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Iteration Process for Fixed Point Problems and Zeros of Maximal Monotone Operators

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Abstract: We introduce an iterative algorithm which converges strongly to a common element of fixed point sets of nonexpansive mappings and sets of zeros of maximal monotone mappings. Our iterative method is quite general and includes a large number of iterative methods considered in recent literature as special cases. In particular, we apply our algorithm to solve a general system of variational inequalities, convex feasibility problem, zero point problem of inverse strongly monotone and maximal monotone mappings, split common null point problem, split feasibility problem, split monotone variational inclusion problem and split variational inequality problem. Under relaxed conditions on the parameters, we derive some algorithms and strong convergence results to solve these problems. Our results improve and generalize several known results in the recent literature.

Keywords: strongly nonexpansive sequence; zero point; fixed point; variational inequality; convex feasibility problem; split feasibility problem

1. Introduction

Fixed point theory has been revealed as a very powerful and effective method for solving a large number of problems which emerge from real world applications and can be translated into equivalent fixed point problems. In order to obtain approximate solution of the fixed point problems various iterative methods have been proposed (see, e.g., [1–10] and the reference therein). One of the important instances of fixed point problems is the problem of solving zero point problem of nonlinear operators. The most popular method for finding zeros of a maximal monotone operator is the proximal point algorithm (PPA). Rockafellar [11] proved the weak convergence of PPA, but it fails to converge strongly (see [12]). To obtain strong convergence, several authors proposed modification of PPA (see: Kamimura and Takahashi [13], Iiduka-Takahashi [14] and reference therein). In [15], Lehdili and Moudafi introduced the prox-Tikhonov regularization method which combined Tikhonov method with PPA to obtain a strongly convergent sequence.

In 2012, Censor, Gibali and Reich [16] (see also [17,18]) introduced a new variational inequality problem, called the common solutions to variational inequality problem (CSVIP) which comprises of finding common solutions to unrelated variational inequalities. The significance of studying the CSVIP lies in the fact that it includes the well-known convex feasibility problem (CFP) as its special case.

The CFP which lies in center of many problems of physical sciences such as sensor networking [19], radiation therapy treatment planning [20], computerized tomography [21], image restoration [22] is to find a point in the intersection of a family of closed convex sets in a Hilbert space.

A special case of the CFP is the split feasibility problem (SFP). In 1994, Censor and Elfving [23] introduced the SFP for modeling phase retrieval problems. This problem has large number of applications in optimization problems, signal processing, image reconstruction, intensity-modulated radiation therapy (IMRT). Starting from SFP, various important split type problems have been introduced and studied in recent years, for example, the split common null point problem (SCNPP), split monotone variational inclusion problem (SMVIP), split variational inequality problem (SVIP).

Motivated and inspired by the above work, we propose an iterative algorithm for finding common element of fixed point sets of nonexpansive mappings and sets of zeros of maximal monotone mappings. As applications, we solve all the problems discussed above under weaker conditions.

2. Preliminaries

Throughout the paper, we assume that $\mathcal{H}$ is a Hilbert space with the inner product $\langle ., . \rangle$ and the norm $\|.\|$ and let I be the identity mapping on $\mathcal{H}$. We denote by $\mathrm{Fix}(T)$ the set of all fixed points of a mapping T. A sequence $\{x_n\}$ in $\mathcal{H}$ converges to $x \in \mathcal{H}$ strongly if $\{\|x_n - x\|\}$ converges to 0 and weakly if $\{\langle x_n - x, y\rangle\}$ converges to 0, for every $y \in \mathcal{H}$. We shall use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to indicate the strong and weak convergence respectively. It is important to note that strong convergence always implies weak convergence, but the converse is not true (see [24]). Let $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$ and $P_{\mathcal{D}}$ denotes the nearest point projection (metric projection) from $\mathcal{H}$ onto $\mathcal{D}$, that is, for each $u \in \mathcal{H}$, $\|u - P_{\mathcal{D}}u\| \leq \|u - v\|$, for all $v \in \mathcal{D}$. Furthermore, $P_{\mathcal{D}}$ is characterized by the fact that $P_{\mathcal{D}}u \in \mathcal{D}$ and

$$\langle u - P_{\mathcal{D}}u, v - P_{\mathcal{D}}u\rangle \leq 0, \quad \forall\, v \in \mathcal{D}. \tag{1}$$

Next, we recall some definitions of well known operators, which we will use in our paper.

Definition 1. *An operator $S\colon \mathcal{H} \to \mathcal{H}$ is said to be*

1. *Nonexpansive if $\|Su - Sv\| \leq \|u - v\|$, $\forall\, u, v \in \mathcal{H}$.*
2. *Contraction if there exists a constant $k \in (0, 1)$ such that $\|Su - Sv\| \leq k\|u - v\|$, $\forall\, u, v \in \mathcal{H}$.*
3. *α-averaged if there exists a constant $\alpha \in (0, 1)$ and a nonexpansive mapping V such that $S = (1 - \alpha)I + \alpha V$.*
4. *β-inverse strongly monotone (for short, β-ism) if there exists $\beta > 0$ such that $\langle Su - Sv, u - v\rangle \geq \beta\|Su - Sv\|^2$, $\forall\, u, v \in \mathcal{H}$.*
5. *Firmly nonexpansive if $\langle Su - Sv, u - v\rangle \geq \|Su - Sv\|^2, \forall\, u, v \in \mathcal{H}$.*

It is known that metric projection $P_{\mathcal{D}}$ is firmly nonexpansive and every firmly nonexpansive is $(1/2)$-averaged.

An operator $\mathcal{M}\colon \mathcal{H} \to 2^{\mathcal{H}}$ is called maximal monotone on $\mathcal{H}$, if $\mathcal{M}$ is monotone, i.e., $\langle u_1 - v_1, u - v\rangle \geq 0\ \forall u, v \in dom(\mathcal{M})$, $u_1 \in \mathcal{M}u$ and $v_1 \in \mathcal{M}v$, and there is no other monotone operator whose graph contains graph of $\mathcal{M}$. Further, a resolvent associated with a maximal monotone operator $\mathcal{M}$ is a single valued operator defined as:

$$J_\lambda^{\mathcal{M}} = (I + \lambda\mathcal{M})^{-1}\colon \mathcal{H} \to \mathcal{H}.$$

It is well known [24] that if $\mathcal{M}\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator and $\lambda > 0$, then $J_\lambda^{\mathcal{M}}$ is firmly nonexpansive and $\mathrm{Fix}(J_\lambda^{\mathcal{M}}) = \mathcal{M}^{-1}0 = \{u \in \mathcal{H} : 0 \in \mathcal{M}u\}$.

A sequence $\{T_n\}$ of mappings is said to be a strongly nonexpansive sequence [25] if each T_n is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \to 0,$$

whenever $\{x_n\}, \{y_n\} \subset \mathcal{H}$ such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$. Note that if we put $T_n = T$, for all $n \in \mathbb{N}$, then we have definition of strongly nonexpansive mapping defined in [26].

In order to establish our results, we collect several lemmas.

Lemma 1. *Let $\mathcal{F}\colon \mathcal{H} \to \mathcal{H}$ be a β-ism operator on $\mathcal{H}$. Then $I - 2\beta\mathcal{F}$ is nonexpansive.*

Proof.

$$\begin{aligned}\|u-v\|^2 - \|(I-2\beta\mathcal{F})u - (I-2\beta\mathcal{F})v\|^2 &= \|u-v\|^2 - \big(\|u-v\|^2 + (2\beta)^2\|\mathcal{F}(u)-\mathcal{F}(v)\|^2 \\ &\quad - 4\beta\langle \mathcal{F}(u)-\mathcal{F}(v), u-v\rangle\big) \\ &= 4\beta\langle \mathcal{F}(u)-\mathcal{F}(u), u-v\rangle - 4\beta^2\|\mathcal{F}(u)-\mathcal{F}(v)\|^2 \\ &= 4\beta\big(\langle \mathcal{F}(u)-\mathcal{F}(v), u-v\rangle - \beta\|\mathcal{F}(u)-\mathcal{F}(v)\|^2\big) \\ &\geq 0.\end{aligned}$$

Thus $I - 2\beta\mathcal{F}$ is nonexpansive. □

Lemma 2. *For all $u, v \in \mathcal{H}$, the following inequality holds:*

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle.$$

Lemma 3 ([27])**.** *Suppose $\{a_n\} \subset [0,\infty)$, $\{\gamma_n\} \subset [0,1]$ and $\{b_n\}$ are three real number sequences satisfying $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n b_n$, $\forall\, n \geq 0$. Assume that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} b_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 4 ([25])**.** *Let $\{V_n\}$ be a sequence of nonexpansive mappings of $\mathcal{D}$ into $\mathcal{H}$, where $\mathcal{D}$ is a nonempty subset of a Hilbert space $\mathcal{H}$. Assume that $\{\gamma_n\} \subset [0,1]$ satisfy the condition $\liminf_{n\to\infty} \gamma_n > 0$. Then a sequence $\{W_n\}$ of mappings of $\mathcal{D}$ into $\mathcal{H}$ defined by $W_n = \gamma_n I + (1-\gamma_n)V_n$, is a strongly nonexpansive sequence, where I is the identity mapping on $\mathcal{D}$.*

Lemma 5 ([25])**.** *Let $\{S_n\}$ be a sequence of firmly nonexpansive mappings of $\mathcal{D}$ into $\mathcal{H}$, where $\mathcal{D}$ is a nonempty subset of $\mathcal{H}$. Then $\{S_n\}$ is a strongly nonexpansive sequence. In particular, $\{J_{\lambda_n}^{\mathcal{M}} = (I + \lambda_n\mathcal{M})^{-1}\}$, resolvent of a maximal monotone operator $\mathcal{M}$ is a strongly nonexpansive sequence.*

Lemma 6 ([25])**.** *Let C and D be two nonempty subsets of a Hilbert space $\mathcal{H}$. Let $\{S_n\}$ be a sequence of mappings of C into $\mathcal{H}$ and $\{T_n\}$ a sequence of mappings of D into $\mathcal{H}$. Suppose that both $\{S_n\}$ and $\{T_n\}$ are strongly nonexpansive sequences such that $T_n(D) \subset C$, for each $n \in \mathbb{N}$. Then $\{S_n T_n\}$ is a strongly nonexpansive sequence.*

Lemma 7 ([26])**.** *If $\{T_i : 1 \leq i \leq k\}$ are strongly nonexpansive mappings and $\bigcap_{i=1}^{k}\{\mathrm{Fix}(T_i) : 1 \leq i \leq k\} \neq \emptyset$, then $\bigcap_{i=1}^{k}\{\mathrm{Fix}(T_i) : 1 \leq i \leq k\} = \mathrm{Fix}(T_1 T_2 \cdots T_k)$.*

Lemma 8 ([28])**.** *The composition of finitely many averaged mappings is averaged. That is, if $\{T_i : 1 \leq i \leq k\}$ are averaged mappings, then so is the composition $T_1 T_2 \cdots T_k$. Furthermore, if $\bigcap_{i=1}^{k}\{\mathrm{Fix}(T_i) : 1 \leq i \leq k\} \neq \emptyset$, then $\bigcap_{i=1}^{k}\{\mathrm{Fix}(T_i) : 1 \leq i \leq k\} = \mathrm{Fix}(T_1 T_2 \cdots T_k)$.*

Lemma 9 ([29]). *Let T be a firmly nonexpansive self-mapping on $\mathcal{H}$ with $\mathrm{Fix}(T) \neq \emptyset$. Then, for any $x \in \mathcal{H}$, one has $\langle x - Tx, w - Tx \rangle \leq 0$, for all $w \in \mathrm{Fix}(T)$.*

Lemma 10 ([30]). *Let $\mathcal{D} \subset \mathcal{H}$ be a nonempty closed convex set and $V\colon \mathcal{D} \to \mathcal{D}$ be a nonexpansive mapping. Then $I - V$ is demiclosed at 0, that is, if $\{x_n\} \subseteq \mathcal{D}$ with $x_n \rightharpoonup w$ and $(I - V)x_n \to 0$, then $w \in \mathrm{Fix}(V)$.*

Lemma 11 (The Resolvent Identity; [31]). *For each $\lambda, \mu > 0$,*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x \right).$$

Lemma 12 ([32]). *Let $\{c_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $c_{n_i} < c_{n_i+1}$, for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_q\} \subset \mathbb{N}$ such that $m_q \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $q \in \mathbb{N}$:*

$$c_{m_q} \leq c_{m_q+1}, c_q \leq c_{m_q+1}.$$

In fact,

$$m_q = \max\{j \leq q : c_j < c_{j+1}\}.$$

3. Main Results

Theorem 1. *Let $\mathcal{H}$ be a real Hilbert space. Let $\{T_i\}_{i=1}^m$ and V be nonexpansive self-mappings on $\mathcal{H}$ and $B_1, B_2\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone mappings such that*

$$\Gamma := \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap \mathrm{Fix}(V) \bigcap B_1^{-1}0 \bigcap B_2^{-1}0 \neq \emptyset.$$

Let $g\colon \mathcal{H} \to \mathcal{H}$ be a contraction with coefficient $k \in (0,1)$ and $\{x_n\}$ a sequence defined by $x_0 \in \mathcal{H}$ and

$$\begin{cases} y_n = \alpha_n g(x_n) + (1 - \alpha_n) J_{\mu_n}^{B_2} V_n x_n, \\ x_{n+1} = J_{\rho_n}^{B_1} T_m^n T_{m-1}^n \dots T_2^n T_1^n y_n, \end{cases}, \tag{2}$$

for all $n \geq 0$, where $V_n = (1 - \beta_n)I + \beta_n V$ and $T_i^n = (1 - \gamma_n^i)I + \gamma_n^i T_i$, for $i = 1, 2, \dots, m$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n^i\}$ are sequences in $(0,1)$ and $\{\rho_n\}$ and $\{\mu_n\}$ are sequences of positive real numbers satisfying the following conditions:

1. $\lim\limits_{n\to\infty} \alpha_n = 0, \ \sum\limits_{n=0}^{\infty} \alpha_n = \infty;$
2. $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1;$
3. $0 < \liminf\limits_{n\to\infty} \gamma_n^i \leq \limsup\limits_{n\to\infty} \gamma_n^i < 1$, *for all* $i = 1, 2, \dots, m;$
4. *for all sufficiently large n, $\min\{\rho_n, \mu_n\} > \varepsilon$ for some $\varepsilon > 0$.*

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where x^* is the unique fixed point of the contraction $P_\Gamma g$.

Proof. Set $W_n = J_{\rho_n}^{B_1} T_m^n \cdots T_2^n T_1^n$ and $S_n = J_{\mu_n}^{B_2} V_n$. Clearly, each W_n and S_n are nonexpansive mappings for each $n \geq 0$. By Lemmas 4 and 5, for each $n \geq 0$, W_n and S_n are composition of strongly nonexpansive mappings. Therefore, from Lemma 7, we get

$$\begin{aligned}\emptyset \neq \Gamma &= \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \bigcap \mathrm{Fix}(V) \bigcap B_1^{-1}0 \bigcap B_2^{-1}0 \\ &= \bigcap_{i=1}^{m} \mathrm{Fix}(T_i^n) \bigcap \mathrm{Fix}(V_n) \bigcap \mathrm{Fix}(J_{\rho_n}^{B_1}) \bigcap \mathrm{Fix}(J_{\mu_n}^{B_2}) \\ &= \mathrm{Fix}(W_n) \bigcap \mathrm{Fix}(S_n).\end{aligned}$$

First, we claim that $\{x_n\}$ is bounded. Take an arbitrary element $x^* \in \Gamma$.

$$\begin{aligned}\|x_{n+1} - x^*\| = \|W_n y_n - x^*\| &\leq \|y_n - x^*\| \\ &= \|\alpha_n(g(x_n) - g(x^*)) + \alpha_n(g(x^*) - x^*) + (1-\alpha_n)(S_n x_n - x^*)\| \\ &\leq \alpha_n\|g(x_n) - g(x^*)\| + \alpha_n\|g(x^*) - x^*\| + (1-\alpha_n)\|S_n x_n - x^*\| \\ &\leq \alpha_n k\|x_n - x^*\| + \alpha_n\|g(x^*) - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &\leq (1 - \alpha_n(1-k))\|x_n - x^*\| + \alpha_n\|g(x^*) - x^*\| \\ &\leq \max\left\{\|x_n - x^*\|, \frac{1}{1-k}\|g(x^*) - x^*\|\right\}.\end{aligned}$$

By induction, we have

$$\|x_{n+1} - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{1}{1-k}\|g(x^*) - x^*\|\right\},$$

which proves the boundedness of $\{x_n\}$ and so we have $\{g(x_n)\}$ and $\{y_n\}$. It is well known that fixed point set of nonexpansive mapping is closed and convex and so their intersection. Hence, the metric projection P_Γ is well defined. In addition, since $P_\Gamma g \colon \mathcal{H} \to \mathcal{H}$ is a contraction mapping, there exist $x^* \in \Gamma$ such that $x^* = P_\Gamma g(x^*)$. In order to prove $x_n \to x^*$ as $n \to \infty$, we examine two possible cases:

Case I. Assume that there exists $n_0 \in \mathbb{N}$ such that the real sequence $\{\|x_n - x^*\|\}$ is nonincreasing for all $n \geq n_0$. Since $\{\|x_n - x^*\|\}$ is bounded, $\{\|x_n - x^*\|\}$ is convergent. We first show that $y_n - W_n y_n \to 0$. Using nonexpansivness of W_n and (2), we obtain

$$\begin{aligned}0 &\leq \|y_n - x^*\| - \|W_n y_n - x^*\| \\ &\leq \alpha_n\|g(x_n) - x^*\| + (1-\alpha_n)\|S_n x_n - x^*\| - \|x_{n+1} - x^*\| \\ &\leq \alpha_n\|g(x_n) - x^*\| + \|x_n - x^*\| - \|x_{n+1} - x^*\|,\end{aligned} \tag{3}$$

since $\{g(x_n)\}$ is bounded, $\alpha_n \to 0$ and $\{\|x_n - x^*\|\}$ is convergent, we obtain

$$\|y_n - x^*\| - \|W_n y_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

Also $\{W_n\}$ is strongly nonexpansive sequence so we conclude that

$$y_n - W_n y_n \to 0 \quad \text{as } n \to \infty. \tag{4}$$

We next show that $x_n - S_n x_n \to 0$. From (2), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \alpha_n\|g(x_n) - x^*\| + (1-\alpha_n)\|S_n x_n - x^*\| \\ &\leq \alpha_n\|g(x_n) - x^*\| + \|S_n x_n - x^*\|,\end{aligned} \tag{5}$$

Now, from the nonexpansiveness of S_n and (5), we observe

$$\begin{aligned} 0 &\le \|x_n - x^*\| - \|S_n x_n - x^*\| \\ &\le \|x_n - x^*\| - \|x_{n+1} - x^*\| + \alpha_n \|g(x_n) - x^*\|, \end{aligned} \tag{6}$$

since $\{g(x_n)\}$ is bounded, $\alpha_n \to 0$ and $\{\|x_n - x^*\|\}$ is convergent, we obtain

$$\|x_n - x^*\| - \|S_n x_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

As $\{S_n\}$ is strongly nonexpansive sequence, we have

$$x_n - S_n x_n \to 0 \quad \text{as } n \to \infty. \tag{7}$$

Again from (2), we observe

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \alpha_n \|g(x_n) - x^*\| + (1 - \alpha_n) \|J_{\mu_n}^{B_2} V_n x_n - x^*\| \\ &\le \alpha_n \|g(x_n) - x^*\| + \|V_n x_n - x^*\|. \end{aligned} \tag{8}$$

Using nonexpansiveness of V_n and (8), we observe

$$\begin{aligned} 0 &\le \|x_n - x^*\| - \|V_n x_n - x^*\| \\ &\le \|x_n - x^*\| - \|x_{n+1} - x^*\| + \alpha_n \|g(x_n) - x^*\|, \end{aligned} \tag{9}$$

so that $\|x_n - x^*\| - \|V_n x_n - x^*\| \to 0$ by boundedness of sequence $\{g(x_n)\}$, $\alpha_n \to 0$ and convergent sequence $\{\|x_n - x^*\|\}$. By Lemma 4, $\{V_n\}$ is strongly nonexpansive sequence, so we have

$$x_n - V_n x_n \to 0 \quad \text{as } n \to \infty. \tag{10}$$

Also, notice that $x_n - V_n x_n = \beta_n (x_n - V x_n)$. Condition (ii) together with (10) implies that

$$x_n - V x_n \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Now consider

$$\begin{aligned} \|x_n - J_{\mu_n}^{B_2} x_n\| &\le \|x_n - J_{\mu_n}^{B_2} V_n x_n\| + \|J_{\mu_n}^{B_2} V_n x_n - J_{\mu_n}^{B_2} x_n\| \\ &\le \|x_n - S_n x_n\| + \|V_n x_n - x_n\|, \end{aligned}$$

in view of (7) and (10), we deduce

$$x_n - J_{\mu_n}^{B_2} x_n \to 0 \quad \text{as } n \to \infty. \tag{12}$$

Notice that $y_n - x_n = \alpha_n (g(x_n) - x_n) + (1 - \alpha_n)(S_n x_n - x_n)$. This together with given condition $\alpha_n \to 0$ and (7) implies that

$$y_n - x_n \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Next, we consider

$$\begin{aligned} \|x_n - W_n x_n\| &\le \|x_n - y_n\| + \|y_n - W_n y_n\| + \|W_n y_n - W_n x_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - W_n y_n\|, \end{aligned}$$

it follows from (4) and (13) that

$$x_n - W_n x_n \to 0 \quad \text{as } n \to \infty. \tag{14}$$

On the other hand, we observe

$$\begin{aligned}\|x_{n+1} - x^*\| = \|W_n y_n - x^*\| &\le \|W_n y_n - W_n x_n\| + \|W_n x_n - x^*\| \\ &\le \|y_n - x_n\| + \|W_n x_n - x^*\|. \end{aligned} \tag{15}$$

Using nonexpansiveness of $T_i^n T_{i-1}^n \cdots T_1^n$ for each $i = 1, 2, \dots, m$ and (15), we obtain

$$\begin{aligned} 0 &\le \|x_n - x^*\| - \|T_i^n T_{i-1}^n \cdots T_1^n x_n - x^*\| \\ &\le \|x_n - x^*\| - \|W_n x_n - x^*\| \\ &\le \|x_n - x^*\| - \|x_{n+1} - x^*\| + \|y_n - x_n\|, \end{aligned} \tag{16}$$

in view of the fact that $\{\|x_n - x^*\|\}$ is convergent and using (13), we obtain

$$\|x_n - x^*\| - \|T_i^n T_{i-1}^n \cdots T_1^n x_n - x^*\| \to 0 \quad \text{as } n \to \infty.$$

Also by using Lemma 6, $\{T_i^n T_{i-1}^n \cdots T_1^n\}$ is strongly nonexpansive sequence for each $i = 1, 2, \dots, m$. Therefore, we have

$$x_n - T_i^n T_{i-1}^n \cdots T_1^n x_n \to 0 \quad \text{as } n \to \infty \text{ for each } i = 1, 2, \dots, m. \tag{17}$$

Now consider

$$\begin{aligned}\|x_n - J_{\rho_n}^{B_1} x_n\| &\le \|x_n - J_{\rho_n}^{B_1} T_m^n T_{m-1}^n \cdots T_1^n x_n\| + \|J_{\rho_n}^{B_1} T_m^n T_{m-1}^m \cdots T_1^n x_n - J_{\rho_n}^{B_1} x_n\| \\ &\le \|x_n - W_n x_n\| + \|T_m^n T_{m-1}^n \cdots T_1^n x_n - x_n\|. \end{aligned}$$

This together with (14) and (17) implies that

$$x_n - J_{\rho_n}^{B_1} x_n \to 0 \quad \text{as } n \to \infty. \tag{18}$$

Choose a fixed number s such that $\varepsilon > s > 0$ and using Lemma 11, for all sufficiently large n, we have

$$\begin{aligned}\|x_n - J_s^{B_1} x_n\| &\le \|x_n - J_{\rho_n}^{B_1} x_n\| + \|J_{\rho_n}^{B_1} x_n - J_s^{B_1} x_n\| \\ &= \|x_n - J_{\rho_n}^{B_1} x_n\| + \left\| J_s^{B_1}\left(\frac{s}{\rho_n} x_n + \left(1 - \frac{s}{\rho_n}\right) J_{\rho_n}^{B_1} x_n\right) - J_s^{B_1} x_n \right\| \\ &\le \|x_n - J_{\rho_n}^{B_1} x_n\| + \left\| \frac{s}{\rho_n} x_n + \left(1 - \frac{s}{\rho_n}\right) J_{\rho_n}^{B_1} x_n - x_n \right\| \\ &= \|x_n - J_{\rho_n}^{B_1} x_n\| + \left(1 - \frac{s}{\rho_n}\right) \|J_{\rho_n}^{B_1} x_n - x_n\| \\ &\le 2\|x_n - J_{\rho_n}^{B_1} x_n\|. \end{aligned}$$

Using (18), we obtain

$$x_n - J_s^{B_1} x_n \to 0 \quad \text{as } n \to \infty. \tag{19}$$

Similarly, using (12) and Lemma 11, we can obtain

$$x_n - J_s^{B_2} x_n \to 0 \quad \text{as } n \to \infty. \tag{20}$$

Next, we show that

$$x_n - T_i^n x_n \to 0 \quad \text{as } n \to \infty \text{ for each } i = 1, 2, \dots, m. \tag{21}$$

Clearly, from (17) for $i = 1$, (21) holds. Now for $i = 2, \ldots, m$, we see that

$$\begin{aligned}
\|x_n - T_i^n x_n\| &\le \|x_n - T_i^n T_{i-1}^n \cdots T_1^n x_n\| + \|T_i^n T_{i-1}^n \cdots T_1^n x_n - T_i^n x_n\| \\
&\le \|x_n - T_i^n T_{i-1}^n \cdots T_1^n x_n\| + \|T_{i-1}^n \cdots T_1^n x_n - x_n\|.
\end{aligned}$$

Thus, in view of (17), (21) holds for all $i = 1, 2, \ldots, m$.
Observe that $x_n - T_i^n x_n = \gamma_n^i (x_n - T_i x_n)$. Condition (iii) and (21) implies that

$$x_n - T_i x_n \to 0 \quad \text{as } n \to \infty \text{ for each } i = 1, 2, \ldots, m. \tag{22}$$

Put $U := \frac{1}{m+3}\Big(\sum_{i=1}^{m} T_i + V + J_s^{B_1} + J_s^{B_2}\Big)$. Clearly, U is a convex combination of nonexpansive mappings, so is itself nonexpansive and

$$\mathrm{Fix}(U) = \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \bigcap \mathrm{Fix}(V) \bigcap B_1^{-1}0 \bigcap B_2^{-1}0 = \Gamma.$$

We observe

$$\begin{aligned}
\|x_n - Ux_n\| &= \Big\| x_n - \frac{1}{m+3}\Big(\sum_{i=1}^{m} T_i x_n + Vx_n + J_s^{B_1} x_n + J_s^{B_2} x_n\Big)\Big\| \\
&= \Big\| \frac{1}{m+3}\Big(m x_n - \sum_{i=1}^{m} T_i x_n\Big) + \frac{1}{m+3}(x_n - Vx_n) + \frac{1}{m+3}(x_n - J_s^{B_1} x_n) \\
&\quad + \frac{1}{m+3}(x_n - J_s^{B_2} x_n)\Big\| \\
&\le \frac{1}{m+3}\sum_{i=1}^{m} \|x_n - T_i x_n\| + \frac{1}{m+3}\|x_n - Vx_n\| + \frac{1}{m+3}\|x_n - J_s^{B_1} x_n\| \\
&\quad + \frac{1}{m+3}\|x_n - J_s^{B_2} x_n\|.
\end{aligned}$$

In view of (11), (19), (20) and (22), we obtain

$$x_n - Ux_n \to 0 \quad \text{as } n \to \infty. \tag{23}$$

Observe that

$$\|y_n - Uy_n\| \le \|y_n - x_n\| + \|x_n - Ux_n\| + \|Ux_n - Uy_n\| \le 2\|y_n - x_n\| + \|x_n - Ux_n\|.$$

This together with (13) and (23) implies that

$$y_n - Uy_n \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Since $\{y_n\}$ is bounded, it has a convergent subsequence $\{y_{n_i}\}$ such that $\{y_{n_i}\}$ converges weakly to some $z \in \mathcal{H}$. Further Lemma 10, and (24) implies that $z \in \mathrm{Fix}(U) = \Gamma$, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle g(x^*) - x^*, y_n - x^* \rangle &= \lim_{i\to\infty} \langle g(x^*) - x^*, y_{n_i} - x^* \rangle = \langle g(x^*) - x^*, z - x^* \rangle \\
&= \langle g(x^*) - P_\Gamma g(x^*), z - P_\Gamma g(x^*) \rangle \le 0,
\end{aligned} \tag{25}$$

where the last inequality follows from (1).

Using Lemma 2, we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &= \|\alpha_n(g(x_n) - x^*) + (1-\alpha_n)(S_n x_n - x^*)\|^2 \\ &\le (1-\alpha_n)^2\|S_n x_n - x^*\|^2 + 2\alpha_n\langle g(x_n) - x^*, y_n - x^*\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle g(x_n) - g(x^*), y_n - x^*\rangle + 2\alpha_n\langle g(x^*) - x^*, y_n - x^*\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k\|x_n - x^*\| \cdot \|y_n - x^*\| + E_n \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha_n k[\|x_n - x^*\|^2 + \|y_n - x^*\|^2] + E_n,\end{aligned}$$

where $E_n = 2\alpha_n\langle g(x^*) - x^*, y_n - x^*\rangle$.

It turns out that

$$\begin{aligned}(1-\alpha_n k)\|y_n - x^*\|^2 &\le [(1-\alpha_n)^2 + \alpha_n k]\|x_n - x^*\|^2 + E_n, \\ \|y_n - x^*\|^2 &\le \Big[\frac{(1-\alpha_n)^2 + \alpha_n k}{1-\alpha_n k}\Big]\|x_n - x^*\|^2 + \frac{E_n}{1-\alpha_n k}.\end{aligned}$$

Next, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|y_n - x^*\|^2 \\ &\le \Big[\frac{(1-\alpha_n)^2 + \alpha_n k}{1-\alpha_n k}\Big]\|x_n - x^*\|^2 + \frac{E_n}{1-\alpha_n k} \\ &\le \Big[1 - \frac{2\alpha_n(1-k)}{1-\alpha_n k}\Big]\|x_n - x^*\|^2 + \frac{\alpha_n^2}{1-\alpha_n k}\|x_n - x^*\|^2 + \frac{E_n}{1-\alpha_n k} \\ &= \Big[1 - \frac{2\alpha_n(1-k)}{1-\alpha_n k}\Big]\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n(1-k)}{1-\alpha_n k}\Big[\frac{1}{1-k}\langle g(x^*) - x^*, y_n - x^*\rangle + \frac{\alpha_n}{2(1-k)}\|x_n - x^*\|^2\Big],\end{aligned}$$

that is,

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n b_n, \tag{26}$$

where $a_n = \|x_n - x^*\|^2$, $\gamma_n = \dfrac{2\alpha_n(1-k)}{1-\alpha_n k}$, $b_n = \dfrac{1}{1-k}\langle g(x^*) - x^*, y_n - x^*\rangle + \dfrac{\alpha_n}{2(1-k)}\|x_n - x^*\|^2$. Using (25), the condition $\alpha_n \to 0$ and boundedness of $\{x_n\}$, we obtain $\limsup_{n\to\infty} b_n \le 0$. Using condition (i), it can be easily proven that $\sum_{n=0}^{\infty} \gamma_n = \infty$. Finally, we apply Lemma 3 to (26) to conclude that $x_n \to x^*$ as $n \to \infty$.

Case II. Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|, \quad \forall j \in \mathbb{N}.$$

Then, by Lemma 12, there exists a nondecreasing sequence of integers $\{m_q\} \subset \mathbb{N}$ such that $m_q \to \infty$ as $q \to \infty$ and

$$\|x_{m_q} - x^*\| \le \|x_{m_q+1} - x^*\| \quad \text{and} \quad \|x_q - x^*\| \le \|x_{m_q+1} - x^*\|, \quad \forall q \in \mathbb{N}. \tag{27}$$

Now, using (27) in (3), we have

$$\begin{aligned}0 \le \|y_{m_q} - x^*\| - \|W_{m_q} y_{m_q} - x^*\| &\le \alpha_{m_q}\|g(x_{m_q}) - x^*\| + \|x_{m_q} - x^*\| - \|x_{m_q+1} - x^*\| \\ &\le \alpha_{m_q}\|g(x_{m_q}) - x^*\|,\end{aligned}$$

since $\{g(x_{m_q})\}$ is bounded and $\alpha_{m_q} \to 0$, we obtain $\|y_{m_q} - x^*\| - \|W_{m_q} y_{m_q} - x^*\| \to 0$ as $q \to \infty$.

As $\{W_{m_q}\}$ is a strongly nonexpansive sequence, we have $y_{m_q} - W_{m_q} y_{m_q} \to 0$ as $q \to \infty$.

Similarly, using (27) in (6) and (9), we obtain

$$x_{m_q} - S_{m_q}x_{m_q} \to 0 \quad \text{and} \quad x_{m_q} - V_{m_q}x_{m_q} \to 0 \quad \text{as} \quad q \to \infty,$$

respectively. Arguing as in case I, we obtain

$$\begin{aligned} x_{m_q} - Vx_{m_q} \to 0, \quad & x_{m_q} - J^{B_2}_{\mu_{m_q}} x_{m_q} \to 0, \\ y_{m_q} - x_{m_q} \to 0, \quad & x_{m_q} - W_{m_q}x_{m_q} \to 0 \quad \text{as } q \to \infty. \end{aligned} \tag{28}$$

Using (27) in (16), we have

$$\begin{aligned} 0 &\le \|x_{m_q} - x^*\| - \|T_i^{m_q} T_{i-1}^{m_q} \cdots T_1^{m_q} x_{m_q} - x^*\| \\ &\le \|x_{m_q} - x^*\| - \|x_{m_q+1} - x^*\| + \|y_{m_q} - x_{m_q}\| \\ &\le \|y_{m_q} - x_{m_q}\|, \end{aligned}$$

it follows from (28) that

$$\|x_{m_q} - x^*\| - \|T_i^{m_q} T_{i-1}^{m_q} \cdots T_1^{m_q} x_{m_q} - x^*\| \to 0 \quad \text{as } q \to \infty \quad \text{for each } i = 1, 2 \ldots, m.$$

Following similar arguments as in Case I, we have

$$\begin{aligned} & x_{m_q} - T_i^{m_q} T_{i-1}^{m_q} \cdots T_1^{m_q} x_{m_q} \to 0, \quad x_{m_q} - J_s^{B_1} x_{m_q} \to 0, \quad x_{m_q} - J_s^{B_2} x_{m_q} \to 0, \\ & x_{m_q} - T_i^{m_q} x_{m_q} \to 0, \qquad\qquad\qquad\quad x_{m_q} - T_i x_{m_q} \to 0, \quad\; x_{m_q} - Ux_{m_q} \to 0, \\ & y_{m_q} - Uy_{m_q} \to 0 \quad \text{as } q \to \infty \end{aligned}$$

$$\limsup_{q\to\infty} \langle g(x^*) - x^*, y_{m_q} - x^* \rangle \le 0. \tag{29}$$

Next, from (26), we have

$$a_{m_q+1} \le (1 - \gamma_{m_q})a_{m_q} + \gamma_{m_q} b_{m_q}, \tag{30}$$

where $a_{m_q} = \|x_{m_q} - x^*\|^2$, $b_{m_q} = \dfrac{1}{1-k}\langle g(x^*) - x^*, y_{m_q} - x^* \rangle + \dfrac{\alpha_{m_q}}{2(1-k)}\|x_{m_q} - x^*\|^2$, $\gamma_{m_q} = \dfrac{2\alpha_{m_q}(1-k)}{1-\alpha_{m_q}k}$. Thus, (30) and (27) implies that

$$\begin{aligned} \gamma_{m_q} a_{m_q} &\le a_{m_q} - a_{m_q+1} + \gamma_{m_q} b_{m_q}, \\ \gamma_{m_q} a_{m_q} &\le \gamma_{m_q} b_{m_q}. \end{aligned}$$

Using the fact that $\gamma_{m_q} > 0$, we obtain $a_{m_q} \le b_{m_q}$, that is,

$$\|x_{m_q} - x^*\|^2 \le \frac{1}{1-k}\langle g(x^*) - x^*, y_{m_q} - x^* \rangle + \frac{\alpha_{m_q}}{2(1-k)}\|x_{m_q} - x^*\|^2.$$

Since $\{x_{m_q}\}$ is bounded, $\alpha_{m_q} \to 0$, it follows from (29) that $\|x_{m_q} - x^*\| \to 0$ as $q \to \infty$.

This together with (30) implies that $\|x_{m_q+1} - x^*\| \to 0$ as $q \to \infty$. But $\|x_q - x^*\| \le \|x_{m_q+1} - x^*\|$, for all $q \in \mathbb{N}$, which gives that $x_q \to x^*$ as $q \to \infty$. □

Remark 1. *A similar approach has been adopted in the study of consensus problems (see the seminal work [33]).*

4. Applications

In this section, we utilize the main result presented in this paper to study many problems in Hilbert spaces.

4.1. Application to a General System of Variational Inequalities

Let $\mathcal{H}$ be a real Hilbert space and let there be given for each $i = 1, 2, \ldots, N$, an operator $A_i \colon \mathcal{H} \to \mathcal{H}$ and a nonempty closed convex subset $C_i \subset \mathcal{H}$. First, we introduce the following general system of variational inequalities in Hilbert space, which aims to find $(x_1^*, x_2^*, \ldots, x_N^*) \in C_1 \times C_2 \times \cdots \times C_N$ such that

$$\begin{cases} \langle \theta_1 A_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C_1, \\ \langle \theta_2 A_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C_2, \\ \vdots \\ \langle \theta_{N-1} A_{N-1} x_N^* + x_{N-1}^* - x_N^*, x - x_{N-1}^* \rangle \geq 0, \quad \forall x \in C_{N-1}, \\ \langle \theta_N A_N x_1^* + x_N^* - x_1^*, x - x_N^* \rangle \geq 0, \quad \forall x \in C_N, \end{cases} \tag{31}$$

where $\theta_i > 0$ for all $i \in \{1, 2, \ldots, N\}$. Here, Ω will be used to denote the solution set of (31). In particular, if $N = 2$ and $C_1 = C_2 = C$, then problem (31) can be reduced to finding $(x_1^*, x_2^*) \in C \times C$ such that

$$\begin{cases} \langle \theta_1 A_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \theta_2 A_2 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases}, \tag{32}$$

which was considered and studied by Ceng et al. [34]. In particular, if $A_1 = A_2 = A$, $\theta_1 = \theta_2 = \theta$ and $x_1^* = x_2^* = x^*$, then the problem (32) reduces to the variational inequality problem for finding $x^* \in C$ such that

$$\langle A x^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{33}$$

Variational inequalities produce effective method to solve several important problems appearing in finance, optimization theory, game theory, mechanics and economics.

Another motivation for introducing (31) is that if we choose $x_1^* = x_2^* = \cdots = x_N^* = x^*$ and $\theta_i = 1$ for all $i \in \{1, 2, \ldots, N\}$, then (31) reduces to an important problem, called the common solutions to variational inequality problem (CSVIP) introduced by Censor, Gibali and Reich [16,17].

Lemma 13. *Let $\{C_i\}_{i=1}^N$ be a finite family of closed convex subsets of a real Hilbert space $\mathcal{H}$. Let $A_i \colon \mathcal{H} \to \mathcal{H}$ be nonlinear mappings, where $i = 1, 2, \ldots, N$. For given $x_i^* \in C_i$, $i = 1, 2, \ldots, N$, $(x_1^*, x_2^*, \ldots, x_N^*)$ is a solution of problem (31) if and only if*

$$x_i^* = P_{C_i}(I - \theta_i A_i) x_{i+1}^*, \quad x_N^* = P_{C_N}(I - \theta_N A_N) x_1^*, \quad i = 1, 2, \ldots, N-1.$$

That is

$$x_1^* = P_{C_1}(I - \theta_1 A_1) P_{C_2}(I - \theta_2 A_2) \cdots P_{C_{N-1}}(I - \theta_{N-1} A_{N-1}) P_{C_N}(I - \theta_N A_N) x_1^*.$$

Proof. We can rewrite (31) as

$$\begin{aligned} &\langle x_1^* - (x_2^* - \theta_1 A_1 x_2^*), x - x_1^* \rangle \geq 0, \quad \forall x \in C_1, \\ &\langle x_2^* - (x_3^* - \theta_2 A_2 x_3^*), x - x_2^* \rangle \geq 0, \quad \forall x \in C_2, \\ &\vdots \\ &\langle x_{N-1}^* - (x_N^* - \theta_{N-1} A_{N-1} x_N^*), x - x_{N-1}^* \rangle \geq 0, \quad \forall x \in C_{N-1}, \\ &\langle x_N^* - (x_1^* - \theta_N A_N x_1^*), x - x_N^* \rangle \geq 0, \quad \forall x \in C_N. \end{aligned} \tag{34}$$

From (1), we find (34) is equivalent to

$$x_i^* = P_{C_i}(I - \theta_i A_i) x_{i+1}^*, \quad x_N^* = P_{C_N}(I - \theta_N A_N) x_1^*, \quad i = 1, 2, \ldots, N-1.$$

Therefore, we have

$$x_1^* = P_{C_1}(I - \theta_1 A_1)P_{C_2}(I - \theta_2 A_2) \cdots P_{C_{N-1}}(I - \theta_{N-1}A_{N-1})P_{C_N}(I - \theta_N A_N)x_1^*. \quad \square$$

Lemma 14. *Let $\{C_i\}_{i=1}^N$ be a finite family of closed convex subsets of a real Hilbert space $\mathcal{H}$. Let A_i be η_i-ism self-mappings on $\mathcal{H}$, where $i \in \{1, 2, \dots, N\}$. Let $T\colon \mathcal{H} \to \mathcal{H}$ be a mapping defined by*

$$T(x) = P_{C_1}(I - \theta_1 A_1)P_{C_2}(I - \theta_2 A_2) \cdots P_{C_{N-1}}(I - \theta_{N-1}A_{N-1})P_{C_N}(I - \theta_N A_N)x, \quad \forall x \in H.$$

If $\theta_i \in (0, 2\eta_i)$, $i = 1, 2, \dots, N$, then T is averaged.

Proof. We first prove that $I - \theta_i A_i$ is averaged for each $i \in \{1, 2, \dots, N\}$.

Note that $I - \theta_i A_i = \left(1 - \frac{\theta_i}{2\eta_i}\right)I + \frac{\theta_i}{2\eta_i}(I - 2\eta_i A_i)$ and $\frac{\theta_i}{2\eta_i} \in (0, 1)$. Thus, applying Lemma 1, $I - 2\eta_i A_i$ is nonexpansive and therefore, $I - \theta_i A_i$ is averaged for $\theta_i \in (0, 2\eta_i)$, $i = 1, 2, \dots, N$. Also, it well known that P_{C_i} is averaged, so the composition $P_{C_i}(I - \theta_i A_i)$ (see Lemma 8). Hence again applying Lemma 8, the mapping T is averaged. $\square$

Theorem 2. *Let $\{C_i\}_{i=1}^N$ be a finite family of closed convex subsets of a real Hilbert space $\mathcal{H}$. Let A_i be η_i-ism self-mappings on $\mathcal{H}$, where $i \in \{1, 2, \dots, N\}$. Assume that $\Omega = \mathrm{Fix}(T) \neq \emptyset$, where T is defined in Lemma 14. Let $\{x_n\}$ be a sequence defined by $x_0 \in \mathcal{H}$ and*

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = P_{C_1}(I - \theta_1 A_1)P_{C_2}(I - \theta_2 A_2) \cdots P_{C_{N-1}}(I - \theta_{N-1}A_{N-1})P_{C_N}(I - \theta_N A_N)y_n, \end{cases} \tag{35}$$

where $\theta_i \in (0, 2\eta_i)$. Suppose $\{\alpha_n\} \subset (0, 1)$ satisfying the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$.

Proof. Applying Lemma 14, we have that T is an averaged mapping on $\mathcal{H}$. Therefore, by definition, $T = (1 - \gamma)I + \gamma T_1$, for some $\gamma \in (0, 1)$ and a nonexpansive mapping T_1, where $\mathrm{Fix}(T_1) = \mathrm{Fix}(T)$. Letting $m = 1$, $B_1 = B_2 = g = 0$, $V = I$ and $\gamma_n^1 = \gamma$ in Theorem 1, the conclusion of Theorem 2 is obtained. $\square$

Remark 2. *In [17], Censor, Gibali and Reich proved the weak convergence theorem for solving the CSVIP. If we take $x_1^* = x_2^* = \dots = x_N^* = z$ and $\theta_i = 1$, for all $i \in \{1, 2, ..., N\}$ in (31), then problem (31) reduces to CSVIP and through algorithm (35), we obtain modification of Algorithm 4.1 in [17] and obtain strong convergence, which is often much more desirable than weak convergence.*

4.2. Convex Feasibility Problem

Let C_i, $i = 1, 2, \dots, m$ be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$ with $\bigcap_{i=1}^m C_i \neq \emptyset$, the convex feasibility problem (CFP) is to find x^* such that $x^* \in \bigcap_{i=1}^m C_i$.

Most common methods to solving CFP are the projection and reflection methods which comprise some well-known methods, such as the so-called alternating projection method [35–37], the Douglas–Rachford (DR) algorithm [38–40] and many extensions [41–43]. Most projection and reflection methods can be extended to solve the convex feasibility problem involving any finite number of sets. An exception is the Douglas–Rachford method, for which only the theory of two set feasibility problems has been investigated. Motivated by this fact, Borwein and Tam [43], introduced the following cyclic Douglas–Rachford method which can be applied directly to many-set convex feasibility problem in a Hilbert space.

For any $x_0 \in \mathcal{H}$, the cyclic Douglas–Rachford method defines a sequence $\{x_n\}$ by setting

$$x_{n+1} = T_{[C_1 C_2 \cdots C_m]} x_n, \forall n \in \mathbb{N}.$$

Here, $T_{[C_1 C_2 \cdots C_m]}$ is a m-set cyclic Douglas–Rachford operator defined as

$$T_{[C_1 C_2 \cdots C_m]} = T_{C_m, C_1} T_{C_{m-1}, C_m} \cdots T_{C_2, C_3} T_{C_1, C_2}$$

and each $T_{C_i,C_j} = \dfrac{I + R_{C_j} R_{C_i}}{2}$ is a two set Douglas-Rachford operator and $R_{C_i} = 2P_{C_i} - I$ and $R_{C_j} = 2P_{C_j} - I$ are the reflection operators into C_i and C_j respectively. However, it is known that cyclic Douglas-Rachford method may fail to converge strongly (see [44]). We introduce a modification of cyclic Douglas-Rachford method in which strong convergence is guaranteed.

Theorem 3. *Let $C_1, C_2, \ldots C_m \subseteq \mathcal{H}$ be closed and convex sets with nonempty intersection and let $\{x_n\}$ be a sequence defined by $x_0 \in \mathcal{H}$ and*

$$\begin{cases} y_n = (1 - \alpha_n) x_n \\ x_{n+1} = T_m^n T_{m-1}^n \cdots T_2^n T_1^n y_n, \end{cases},$$

where $T_i^n = (1 - \gamma_n^i) I + \gamma_n^i R_{C_{i+1}} R_{C_i}$, for $i = 1, 2, \ldots, m$ and $C_{m+1} := C_1$. Suppose $\{\alpha_n\}$ and $\{\gamma_n^i\} \subset (0,1)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} = \infty$;

(ii) $0 < \liminf_{n\to\infty} \gamma_n^i \leq \limsup_{n\to\infty} \gamma_n^i < 1$, for all $i = 1, 2, \ldots, m$.

Then the sequence $\{x_n\}$ converges strongly to a point x^* such that $P_{C_i} x^* \in \bigcap_{i=1}^m C_i$ for $i = 1, 2, \ldots, m$.

Proof. Set $T_i = R_{C_{i+1}} R_{C_i}$, for $i = 1, 2, \ldots, m$. By Proposition 4.2, in [24], R_{C_i} and $R_{C_{i+1}}$ are nonexpansive. Therefore, their combination T_i is nonexpansive.

Further $\emptyset \neq \bigcap_{i=1}^m C_i \subseteq \bigcap_{i=1}^m \mathrm{Fix}(T_i)$. Put $B_1 = B_2 = g = 0$ and $V = I$ in Theorem 1, the sequence $\{x_n\}$ converges strongly to a point x^* in $\bigcap_{i=1}^m \mathrm{Fix}(T_i)$. By Corollary 4.3.17 (iii) in [45], $P_{C_i} x^* \in C_i \bigcap C_{i+1}$, for each $i = 1, 2, \ldots, m$. So, $P_{C_i} x^* \in C_{i+1}$ for each $i = 1, 2, \ldots, m$. Further, using inequality (1), we have

$$\begin{aligned} 0 &\geq \sum_{i=1}^m \langle x^* - P_{C_{i+1}} x^*, P_{C_i} x^* - P_{C_{i+1}} x^* \rangle \\ &= \frac{1}{2} \sum_{i=1}^m (\|x^* - P_{C_{i+1}} x^*\|^2 + \|P_{C_i} x^* - P_{C_{i+1}} x^*\|^2 - \|x^* - P_{C_i} x^*\|^2) \\ &= \frac{1}{2} \sum_{i=1}^m \|P_{C_i} x^* - P_{C_{i+1}} x^*\|^2 \geq 0. \end{aligned}$$

Thus, $P_{C_i} x^* = P_{C_{i+1}} x^*$, for each i and therefore, $P_{C_i} x^* \in \bigcap_{i=1}^m C_i$ for each i. $\square$

Remark 3. *By taking $\gamma_n^i = \frac{1}{2}$, for all $i = 1, 2, \ldots, m$ in the operator $T_m^n T_{m-1}^n \cdots T_2^n T_1^n$, we obtain the cyclic Douglas–Rachford operator.*

4.3. Zeros of Ism and Maximal Monotone

Very recently, based on Yamada's hybrid steepest descent method, Tian and Jiang [46] introduced an iterative algorithm and proved a weak convergence theorem for zero points of ism and fixed points of a nonexpansive mapping in Hilbert space. Moreover, using this algorithm, they also constructed following algorithm to obtain weak convergence theorem for common zeros of ism and maximal monotone mapping:

$$\begin{cases} z_n = (1-\lambda_n)x_n + \lambda_n J_r^{B_1} x_n, \\ x_{n+1} = (I - \mu\delta_n F)z_n. \end{cases} \tag{36}$$

Now, we combine hybrid steepest descent method, proximal point algorithm and viscosity approximation method to obtain following strong convergence result.

Theorem 4. *Let $M\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone mapping and F be an θ-ism of $\mathcal{H}$ into itself such that $M^{-1}0 \bigcap F^{-1}0 \neq \emptyset$. Let $g\colon H \to H$ be a contraction with coefficient $k \in (0,1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in \mathcal{H}$ and*

$$\begin{cases} y_n = \alpha_n g(x_n) + (1-\alpha_n)x_n, \\ z_n = (1-\lambda_n)y_n + \lambda_n J_r^M y_n, \qquad \forall n \geq 0. \\ x_{n+1} = (I - \eta\delta_n F)z_n, \end{cases} \tag{37}$$

Suppose that $\{\lambda_n\} \subset (0,2)$, $\{\eta\delta_n\} \subset (0,2\theta)$ and $\{\alpha_n\} \subset (0,1)$ satisfying

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 2$;

(iii) $0 < \liminf\limits_{n\to\infty} \eta\delta_n \leq \limsup\limits_{n\to\infty} \eta\delta_n < 2\theta$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in M^{-1}0 \bigcap F^{-1}0$.

Proof. First, we rewrite $I - \eta\delta_n F$ as

$$I - \eta\delta_n F = \left(1 - \frac{\eta\delta_n}{2\theta}\right)I + \frac{\eta\delta_n}{2\theta}(I - 2\theta F).$$

Using Lemma 1, $I - 2\theta F$ is nonexpansive. Also, it can be easily proven that $\mathrm{Fix}(I - 2\theta F) = F^{-1}0$.
Further, we observe that

$$(1-\lambda_n)I + \lambda_n J_r^M = \left(1 - \frac{\lambda_n}{2}\right)I + \frac{\lambda_n}{2}(2J_r^M - I).$$

By Proposition 4.2, in [24], $2J_r^M - I$ is nonexpansive. Also note that $\mathrm{Fix}(2J_r^M - I) = M^{-1}0$. Now, take $m = 2$, $T_1 = 2J_r^M - I$, $T_2 = I - 2\theta F$, $\gamma_n^1 = \dfrac{\lambda_n}{2}$, $\gamma_n^2 = \dfrac{\eta\delta_n}{2\theta}$, $B_1 = B_2 = 0$ and $V = I$ in Theorem 1, which yields the conclusion of Theorem 4. □

Remark 4. *Theorem 4 improves the Tian and Jiang's result ([46] Theorem 4.4) from weak to strong convergence theorem. Also $\{\lambda_n\}$ is bounded in $(0,1)$ in ([46] Theorem 4.4), but in Theorem 4, we relax $\{\lambda_n\} \subset (0,1)$ to $\{\lambda_n\} \subset (0,2)$.*

Theorem 5. *Let S be an θ-ism of $\mathcal{H}$ into itself and let $B_1, B_2\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone mappings such that $S^{-1}0 \bigcap B_1^{-1}0 \bigcap B_2^{-1}0 \neq \emptyset$. Let $g\colon \mathcal{H} \to \mathcal{H}$ be a contraction with coefficient $k \in (0,1)$ and let $\{x_n\}$ be a sequence defined by $x_0 \in \mathcal{H}$ and*

$$x_{n+1} = J_{\rho_n}^{B_1}\left(\alpha_n g(x_n) + (1-\alpha_n)J_{\mu_n}^{B_2}(x_n - \lambda_n S x_n)\right), \quad \forall n \geq 0. \tag{38}$$

Suppose that $\{\alpha_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,2\theta)$ *and* $\{\rho_n\}, \{\mu_n\} \subset (0,\infty)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 2\theta$;

(iii) *for all sufficiently large* n, $\min\{\rho_n, \mu_n\} > \varepsilon$ *for some* $\varepsilon > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $x^* \in S^{-1}0 \bigcap B_1^{-1}0 \bigcap B_2^{-1}0$.

Proof. First, we rewrite that

$$I - \lambda_n S = \left(1 - \frac{\lambda_n}{2\theta}\right) I + \frac{\lambda_n}{2\theta}(I - 2\theta S).$$

By using Lemma 1, $I - 2\theta S$ is nonexpansive and it can be easily proven that $\mathrm{Fix}(I - 2\theta S) = S^{-1}0$. Putting $V = I - 2\theta S$, $\beta_n = \frac{\lambda_n}{2\theta}$, $T_i = I$, for all $i = 1, 2, \ldots, m$, in Theorem 1, the conclusion of Theorem 5 is obtained. □

Remark 5.

1. *Theorem 5 improves and extends Iiduka–Takahashi's result ([14] Theorem 4.3). By taking* $B_1 = 0$, $B_2 = B$, $\mu_n = r$, $g = x_0$ *in Theorem 5, we obtain ([14] Theorem 4.3) without assuming extra conditions* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *assumed in ([14] Theorem 4.3).*
2. *If we take* $B_1 = S = 0$, $g = x_0$ *in Theorem 5, we obtain Kamimura and Takahashi's result ([13] Theorem 1). Also we remove the superfluous condition* $\lim_{n\to\infty} r_n = \infty$ *assumed in ([13] Theorem 1). Hence our result improves the result of Kamimura and Takahashi.*
3. *The alternating resolvent method studied in Bauschke et al. [47] deals essentially with a special case of the algorithm (38). In fact, if we take* $g = S = 0$, *then (38) becomes*

$$x_{n+1} = J_{\rho_n}^{B_1}\big((1-\alpha_n) J_{\mu_n}^{B_2}(x_n)\big), \quad n \ge 0. \tag{39}$$

We can rewrite (39) as

$$x_{n+1} = J_{\gamma_n}^{A_n} J_{\mu_n}^{B_2} x_n, \quad n \ge 0, \tag{40}$$

where $\gamma_n = \frac{\rho_n}{1-\alpha_n}$ *and* $A_n = B_1 + \frac{\alpha_n}{\rho_n} I$ *is the Tikhonov regularization of* B_1. *Thus Theorem 5 extends and improves the result of Bauschke et al. [47] from weak to strong convergence theorem by using prox-Tikhonov method.*
4. *Theorem 5 also improves the convergence result studied in Lehdili and Moudafi [15]. In fact, if we take* $B_2 = 0$ *in (40), then (40) becomes*

$$x_{n+1} = J_{\gamma_n}^{A_n} x_n, \quad n \ge 0, \tag{41}$$

which is prox-Tikhonov algorithm presented by Lehdili and Moudafi [15].

4.4. Split Common Null Point Problem

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given two set-valued operators $A_1 \colon \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2 \colon \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ and a bounded linear operator $U \colon \mathcal{H}_1 \to \mathcal{H}_2$, the split common null point problem (SCNPP) is the problem of finding

$$\widehat{x} \in \mathcal{H}_1 \quad \text{such that} \quad 0 \in A_1(\widehat{x}) \quad \text{and} \quad 0 \in A_2(U\widehat{x}). \tag{42}$$

In [48], Byrne et al. introduced this problem for finding such a solution $\widehat{x}$ when A_1 and A_2 are maximal monotone.

Using the fact $0 \in A(x)$ if and only if $x \in \mathrm{Fix}(J_\mu^A)$, the problem (42) is equivalent to the problem of finding

$$\widehat{x} \in \mathcal{H}_1 \quad \text{such that} \quad \widehat{x} \in \mathrm{Fix}(J_\mu^{A_1}) \quad \text{and} \quad U\widehat{x} \in \mathrm{Fix}(J_\mu^{A_2}),$$

where $\mu > 0$. Here, Ψ will be used to denote the solution set of (42).

Lemma 15. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $S\colon \mathcal{H}_2 \to \mathcal{H}_2$ be a firmly nonexpansive maping. Then $U^*(I-S)U$ is $1/\|U\|^2$ -ism.*

Proof. Since S is firmly nonexpansive, using Proposition 4.2, in [24], $I - S$ is firmly nonexpansive. Therefore, for all $x, y \in \mathcal{H}_1$, we obtain

$$\begin{aligned}\langle U^*(I-S)Ux - U^*(I-S)Uy, x-y\rangle &= \langle U^*((I-S)Ux - (I-S)Uy), x-y\rangle \\ &= \langle (I-S)Ux - (I-S)Uy, Ux - Uy\rangle \\ &\geq \|(I-S)Ux - (I-S)Uy\|^2.\end{aligned}$$

Also,

$$\begin{aligned}\|U^*(I-S)Ux - U^*(I-S)Uy\|^2 &= \langle U^*((I-S)Ux-(I-S)Uy), U^*((I-S)Ux-(I-S)Uy)\rangle \\ &= \langle (I-S)Ux - (I-S)Uy, UU^*((1-S)Ux - (I-S)Uy)\rangle \\ &\leq \|U\|^2\|(I-S)Ux - (I-S)Uy\|^2.\end{aligned}$$

Combining the above inequalities, we obtain

$$\langle U^*(I-S)Ux - U^*(I-S)Uy, x-y\rangle \geq (1/\|U\|^2)\|U^*(I-S)Ux - U^*(I-S)Uy\|^2.$$

Thus $U^*(I-S)U$ is $1/\|U\|^2$-ism. □

Theorem 6. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $A_1\colon \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $A_2\colon \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two set-valued maximal monotone operators. Let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $g\colon \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction with coefficient $k \in (0,1)$. Let $\Psi \neq \emptyset$ and let $\{x_n\}$ be a sequence defined by $x_0 \in \mathcal{H}_1$ and*

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)J_\mu^{A_1}(I + \lambda_n U^*(J_\mu^{A_2} - I)U)x_n, \quad \forall n \geq 0.$$

Suppose that $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2/\|U\|^2)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2/\|U\|^2.$

Then the sequence $\{x_n\}$ converges strongly to a point in Ψ.

Proof. Let $\widehat{x}$ solves SCNPP i.e. $\widehat{x} \in \Psi$, then we have $\widehat{x} \in \mathcal{H}_1$ such that $0 \in A_1(\widehat{x})$ and $0 \in A_2(U\widehat{x})$. Note that $0 \in A_2(U\widehat{x})$ if and only if $U\widehat{x} \in \mathrm{Fix}(J_\mu^{A_2})$.

Therefore, $(I - J_\mu^{A_2})U\widehat{x} = 0$ and so $U^*(I - J_\mu^{A_2})U\widehat{x} = 0$, means $\widehat{x} \in (U^*(I - J_\mu^{A_2})U)^{-1}0$. Thus $\Psi \subseteq A_1^{-1}0 \bigcap (U^*(I - J_\mu^{A_2})U)^{-1}0$.

Now let $\widehat{x} \in A_1^{-1}0 \bigcap (U^*(I - J_\mu^{A_2})U)^{-1}0$, which implies

$$U^*(I - J_\mu^{A_2})U\widehat{x} = 0. \tag{43}$$

Choose $z \in \Psi$. Therefore, $Uz \in \mathrm{Fix}(J_\mu^{A_2})$. An application of Lemma 9, yields

$$\langle (I - J_\mu^{A_2})U\widehat{x}, Uz - J_\mu^{A_2}U\widehat{x}\rangle \leq 0. \tag{44}$$

Using (43) and (44), we have

$$\begin{aligned}
\|(I - J_\mu^{A_2})U\widehat{x}\|^2 &= \langle (I - J_\mu^{A_2})U\widehat{x}, U\widehat{x} - Uz\rangle + \langle (I - J_\mu^{A_2})U\widehat{x}, Uz - J_\mu^{A_2}U\widehat{x}\rangle \\
&\le \langle (I - J_\mu^{A_2})U\widehat{x}, U\widehat{x} - Uz\rangle \\
&= \langle U^*(I - J_\mu^{A_2})U\widehat{x}, \widehat{x} - z\rangle = 0.
\end{aligned}$$

Therefore, $U\widehat{x} \in \mathrm{Fix}(J_\mu^{A_2})$ i.e., $0 \in A_2(U\widehat{x})$. Thus $\widehat{x} \in \Psi$. Hence $\Psi = A_1^{-1}0 \bigcap (U^*(I - J_\mu^{A_2})U)^{-1}0$. Also, using Lemma 15, $U^*(I - J_\mu^{A_2})U$ is $1/\|U\|^2$-ism.

Now, putting $B_1 = 0$, $B_2 = A_1$, $S = U^*(I - J_\mu^{A_2})U$ and $\mu_n = \mu$ in Theorem 5, the conclusion of Theorem 6 is obtained. □

Remark 6.

1. *Theorem 6 generalizes and improves the result in ([49] Theorem 5.1). Indeed, the result in ([49] Theorem 5.1) considers the special case $\lambda_n = \gamma$, for all n. Moreover, we assume that $\lambda_n \in (0, 2/\|U\|^2)$, while in ([49], Theorem 5.1) γ was assumed to be in $(0, 1/\|U\|^2)$, which is a more restrictive condition.*
2. *If we take $g = x_0$ and $\lambda_n = \gamma$ in Theorem 6, we obtain the result of Byrne et al. ([48] Theorem 4.5).*

4.5. Split Feasibility Problem

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The split feasibility problem (SFP) [23] is defined as finding a point $\widehat{x}$ satisfying:

$$\widehat{x} \in C \quad \text{and} \quad U\widehat{x} \in Q, \tag{45}$$

where $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. In [50], Byrne gave the following algorithm called CQ algorithm for solving the SFP (45):

$$x_{n+1} = P_C(I - \gamma U^*(I - P_Q)U)x_n,$$

where $\gamma \in (0, 2/\|U\|^2)$. Let $h\colon \mathcal{H} \to (-\infty, \infty]$ be a proper lower semicontinuous convex function.

Then subdifferential of h can be defined as

$$\partial h(x) = \{y \in \mathcal{H} : h(x) + \langle z - x, y\rangle \le h(z),\ \forall z \in \mathcal{H}\}, \quad \forall x \in \mathcal{H}.$$

By Rockafellar Theorem [51], ∂h is a maximal monotone operator of $\mathcal{H}$ into itself. For a closed convex subset C of $\mathcal{H}$, the indicator function i_C can be defined as

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Also recall, the normal cone of C at a point $x \in C$ can be defined as

$$N_C(x) = \{y \in \mathcal{H} : \langle y, z - x\rangle \le 0,\ \forall\, z \in C\}.$$

Since $i_C\colon \mathcal{H} \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, ∂i_C is a maximal monotone operator. Also it is known that $\partial i_C = N_C$ (see [24] Ex. 16.12). Using Theorem 1 and the equality

$$(I + r\partial i_C)^{-1} = (I + rN_C)^{-1} = P_C,$$

for all closed convex subset C in $\mathcal{H}$ and for all $r > 0$, we solve the SFP as follows:

Theorem 7. *Let the solution set of SFP (45) is nonempty. Let $g\colon \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction with coefficient $k \in (0,1)$ and let $\{x_n\}$ be a sequence defined by $x_o \in \mathcal{H}_1$ and*

$$x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) P_C(I - \lambda_n U^*(I - P_Q)U)x_n, \quad \forall n \geq 0.$$

Suppose that $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2/\|U\|^2)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2/\|U\|^2.$

Then the sequence $\{x_n\}$ converges strongly to a point in the solution set of SFP (45).

Proof. Put $A_1 = N_C$ and $A_2 = N_Q$ in Theorem 6, which yields the conclusion of Theorem 7. □

Remark 7.

1. *Theorem 7 extends and improves the result in ([52] Corollary 3.7). In fact, in Theorem 7 taking $g = u$ (constant) and $\lambda_n = \gamma$, for all n, we obtain the result in ([52] Corollary 3.7) without assuming an extra condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ which was assumed in ([52] Corollary 3.7).*
2. *Theorem 7 also improves the result in ([53], Theorem 1).*

4.6. Split Monotone Variational Inclusion Problem and Fixed Point Problem for Strictly Pseudocontractive Maps

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $M_1\colon \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $M_2\colon \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two set-valued maximal monotone operators.

Let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and $f_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $f_2\colon \mathcal{H}_2 \to \mathcal{H}_2$ be two ism mappings. The split monotone variational inclusion problem (SMVIP) is to find $\hat{x} \in \mathcal{H}_1$ such that

$$0 \in f_1(\hat{x}) + M_1(\hat{x}) \tag{46}$$

and

$$\hat{y} = U\hat{x} \in \mathcal{H}_2 \quad \text{such that} \quad 0 \in f_2(\hat{y}) + M_2(\hat{y}). \tag{47}$$

Also, it can be easily proven that (see, e.g., Moudafi [54])

$$0 \in f_1(\hat{x}) + M_1(\hat{x}) \Leftrightarrow \hat{x} = J_\lambda^{M_1}(I - \lambda f_1)\hat{x}$$

and

$$0 \in f_2(\hat{y}) + M_2(\hat{x}) \Leftrightarrow \hat{y} = J_\lambda^{M_2}(I - \lambda f_2)\hat{y}.$$

Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. A mapping $S\colon K \to K$ is said to be θ-strictly pseudocontractive if there exist θ with $0 \leq \theta < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x-y\|^2 + \theta\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in K.$$

It can be observed that $I - S$ is $\dfrac{1-\theta}{2}$-ism. In fact, in a Hilbert space, we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \|(x-y) - ((I-S)x - (I-S)y)\|^2 \\ &= \|x-y\|^2 + \|(I-S)x - (I-S)y\|^2 - 2\langle x-y, (I-S)x - (I-S)y\rangle. \end{aligned}$$

Hence, we have

$$\langle x-y,(I-S)x-(I-S)y\rangle \geq \frac{1-\theta}{2}\|(I-S)x-(I-S)y\|^2.$$

Moudafi [54] introduced the SMVIP (46) and (47) and gave an iterative algorithm for solving this problem. Very recently, Shehu and Ogbuisi [55] proposed an iterative algorithm for solving SMVIP which also solves a fixed point problem for strictly pseudocontractive maps in a real Hilbert space.

The following result of Shehu and Ogbuisi [55] is a consequence of our Theorem 1.

Theorem 8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $M_1\colon \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $M_2\colon \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two set-valued maximal monotone operators. Let $U\colon \mathcal{H}_1 \to \mathcal{H}_1$ be a bounded linear operator. Let $f_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ be ν_1-ism and $f_2\colon \mathcal{H}_2 \to \mathcal{H}_2$ be ν_2-ism. Let $S\colon \mathcal{H}_1 \to \mathcal{H}_1$ be a θ-strictly pseudocontractive mapping and $\mathrm{Fix}(S)\bigcap\Lambda \neq \emptyset$, where Λ is a solution set of (46) and (47). Let $\{x_n\}$ be a sequence defined by $x_o \in \mathcal{H}_1$ and*

$$\begin{cases} z_n = (1-\alpha_n)x_n, \\ y_n = J_\lambda^{M_1}(I-\lambda f_1)(z_n + \eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)Uz_n), \qquad \forall n \geq 0, \ , \\ x_{n+1} = (1-\delta_n)y_n + \delta_n S y_n, \end{cases}$$

where $\lambda \in (0, 2\nu)$, $\nu = \min\{\nu_1,\nu_2\}$ and $\eta \in \left(0, \frac{1}{L}\right)$ with L being the spectral radius of the operator U^*U and U^* is the adjoint of U. Suppose $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset (0, 1-\theta)$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1-\theta.$

Then the sequence $\{x_n\}$ converges strongly to point in $\mathrm{Fix}(S)\bigcap\Lambda$.

Proof. With similar arguments as in the proof of Lemma 14, we can easily show that $I-\lambda f_1$ and $I-\lambda f_2$ are averaged mappings on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Further, in view of Lemma 3.3 in [56], $(I+\eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)U)$ is averaged mapping on H_1. Also, applying Lemma 8, the operator $J_\lambda^{M_1}(I-\lambda f_1)$ is averaged. Therefore, the composition R is averaged, where $R := J_\lambda^{M_1}(I-\lambda f_1)(I+\eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)U)$. Thus, by definition, $R = (1-\gamma^1)I + \gamma^1 T_1$ for some $\gamma_1 \in (0,1)$ and a nonexpansive mapping T_1, where $\mathrm{Fix}(T_1) = \mathrm{Fix}(R)$.

Also, we note that

$$(1-\delta_n)I + \delta_n S = (1-\gamma_n^2)I + \gamma_n^2 T_2,$$

where $\gamma_n^2 = \dfrac{\delta_n}{1-\theta}$ and $T_2 = I-(1-\theta)(I-S)$.

Note that $I-S$ is $\dfrac{1-\theta}{2}$-ism. Therefore, using Lemma 1, T_2 is nonexpansive. Also, it can be easily proven that $\mathrm{Fix}(T_2) = \mathrm{Fix}(S)$.

Now let $\hat{x} \in \Lambda$, then we have $\hat{x} \in \mathrm{Fix}(J_\lambda^{M_1}(I-\lambda f_1))$ and $U\hat{x} \in \mathrm{Fix}(J_\lambda^{M_2}(I-\lambda f_2))$.

It is obvious that $U\hat{x} \in \mathrm{Fix}(J_\lambda^{M_2}(I-\lambda f_2))$ implies $\hat{x} \in \mathrm{Fix}(I+\eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)U)$. Therefore, $\hat{x} \in \mathrm{Fix}(J_\lambda^{M_1}(I-\lambda f_1))\bigcap \mathrm{Fix}(I+\eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)U)$. Using Lemma 8, $\hat{x} \in \mathrm{Fix}(R)$. Thus $\Lambda \subseteq \mathrm{Fix}(R)$.

Now let $\hat{x} \in \mathrm{Fix}(R)$. Using Lemma 8, $\hat{x} \in \mathrm{Fix}(J_\lambda^{M_1}(I-\lambda f_1))\bigcap \mathrm{Fix}(I+\eta U^*(J_\lambda^{M_2}(I-\lambda f_2)-I)U)$. It follows from Lemma 3.3 in [57] that

$$\hat{x} \in \mathrm{Fix}(J_\lambda^{M_1}(I-\lambda f_1)) \quad \text{and} \quad U\hat{x} \in \mathrm{Fix}(J_\lambda^{M_2}(I-\lambda f_2)).$$

Therefore, $\hat{x} \in \Lambda$. Hence $\Lambda = \mathrm{Fix}(R)$. Thus $\emptyset \neq \mathrm{Fix}(S)\bigcap\Lambda = \mathrm{Fix}(S)\bigcap\mathrm{Fix}(R) = \mathrm{Fix}(T_2)\bigcap\mathrm{Fix}(T_1)$.

Now taking $B_1 = B_2 = g = 0$, $V = I$, $m = 2$, $\gamma_n^1 = \gamma^1$ and $\gamma_n^2 = \dfrac{\delta_n}{1-\theta}$ in Theorem 1, which yields the desired result. $\square$

4.7. Split Variational Inequality Problem (SVIP)

The SVIP [16] can be formulated as follows:

$$\text{find a point } \hat{x} \in C \text{ such that } \langle f_1(\hat{x}), x - \hat{x}\rangle \geq 0, \text{ for all } x \in C, \tag{48}$$

and such that

$$\hat{y} = U\hat{x} \in Q \text{ solves } \langle f_2(\hat{y}), y - \hat{y}\rangle \geq 0, \text{ for all } y \in Q, \tag{49}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator and $f_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $f_2\colon \mathcal{H}_2 \to \mathcal{H}_2$ are two given operators. If we denote the solution sets of VIPs in (48) and (49) by $\mathrm{SOL}(f_1, C)$ and $\mathrm{SOL}(f_2, Q)$ respectively, then the solution set of SVIP can be written as:

$$\Phi = \{\hat{x} \in \mathrm{SOL}(f_1, C) \text{ such that } U\hat{x} \in \mathrm{SOL}(f_2, Q)\}. \tag{50}$$

As mentioned in [54], if we choose $M_1 = N_C$ and $M_2 = N_Q$ in SMVIP (46) and (47), respectively, then we recover SVIP (48,49), where N_C and N_Q are normal cones of closed and convex sets C and Q respectively.

Theorem 9. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $f_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ be ν_1-ism and $f_2\colon \mathcal{H}_2 \to \mathcal{H}_2$ be ν_2-ism. Assume that $\Phi \neq \emptyset$ and let $\{x_n\}$ be a sequence defined by $x_o \in \mathcal{H}_1$ and*

$$\begin{cases} y_n = (1-\alpha_n)x_n, \\ x_{n+1} = P_C(I - \lambda f_1)(y_n + \eta U^*(P_Q(I-\lambda f_2) - I)Uy_n), \end{cases} \quad \forall n \geq 0,$$

where $\lambda \in (0, 2\nu)$, $\nu = \min\{\nu_1, \nu_2\}$ and $\eta \in \left(0, \dfrac{1}{L}\right)$ with L being the spectral radius of the operator U^*U and U^* is the adjoint of U. Suppose $\{\alpha_n\}$ is a real sequence in $(0,1)$ satisfying the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point in Φ.

Proof. Put $M_1 = N_C$, $M_2 = N_Q$ and $S = I$ in Theorem 8, which yields the desired result. $\square$

Remark 8. *Theorem 9 improves and extends the Censor et al.'s result ([16] Theorem 6.3), where it was assumed that for all $\hat{x} \in \mathrm{SOL}(f_1, C)$,*

$$\langle f_1(x), P_C(I - \lambda f_1)(x) - \hat{x}\rangle \geq 0, \quad \forall x \in \mathcal{H}_1.$$

We drop this assumption in our result. Furthermore, our result extends Censor et al.'s result ([16] Theorem 6.3) from weak to strong convergence.

5. Concluding Remarks

In this article, we present a new iterative algorithm for finding a common point of fixed point sets of nonexpansive mappings and sets of zeros of maximal monotone mappings. Further, we introduced a new general system of variational inequalities which comprises some existing general system of variational inequalities and it is shown that our algorithm converges strongly to a solution of this variational inequality problem. Also, we give modification of cyclic Douglas–Rachford method to solve convex feasibility problem in such a way that strong convergence is guaranteed. In addition, we combine hybrid steepest descent method, proximal point algorithm and viscosity approximation method to obtain a common zero point of maximal monotone and inverse strongly monotone mappings. Further, we improve and extend many results related to different split type problems like split common null point problem, split feasibility problem, split monotone variational inclusion problem and split variational inequality problem. Applicability of our algorithm is not limited to the problems discussed above, it can be further used to solve many important problems, for instance, quasi variational inclusion problem, convex minimization problem, lasso problem, equilibrium problem and many more. Since in this paper, we have worked in a Hilbert space, it should be a natural question for the next research to extend our result in Banach spaces.

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