrum σ_S is about gradient operators F that are "one step closer" to linear in the sense that they are also *positively homogeneous*, meaning that $F(tx) = tF(x)$ for every $x \in H$ and every real $t > 0$. It is clear that the behaviour of such operators is entirely determined by their properties on the unit sphere S of H,

$$S = \{x \in H : ||x|| = 1\}.$$

As to their spectrum, note in particular that

$$b(F - \lambda I) = \inf_{x \neq 0} \frac{\|F(x) - \lambda x\|}{\|x\|} = \inf_{x \in S} \|F(x) - \lambda x\|. \tag{42}$$

Likewise, we have

$$m(F) = \inf_{x \neq 0} \frac{\langle F(x), x\rangle}{\|x\|^2} = \inf_{x \in S} \langle F(x), x\rangle \tag{43}$$

and similarly for $M(F)$. Moreover as to the point spectrum $\sigma_p(F)$, it suffices evidently to consider only normalized eigenvectors corresponding to a given eigenvalue, and one more definition seems here to be useful: by a *compact eigenvalue* of a nonlinear operator F we mean a $\lambda \in \sigma_p(F)$ such that the corresponding set of normalized eigenvectors,

$$N_S(F-\lambda I) \equiv \{x \in H : ||x|| = 1, F(x) = \lambda x\} \tag{44}$$

is compact. For a linear F, we have

$$N_S(F-\lambda I) = S \cap \mathrm{Ker}(F-\lambda I)$$

so that, on the basis of Riesz' theorem characterizing finite-dimensional normed spaces (see, e.g., [12], Theorem 6.5) "compact eigenvalue" is just a synonymous of "eigenvalue of finite (geometric) multiplicity".

Theorem 6. *Let $F : H \to H$ be sublinear, continuous and $\alpha-$Lipschitz. Suppose moreover that F is a gradient and is positively homogeneous. Then in addition to (40), we have:*

- *$m(F), M(F) \in \sigma_S(F)$;*
- *If moreover $m(F) < -\alpha(F)$, then $m(F) \in \sigma_p(F)$. Furthermore, $m(F)$ is the* smallest *eigenvalue of F and is a* compact *eigenvalue. A similar conclusion holds for $M(F)$ in case $M(F) > \alpha(F)$.*

The content of Theorem 6 is very much the same as that of Theorems 1.1 and 1.2 of [8]. However, the above statement is consistent with the introduction of $\sigma_S(F)$ and with the discussion made so far about it: thus, we give here a complete proof of Theorem 6, slightly different and hopefully simplified as compared with the arguments used in [8]. The starting point is again the Ekeland Variational Principle, of which we state and prove below two consequences. The first (Corollary 3) is the "constrained" version of Corollary 2 referred to the unit sphere S of H; the second (Corollary 4) is a reformulation of the same fact in terms of a given gradient operator F.

Corollary 3. *Let f be a C^1 functional defined on the Hilbert space H and suppose that f is bounded below on the sphere S. Let $C = \inf_{x \in S} f(x)$. Then given any $\epsilon > 0$, there exists $x_\epsilon \in S$ such that*

$$\begin{cases} f(x_\epsilon) < C + \epsilon \\ \|f'_S(x_\epsilon)\| \le \epsilon \end{cases} \tag{45}$$

where for $x_0 \in S$, $f'_S(x_0)$ denotes the restriction of $f'(x_0)$ to $T_{x_0}(S)$, the tangent space to S at x_0:

$$T_{x_0}(S) = \{y \in H : \langle x_0, y\rangle = 0\}. \tag{46}$$

Proof. We use the Ekeland principle, Theorem 3, taking as complete metric space X the unit sphere S of H and working as in the proof of Corollary 2; consider that as $f'_S(x_0)$ is a bounded linear form on $T_{x_0}(S)$, its norm in the dual space $(T_{x_0}(S))^*$ is

$$\|f'_S(x_0)\| = \sup\{|f'(x_0)v| : v \in T_{x_0}(S), \|v\| = 1\}. \tag{47}$$

Thus for $\epsilon > 0$, let $x_\epsilon \in S$ be as in (27). Given any $v \in T_{x_\epsilon}(S)$, by definition there exists a C^1 curve γ_v, defined in some neighborhood I of $t = 0$, such that $\gamma_v(t) \in S$ for all $t \in I$, $\gamma_v(0) = x_\epsilon$ and $\gamma_v(t) \neq x_\epsilon$ for $t \neq 0$, $\gamma'_v(0) = v$. Then putting $x = \gamma_v(t)$ in the second inequality of (27) yields

$$f(x_\epsilon) < f(\gamma_v(t)) + \epsilon||\gamma_v(t) - x_\epsilon|| \qquad (t \in I, t \neq 0). \tag{48}$$

By the properties of γ_v, we have

$$\gamma_v(t) = \gamma_v(0) + \gamma_v'(0)t + o(t) = x_\epsilon + tv + o(t) \quad (t \to 0) \tag{49}$$

whence, as $||v|| = 1$, it follows that

$$\|\gamma_v(t) - x_\epsilon\| = \|tv + o(t)\| \leq |t|(1 + o(1)) \qquad (t \to 0). \tag{50}$$

Putting (50) into (48) we then have

$$f(x_\epsilon) < f(\gamma_v(t)) + \epsilon|t|(1 + o(1)) \qquad (t \to 0). \tag{51}$$

Now if $t > 0$, this yields

$$\frac{f(\gamma_v(t)) - f(x_\epsilon)}{t} > -\epsilon(1 + o(1)) \qquad (t \to 0) \tag{52}$$

whence, letting $t \to 0^+$, we get $(f \circ \gamma_v)'(0) = f'(x_\epsilon)v \geq -\epsilon$. While considering $t < 0$, we obtain in a similar way $f'(x_\epsilon)v \leq \epsilon$. Therefore,

$$|f'(x_\epsilon)v| \leq \epsilon \tag{53}$$

and since this holds for any $v \in T_{x_\epsilon}(S)$ with $\|v\| = 1$, it follows from (47) that $\|f_S'(x_\epsilon)\| \leq \epsilon$. This ends the proof of Corollary 3. $\square$

Corollary 4. *Let $F : H \to H$ be a sublinear, continuous, gradient operator. Let $C = \inf_{x \in S} f(x)$, where f is the potential of F. Then given any $\epsilon > 0$, there exists $x_\epsilon \in S$ such that*

$$\begin{cases} f(x_\epsilon) < C + \epsilon \\ ||F(x_\epsilon) - \langle F(x_\epsilon), x_\epsilon\rangle x_\epsilon|| \leq \epsilon. \end{cases} \tag{54}$$

Proof. First notice that f is of class C^1 and is bounded on S, for by (26) we have, using the sublinearity assumption (7),

$$|f(x)| \leq \int_0^1 |\langle F(tx), x\rangle|\, dt \leq A \int_0^1 ||tx||||x||\, dt = \frac{A}{2}||x||^2$$

for every $x \in S$. Thus by Corollary 3, for any $\epsilon > 0$ there is an $x_\epsilon \in S$ satisfying (45); and so to finish the proof of Corollary 4, it is enough to verify the equality

$$\|f_S'(x)\| = ||F(x) - \langle F(x), x\rangle x|| \qquad (x \in S). \tag{55}$$

Indeed (keeping the notations used in the proof of Corollary 3) we know that for $x \in S$, $f_S'(x)$ is a bounded linear form on the Hilbert space $T_x(S)$, hence there exists a unique vector $G(x) \in T_x(S)$ such that

$$f_S'(x)v = \langle G(x), v\rangle \qquad (v \in T_x(S)) \tag{56}$$

and moreover

$$\|f_S'(x)\| = \|G(x)\|. \tag{57}$$

Let us check that

$$G(x) = F(x) - \langle F(x), x\rangle x \qquad (x \in S). \tag{58}$$

Indeed we have, for every $x \in S$,

$$\langle G(x), x\rangle = \langle F(x), x\rangle - \langle F(x), x\rangle = 0$$

so that $G(x) \in T_x(S)$ by the expression (46) of $T_x(S)$; this same expression implies, using also (25), that for $v \in T_x(S)$ we have

$$f'_S(x)v = f'(x)v = \langle F(x), v\rangle = \langle F(x), v - \langle v, x\rangle x\rangle =$$

$$= \langle F(x), v\rangle - \langle v, x\rangle\langle F(x), x\rangle = \langle F(x) - \langle F(x), x\rangle x, v\rangle. \tag{59}$$

This proves (58) and so—by virtue of (57)—also proves our claim (55). □

Equipped with these preparatory results, we can now readily prove Theorem 6. Indeed letting f be the potential of F and using Corollary 4, we find a sequence $(x_n) \subset S$ such that

$$\begin{cases} f(x_n) \to C = \inf_{x\in S} f(x) \\ F(x_n) - \langle F(x_n), x_n\rangle x_n \to 0. \end{cases} \tag{60}$$

However, the assumption that F is positively homogeneous employed in (26) yields

$$f(x) = \int_0^1 \langle F(tx), x\rangle \, dt = \frac{1}{2}\langle F(x), x\rangle \tag{61}$$

so that the first relation in (60) is equivalent to

$$\langle F(x_n), x_n\rangle \to \inf_{x\in S} \langle F(x), x\rangle = m \tag{62}$$

where we have put for notational convenience $m = m(F)$. Using this and the second relation in (60), we obtain

$$F(x_n) - mx_n \to 0 \tag{63}$$

and this finally implies that

$$b(F - mI) = \inf_{x\in S} \|F(x) - mx\| = 0$$

so that $m = m(F) \in \sigma_F(F)$, as claimed in the first statement of Theorem 6. To prove the second statement, suppose now that $m = m(F) < -\alpha(F)$. Then by (18),

$$\omega(F - mI) = \omega(mI - F) \geq -m - \alpha(F) > 0, \tag{64}$$

so that (by Proposition 1) $F - mI$ is proper on closed bounded sets and in particular, as (x_n) is bounded and $F(x_n) - mx_n$ converges as shown by (63), it follows that (x_n) contains a subsequence (x_{n_k}) such that $x_{n_k} \to x_0$, say. Then $x_0 \in S$ and it follows by (63) and the continuity of F that

$$F(x_0) = mx_0 \tag{65}$$

so that m is an eigenvalue of F with eigenvector x_0. Of course, (65) also implies that

$$\langle F(x_0), x_0\rangle = m \leq \langle F(x), x\rangle$$

for every $x \in S$, so that m is in fact the minimum of the Rayleigh quotient of F and therefore the minimum of all possible eigenvalues of F. Finally, that m is a compact eigenvalue is again a consequence of (64) and of Proposition 1: indeed—see (44)—we have

$$N_S(F - mI) = S \bigcap (F - mI)^{-1}(\{0\}).$$

4. Compact Operators

A final remark is useful to illustrate the above results in the case that the operator F is compact. As already recalled, this happens if and only if $\alpha(F) = 0$, and it is thus enough to insert this condition in the various statements. Thus for a sublinear, continuous and compact operator in a Banach space we have from Theorem 1 that

$$\sigma_S(F) \subset [-\|F\|, \|F\|]. \tag{66}$$

As to compact gradient operators in Hilbert space, we collect in a single formal statement the features of their spectrum that follow from Theorems 5 and 6.

Theorem 7. *Suppose that $F : H \to H$ is sublinear, continuous and compact. Suppose moreover that F is the gradient of a functional. Then*

$$\sigma_S(F) \subset \{0\} \bigcup [m(F), MF)]. \tag{67}$$

If in addition F is positively homogeneous, then $m(F), M(F) \in \sigma_S(F)$; moreover if $m(F) < 0$, then it is an eigenvalue of F, the smallest eigenvalue, and a compact eigenvalue. A similar statement holds for $M(F)$ in case that $M(F) > 0$.

Theorem 7 has the following interesting consequence.

Corollary 5. *If $F : H \to H$ is as in Theorem 7 and $F \neq 0$, then F has at least one nonzero eigenvalue.*

Proof. If $F \neq 0$, then necessarily $m(F) < 0$ or $M(F) > 0$ (or both). For otherwise, we would have $\langle F(x), x\rangle = 0$ for every $x \in S$ and therefore (by homogeneity) for every $x \in H$. Again by homogeneity and in particular by virtue of (61), it would then follow that the potential $f \equiv 0$, and this in turn would imply that $F \equiv 0$, contradicting our assumption. □

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