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Basic Concepts of Riemann–Liouville Fractional Differential Equations with Non-Instantaneous Impulses

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Received: 27 March 2019; Accepted: 22 April 2019; Published: 2 May 2019

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Abstract: In this paper a nonlinear system of Riemann–Liouville (RL) fractional differential equations with non-instantaneous impulses is studied. The presence of non-instantaneous impulses require appropriate definitions of impulsive conditions and initial conditions. In the paper several types of initial value problems are considered and their mild solutions are given via integral representations. In the linear case the equivalence of the solution and mild solutions is established. Conditions for existence and uniqueness of initial value problems are presented. Several examples are provided to illustrate the influence of impulsive functions and the interpretation of impulses in the RL fractional case.

Keywords: Riemann–Liouville fractional derivative; non-instantaneous impulses; initial value problems; integral representation; existence and uniqueness of solution

1. Introduction

Fractional differential equations model nonlocal phenomena in time. One of the basic fractional derivatives is the Riemann–Liouville one which arises naturally in real world phenomena. For example, Heymans and Podlubny [1] provide several examples from the field of viscoelasticity. Several applications of fractional calculus to control theory, electrical circuits, fractional-order multipoles in electromagnetism, electrochemistry, and the neurons in biology are provided in [2–5]. In particular, several applications of fractional derivatives in physics are given in the book [6].

Many physical phenomena have short-term perturbations at some points caused by external interventions during their evolution. Adequate models for this kind of phenomena are impulsive differential equations. Two types of impulses are popular in the literature: instantaneous impulses (whose duration is negligible small) and non-instantaneous impulses (these changes start impulsively and remain active on finite initially given time intervals). There are mainly two approaches for the interpretation of the solutions of impulsive fractional differential equations: one by keeping the lower bound of the fractional derivative at the fixed initial time and the other by switching the lower limit of the fractional derivative at the impulsive points. The statement of the problem depends significantly on the type of fractional derivative. Caputo fractional derivatives have some properties similar to ordinary derivatives (such as the derivative of a constant) which lead to similar initial value problems as well as similar impulsive conditions (instantaneous and non-instantaneous). In the literature many types of initial value problems and boundary value problems for Caputo fractional differential equations with instantaneous and non-instantaneous impulses are studied (see, for example, [7–9]). For Riemann–Liouville (RL) fractional differential equations with instantaneous impulses several results are obtained in [8,10–12]. However for the RL fractional derivative, the physical interpretation of the initial condition (see [1]) requires different impulsive conditions, so as a result the statement of the problem considered is crucial. To the best of our knowledge this is the first paper concerning Riemann–Liouville fractional differential equations with non-instantaneous impulses.

In this paper the basic ideas in introducing non-instantaneous impulses for RL fractional differential equations are presented. The appropriate definition of both the initial conditions and the impulsive conditions for RL fractional differential equations is an important starting point for their qualitative investigation. We set up and discuss several types of initial conditions and impulsive conditions which are deeply connected with the RL fractional derivative. We use both the RL integral and the weighted limit to present the initial condition and the impulsive conditions. We consider both approaches in the literature in the fractional case with the presence of impulses in the equations: when the lower bound of the fractional derivative is fixed at the initial time and when the lower bound of the fractional derivative is changed at any point of impulse. In all cases mild solutions are defined by appropriate Volterra integro–algebraic representations and some conditions for existence and uniqueness are given. Note since instantaneous impulses are a special case of non-instantaneous ones, a brief overview of impulsive conditions in the instantaneous case is given.

2. Some Preliminary Results From Fractional Calculus

Let $t_0 \in \mathbb{R}_+ = [0, \infty)$ be the initial time. Let $L_1^{loc}(J, \mathbb{R}^n)$ be the linear space of all locally Lebesgue integrable functions $m : J \to \mathbb{R}^m,\ J \subset \mathbb{R}$. Let $||.||$ be a norm in $\mathbb{R}^n$.

In this paper we will use the following definitions for fractional derivatives and integrals:

- Riemann–Liouville fractional integral of order $q \in (0,1)$ ([13])

$$ {}_{t_0}I_t^q m(t) = \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} \frac{m(s)}{(t-s)^{1-q}} ds, \quad t \geq t_0, $$

  where $m \in L_1^{loc}([t_0, \infty), \mathbb{R})$ and $\Gamma(.)$ is the Gamma function.

  This is called by some authors the left Riemann–Liouville fractional integral of order q.

  Note sometimes the notation ${}_{t_0}D_t^{-q} m(t) = {}_{t_0}I_t^q m(t)$ is used.
- Riemann–Liouville fractional derivative of order $q \in (0,1)$ ([13])

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{d}{dt}\left({}_{t_0}I_t^{1-q} m(t) \right) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int\limits_{t_0}^{t} (t-s)^{-q} m(s) ds, \quad t \geq t_0 $$

  where $m \in L_1^{loc}([t_0, \infty), \mathbb{R})$.

  This is also called the left Riemann–Liouville fractional derivative.
- Caputo fractional derivative of order $q \in (0,1)$ ([13])

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int\limits_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0 $$

  where $m \in AC^1([t_0, \infty), \mathbb{R})$.

Proposition 1. *([13]) For $q \in (0,1), \beta > 0$ the following hold:*

$$_{t_0}I_t^q (t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+q)}(t-t_0)^{\beta+q-1},$$

$$^{RL}_{t_0}D_t^q (t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-q)}(t-t_0)^{\beta-q-1}.$$

From Proposition 1 we have:

Corollary 1. *([13]) For $q \in (0,1)$:*

$$\begin{aligned} &_{t_0}I_t^q (t-t_0)^{-q} = \Gamma(1-q), \\ &^{RL}_{t_0}D_t^q 1 = \frac{1}{\Gamma(1-q)}(t-t_0)^{-q}, \\ &^{RL}_{t_0}D_t^q (t-t_0)^{q-1} = 0. \end{aligned}$$

Proposition 2. *([13]) For any $m \in AC^1([t_0,\infty),\mathbb{R}),\ m(t_0) \geq 0$ we get*

$$^{C}_{t_0}D_t^q m(t) \leq\ ^{RL}_{t_0}D_t^q m(t).$$

Corollary 2. *([13]) The unique solution of $^{RL}_{0}D_t^q u(t) = 0$ is $u(t) = ct^{q-1}$.*

The definitions of the initial condition of fractional differential equations with RL-derivatives are based on the following result:

Lemma 1. *(Lemma 3.2 [14]). Let $q \in (0,1)$ and $b \in \mathbb{R}:\ t_0 < b,\ m : [t_0,b] \to \mathbb{R}$ be a Lebesgue measurable function.*

(a) If there exists a limit $\lim_{t\to t_0+}[(t-t_0)^{q-1}m(t)] = c \in \mathbb{R}$, then there also exists a limit

$$_{t_0}I_t^{1-q} m(t)|_{t=t_0} := \lim_{t\to t_0+}\ _{t_0}I_t^{1-q} m(t) = c\Gamma(q).$$

(b) If there exists a limit $_{t_0}I_t^{1-q} m(t)|_{t=t_0} = b \in \mathbb{R}$, and if the limit $\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)]$ exists, then

$$\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)] = \frac{b}{\Gamma(q)}.$$

Let $a < b \leq \infty$ be real numbers and consider the nonlinear RL fractional differential equation (RLFrDE)

$$^{RL}_{a}D_t^q x(t) = F(t, x(t)), \quad t \in (a,b]. \tag{1}$$

Note that according to [14] the initial conditions to (1) could be:

- the integral form (see (3.1.6) [14])

$$_{a}I_t^{1-q} x(t)|_{t=a} = B \in \mathbb{R} \tag{2}$$

- a weighted Cauchy type problem (see (3.1.7) [14])

$$\lim_{t\to a}\left((t-a)^{1-q}x(t)\right) = C \in \mathbb{R} \tag{3}$$

- the initial condition at the inner point of a finite interval (see (3.4.71) [14])

$$x(\xi) = B, \quad \xi \in (a,b). \tag{4}$$

Remark 1. *According to Lemma 1 if the function $x(t)$ satisfies the initial condition (3), then, $x(t)$ also satisfies the condition (2) with $B = C\,\Gamma(q)$.*

Any of the above initial value problem (IVP)'s of RLFrDE (1) has an equivalent integral representation.

Lemma 2. *Let $\mathcal{G}$ be an open set in $\mathbb{R}$ and $F(t,x) : (a,b] \times \mathcal{G} \to \mathbb{R}$ be such that $F \in L(a,b)$ for any $x \in \mathcal{G}$. Then*

- *the IVP for RLFrDE (1), (2) is equivalent to (see Corollary 3.1 [14])*

$$x(t) = \frac{B}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t \frac{F(s,x(s))}{(t-s)^{1-q}}ds, \quad t \in (a,b]. \tag{5}$$

- *the weighted IVP for RLFrDE (1), (3) is equivalent to (see Corollary 3.1 [14] and Remark 1)*

$$x(t) = C(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t \frac{F(s,x(s))}{(t-s)^{1-q}}ds, \quad t \in (a,b]. \tag{6}$$

- *the IVP for RLFrDE (1), (4) is equivalent to (see (3.4.27) [14])*

$$x(t) = \Big(\frac{\xi-a}{t-a}\Big)^{1-q}\Big(B - \frac{1}{\Gamma(q)}\int_a^\xi \frac{F(s,x(s))}{(\xi-s)^{1-q}}ds\Big) + \frac{1}{\Gamma(q)}\int_a^t \frac{F(s,x(s))}{(t-s)^{1-q}}ds, \quad t \in (\xi,b]. \tag{7}$$

Lemma 3. *(Corollary 3.12 [14]). Let $\mathcal{G}$ be an open set in $\mathbb{R}$ and let $F : (a,b] \times \mathcal{G} \to \mathbb{R}]$ $F(t,x) \in C_{1-q}[a,b]$ (see Section 3) and for all $t \in (a,b]$ and $x,y \in \mathcal{G}$ the inequality $|F(t,x) - F(t,y)| \le L|x-y|$, $L > 0$ holds.*

Then there exists a unique solution to the integral type initial value problem for RLFrDE (1), (2) in the space $C_{1-q}[a,b]$.

Remark 2. *Note that the global existence result is true for the weighted Cauchy type problem for RLFrDE (1), (3) (see Theorem 3.12 [14]).*

Lemma 4. *([14])*

(i) *If $g \in C(t_0,T]$, then for any point $t \in (t_0,T]$*

$${}^{RL}_{t_0}D^q_t\left({}_{t_0}I^q_t g(t)\right) = g(t).$$

(ii) *If $g \in C(t_0,T]$ and $I^{1-q}_{t_0} g(t) \in C(t_0,T]$, then for any point $t \in (t_0,T]$*

$${}_{t_0}I^q_t\left({}^{RL}_{t_0}D^q_t g(t)\right) = g(t) - \frac{{}_{t_0}I^{1-q}_t g(t)|_{t=t_0}}{\Gamma(q)}(t-t_0)^{q-1}.$$

Remark 3. *In the vector case, the fractional derivatives with the same fractional order are taken for all components.*

3. Non-Instantaneous Impulses in RL Fractional Differential Equations

The appropriate definition of both the initial conditions and the impulsive conditions for RL fractional differential equations is an important starting point. We will discuss the initial value problem for nonlinear Riemann–Liouville fractional differential equations with non-instantaneous impulses (NIRLFrDE). We will set up in an appropriate way both the initial conditions and the impulsive conditions.

Note RL fractional functional differential equations with non-instantaneous impulses were studied in [15] but the impulsive conditions as well as the initial condition do not depend on the fractional order.

Let $t_0, T \in \mathbb{R}_+ :\ t_0 < T < \infty$ and points $\{t_i\}_{i=1}^{p}$ and $\{s_i\}_{i=0}^{p}$ be given such that $0 \le t_0 < s_0 < t_i < s_i < t_{i+1}$, $i = 1, 2, p-1,\ T = s_p$, and $t_0 \in \mathbb{R}_+$ with p a natural number.

Definition 1. *The intervals $(s_{k-1}, t_k]$, $k = 1, 2, \dots, p$ are called intervals of non-instantaneous impulses.*

Remark 4. *If $t_k = s_{k-1}$, $k = 1, 2, \dots, p$, then the intervals of non-instantaneous impulses are reduced to points of instantaneous impulses.*

We consider the following sets:

$$
\begin{aligned}
&C_{1-q}[t_0, T] = \{y : [t_0, T] \to \mathbb{R}^n :\ y(t) \in C[t_0, T] \text{ and there exists } {}_{t_0}I_t^q y(t) \text{ for } t \in [t_0, T]\},\\
&PC[t_0, T] = \Big\{x : [t_0, T] \to \mathbb{R}^n :\ x \in C([t_0, s_0) \bigcup \cup_{k=1}^{p-1}(s_k, s_{k+1}))\\
&\qquad \text{and } x(s_k + 0),\ x(s_k - 0) \text{ exist for } \ k = 0, 1, \dots, p-1\Big\},\\
&PC_{1-q}[t_0, T] = \Big\{y(t) \in PC[t_0, T] : (t - t_0)^{1-q} y(t) \in PC[t_0, T] \ \text{ for } t \in \cup_{k=0}^{p}(t_k, s_k]\Big\},\\
&\mathcal{PC}_{1-q}[t_0, T] = \Big\{y(t) \in \cup_{k=0}^{p} C(t_k, s_k) \bigcup \cup_{k=0}^{p} C(s_k, t_{k+1}) :\\
&\qquad (t - t_k)^{1-q} y(t) \in C(t_k, s_k) \ \text{ for } k = 0, 1, \dots, p\\
&\qquad \text{and } \lim_{t \to t_k+} \left({}_{t_k}I_t^{1-q} y(t) \right) = \lim_{t \to t_k-} y(t),\ k = 1, 2, \dots, p\Big\}
\end{aligned}
$$

with the norms

$$
||y||_{PC_{1-q}[t_0,T]} = \max\{ \max_{k=0,1,\dots,p} \Big(\sup_{t \in (t_k, s_k]} ||(t - t_0)^{1-q} y(t)|| \Big), \max_{k=0,1,\dots,p-1} \Big(\sup_{t \in (s_k, t_{k+1}]} ||y(t)|| \Big) \}
$$

and

$$
||y||_{\mathcal{PC}_{1-q}[t_0,T]} = \max\{ \max_{k=0,1,\dots,p} \Big(\sup_{t \in (t_k, s_k]} ||(t - t_k)^{1-q} y(t)|| \Big), \max_{k=0,1,\dots,p-1} \Big(\sup_{t \in (s_k, t_{k+1}]} ||y(t)|| \Big) \}.
$$

When impulses are involved in fractional differential equations there are two main approaches for interpretation of the solutions. We will consider both approaches.

3.1. Fixed Lower Bound of the RL Fractional Derivative at the Given Initial Time

For our considerations in this case we need the function $f(t, x)$ to be defined for all $t \in [t_0, T],\ x \in \mathbb{R}^n$.

As is mentioned in Section 2 there are two types of initial conditions to RL fractional differential equations: the integral form (2) and the weighted form (3). Following this idea we will define two different types of initial value problems for NIRLFrDE.

3.1.1. Integral Form of the Initial Value Problem

Consider the nonlinear non-instantaneous impulsive Riemann–Liouville fractional differential equation (NIRLFrDE)

$$
{}_{t_0}^{RL}D_t^q x(t) = f(t, x) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \dots, p, \tag{8}
$$

with impulsive conditions

$$x(t) = \phi_k(t, x(t), x(s_{k-1} - 0)) \quad \text{for } t \in (s_{k-1}, t_k],\ k = 1, 2, \ldots, p, \tag{9}$$

and the initial condition in integral form

$${}_{t_0}I_t^{1-q} x(t)|_{t=t_0} = x_0, \tag{10}$$

where $x_0 \in \mathbb{R}^n,\ f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_{k-1}, t_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 1, 2, 3, \ldots, p)$.

The impulses in problem (8), (9) start abruptly at the points $s_k,\ k = 0, 1, 2, \ldots, p-1$ and their action continues on the interval $(s_k, t_{k+1}]$. The function x takes an impulse at $s_k,\ k = 0, 1, 2, \ldots, p-1$ and it follows different rules in the two consecutive intervals $(s_k, t_{k+1}]$ and $(t_{k+1}, s_{k+1}]$. At the point $t_{k+1},\ k = 0, 1, 2, \ldots, p-1$, the function x is continuous.

We will define a mild solution of the initial value problem (8), (9) by properly handling the RL fractional derivative and both the initial condition and the impulsive conditions.

Next we prove some auxiliary results.

Lemma 5. *Let $h : [t_0, T] \to \mathbb{R} :\ h \in C_{1-q}[t_0, T]$, for all $k = 1, 2, \ldots, p$ the functions $\psi_k \in C([t_k, s_k] \times \mathbb{R})$ and $x : [t_0, T] \to \mathbb{R}$ be a function defined by*

$$x(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t - t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} h(s) ds, \quad t \in (t_0, s_0] \\ \psi_k(t, x(s_{k-1} - 0)),\ \ t \in (s_{k-1}, t_k],\ k = 1, 2, 3, \ldots, p \\ \left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \psi_k(t_k, x(s_{k-1} - 0)) \\ \quad -\left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} h(s) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} h(s) ds,\ \ t \in (t_k, s_k],\ k = 1, 2, \ldots, p. \end{cases} \tag{11}$$

Then:

(i) the function x satisfies the linear problem

$$\begin{aligned} &{}_{t_0}^{RL}D_t^q x(t) = h(t) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \ldots, p, \\ &x(t) = \psi_k(t, x(s_{k-1} - 0)) \ \text{ for } t \in (s_{k-1}, t_k],\ k = 1, 2, \ldots, p, \\ &{}_{t_0}I_t^{1-q} x(t)|_{t=t_0} = x_0. \end{aligned} \tag{12}$$

(ii) The function x is continuous at $t_i,\ i = 1, 2, \ldots, p$.

Proof. (i). Let $t \in (t_k, s_k],\ k = 1, 2, \ldots, p$ (the proof in the case $t \in (t_0, s_0]$ is obvious and we omit it). According to Lemma 4i and Proposition 1 the derivative ${}_{t_0}^{RL}D_t^q x(t)$ exists. Apply the operator ${}_{t_0}^{RL}D_t^q$ to both sides of (11), use Proposition 1, Lemma 4i and we obtain

$$\begin{aligned} {}_{t_0}^{RL}D_t^q x(t) = &(t_k - t_0)^{1-q} \psi_k(t_k, x(s_{k-1} - 0))\, {}_{t_0}^{RL}D_t^q (t - t_0)^{q-1} \\ &- \frac{1}{\Gamma(q)} (t_k - t_0)^{1-q} \int_{t_0}^{t_k} (t_k - s)^{q-1} h(s) ds\, {}_{t_0}^{RL}D_t^q (t - t_0)^{q-1} \\ &+ \frac{1}{\Gamma(q)}\, {}_{t_0}^{RL}D_t^q \int_{t_0}^{t} (t - s)^{q-1} h(s) ds = h(t). \end{aligned} \tag{13}$$

Therefore the function $x(t)$ satisfies the first equation in (12).

Applying the operator ${}_{t_0}I_t^{1-q}x(t)$ to the first equation of (11), and using Proposition 1 and the equality ${}_{t_0}I_t^{1-q}\ {}_{t_0}I_t^{q} = {}_{t_0}I_t^{1}$ (see p. 10, (1.10) [16]), we get

$$ {}_{t_0}I_t^{1-q}x(t) = \frac{x_0}{\Gamma(q)}\ {}_{t_0}I_t^{1-q}(t-t_0)^{q-1} + {}_{t_0}I_t^{1-q}\ {}_{t_0}I_t^{q}h(t) = x_0 + \frac{1}{\Gamma(1)}\int_{t_0}^{t} h(s)ds. $$

Therefore, $\lim_{t\to t_0}\ {}_{t_0}I_t^{1-q}x(t) = x_0$ and the function $x(t)$ satisfies the initial condition in (12).
(ii) Let $k = 1, 2, \ldots, p$. Taking the limit in (11), we obtain

$$ \begin{aligned} \lim_{t\to t_k+} x(t) &= \lim_{t\to t_k+} \Big(\frac{t_k-t_0}{t-t_0}\Big)^{1-q}\psi_k(t_k, x(s_{k-1}-0)) \\ &\quad - \lim_{t\to t_k+} \Big(\frac{t_k-t_0}{t-t_0}\Big)^{1-q}\frac{1}{\Gamma(q)}\int_{t_0}^{t_k}(t_k-s)^{q-1}h(s)ds \\ &\quad + \lim_{t\to t_k+}\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)ds \\ &= \psi_k(t_k, x(s_{k-1}-0)) = x(t_k-). \end{aligned} \tag{14} $$

□

Lemma 6. *Let $h \in C_{1-q}[t_0, T]$. If $x \in \mathcal{PC}_q[t_0, T]$ and satisfies (12), then x satisfies (11).*

Proof. Let $t \in (t_0, s_0]$ and ${}_{t_0}^{RL}D_t^q x(t) = h(t)$. Then according to Lemma 4ii we get

$$ {}_{t_0}I_t^{q}h(t) = {}_{t_0}I_t^{q}\Big({}_{t_0}^{RL}D_t^q x(t)\Big) = x(t) - \frac{{}_{t_0}I_t^{1-q}x(t)|_{t=t_0}}{\Gamma(q)}(t-t_0)^{q-1}, $$

i.e.,

$$ x(t) = {}_{t_0}I_t^{q}h(t) + \frac{{}_{t_0}I_t^{1-q}x(t)|_{t=t_0}}{\Gamma(q)}(t-t_0)^{q-1} = \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + {}_{t_0}I_t^{q}h(t), $$

i.e., the function $x(t)$ satisfies the first equation of (11).

Let $t \in (t_1, s_1]$ and ${}_{t_0}^{RL}D_t^q x(t) = h(t)$. According to Lemma 2 with $a = t_0$, $\xi = t_1, b = s_1$, $B = \psi_1(t_1, x(s_0-0))$ we get

$$ x(t) = \Big(\frac{t_1-t_0}{t-t_0}\Big)^{1-q}\Big(B - \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}\frac{h(s)}{(t_1-s)^{1-q}}ds\Big) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}\frac{h(s)}{(t-s)^{1-q}}ds, \tag{15} $$

i.e., the last equality in (11) is satisfied.

Similarly, we can prove that x satisfies (11) for the other subintervals. □

Now, we introduce the concept of a mild solution for IVP for NIRLFrDE (8)–(10).

Definition 2. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (8)–(10) if it satisfies the following Volterra integral–algebraic equation*

$$ x(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s, x(s))ds, \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \ \ t \in (s_{k-1}, t_k],\ k = 1, 2, 3, \ldots, p, \\ \Big(\frac{t_k-t_0}{t-t_0}\Big)^{1-q}\phi_k(t_k, x(t_k), x(s_{k-1}-0)) \\ \quad -\Big(\frac{t_k-t_0}{t-t_0}\Big)^{1-q}\frac{1}{\Gamma(q)}\int_{t_0}^{t_k}(t_k-s)^{q-1}f(s, x(s))ds \\ \quad +\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s, x(s))ds, \ \ t \in (t_k, s_k],\ k = 1, 2, \ldots, p. \end{cases} \tag{16} $$

Remark 5. *Note in formula (16) the function $f(t,x)$ has to be defined on the whole interval $[t_0,T]$.*

Now we will establish existence results for mild solutions to the integral form of IVP for NIRLFrDE (8)–(10) on a finite interval.

Theorem 1. *(Existence and uniqueness). Let the following assumptions be satisfied:*

1. *The function $f(t,x) \in C_{1-q}[t_0,T]$ for any $x \in \mathbb{R}^n$ and the inequality $||f(t,x) - f(t,y)|| \leq L||x-y||$ holds for all $t \in [0,T]$ and $x,y \in \mathbb{R}^n$, where $L > 0$.*
2. *For all $k = 1,2,\dots,p$ the functions $\phi_k(t,x,y) \in C[s_{k-1},t_k]$ for any $x,y \in \mathbb{R}^n$ and the inequality $||\phi_k(t,x_1,y_1) - \phi_k(t,x_2,y_2)|| \leq l_k(||x_1-x_2|| + ||y_1-y_2||)$ holds for $t \in [s_{k-1},t_k]$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}^n$ where $l_k > 0$, $k = 1,2,\dots,p$.*
3. *The inequality $K < 1$ holds where*

$$
\begin{aligned}
K = \max\Bigg\{ & \max_{k=1,2,\dots p} l_k, \frac{2^{1-2q} L\sqrt{\pi}(s_0-t_0)^q}{\Gamma(0.5+q)}, \\
& \left(l_k\left((t_k-t_0)^{1-q} + \left(\frac{t_k-t_0}{s_{k-1}-t_0}\right)^{1-q}\right) + \frac{LP_k}{\Gamma(q)} + \frac{L(s_k-t_k)^q}{\Gamma(q+1)}\left(\frac{s_k-t_0}{t_k-t_0}\right)^{1-q}\right)\Bigg\}, \\
P_k = \max_{t\in[t_k,s_k]} & \Bigg(\frac{(t_k-t_0)^{1-q}}{q} \sum_{i=0,1,\dots,k} \frac{(t_k-t_i)^q - (t_k-s_i)^q}{(t_i-t_0)^{1-q}} \\
& - \frac{(t-t_0)^{1-q}}{q} \sum_{i=0,1,\dots,k} \frac{(t-t_i)^q - (t-s_i)^q}{(t_i-t_0)^{1-q}} \\
& + \frac{(t_k-t_0)^{1-q}}{q} \sum_{i=0,1,\dots,k-1} \Big((t_k-s_i)^q - (t_k-t_{i+1})^q \Big) \\
& - \frac{(t-t_0)^{1-q}}{q} \sum_{i=0,1,\dots,k-1} \Big((t-s_i)^q - (t-t_{i+1})^q \Big)\Bigg), \ k = 1,2,\dots,p.
\end{aligned}
$$

Then there exists a unique mild solution to the integral form of IVP for NIRLFrDE (8)–(10) in the space $PC_{1-q}[t_0,T]$.

We will apply the Banach contraction principle. For any function $x \in PC_{1-q}[t_0,T]$ we define the operator

$$
Tx(t) = \begin{cases}
\frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \quad t \in (t_0,s_0] \\
\phi_k(t,x(t),x(s_{k-1}-0)), \ \ t \in (s_{k-1},t_k], \ k = 1,2,3,\dots,p, \\
\left(\frac{t_k-t_0}{t-t_0}\right)^{1-q} \phi_k(t_k,x(t_k),x(s_{k-1}-0)) \\
\quad -\left(\frac{t_k-t_0}{t-t_0}\right)^{1-q} \frac{1}{\Gamma(q)}\int_{t_0}^{t_k} (t_k-s)^{q-1} f(s,x(s))ds \\
\quad + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \ \ t \in (t_k,s_k], \ k = 1,2,\dots,p.
\end{cases} \tag{17}
$$

From condition 1 it follows that the operator T is well defined.
Step 1. We prove that $Tx(t) \in PC_{1-q}[t_0,T]$ for $x \in PC_{1-q}[t_0,T]$.
From (17) and $\lim_{t\to t_k+0} Tx(t) = \lim_{t\to t_k-0} Tx(t)$ it follows that $Tx(t) \in C(t_0,T]/\cup\{s_k\}$.
Let $t \in (t_0,s_0]$. Then from condition 1 we get

$$
(t-t_0)^{1-q}Tx(t) = \frac{x_0}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^t \left(\frac{t-t_0}{t-s}\right)^{1-q} f(s,x(s))ds \in C(t_0,s_0].
$$

Let $t \in (t_k, s_k]$, $k = 1, 2, \dots, p$. Then from (17) and condition 1 we get

$$(t - t_0)^{1-q} T x(t) \in C[t_k, s_k).$$

Step 2. The operator T is a contraction in $PC_{1-q}[t_0, T]$.
Let $x_1, x_2 \in PC_{1-q}[t_0, T]$. First let $t \in (t_0, s_0]$. Then we obtain

$$\begin{aligned}
&\sup_{t\in[t_0,s_0]} ||(t-t_0)^{1-q} T x_1(t) - (t-t_0)^{1-q} T x_2(t)|| \\
&\le \sup_{t\in[t_0,s_0]} \frac{L}{\Gamma(q)} \int_{t_0}^{t} \Big(\frac{t-t_0}{t-s}\Big)^{1-q} ||x_1(s) - x_2(s))|| ds \\
&= \sup_{t\in[t_0,s_0]} \frac{L}{\Gamma(q)} \int_{t_0}^{t} \Big(\frac{t-t_0}{(t-s)(s-t_0)}\Big)^{1-q} ||(s-t_0)^{1-q} x_1(s) - (s-t_0)^{1-q} x_2(s))|| ds \\
&\le \sup_{t\in[t_0,s_0]} \frac{L}{\Gamma(q)} \int_{t_0}^{t} \Big(\frac{t-t_0}{(t-s)(s-t_0)}\Big)^{1-q} ||(s-t_0)^{1-q} x_1(s) - (s-t_0)^{1-q} x_2(s))|| ds \\
&\le ||x_1 - x_2||_{PC_{1-q}[t_0,T]} \frac{L}{\Gamma(q)} \int_{t_0}^{t} \Big(\frac{t-t_0}{(t-s)(s-t_0)}\Big)^{1-q} ds \\
&= ||x_1 - x_2||_{PC_{1-q}[t_0,T]} \frac{2^{1-2q} L \sqrt{\pi} (t-t_0)^q}{\Gamma(0.5+q)} \le K ||x_1 - x_2||_{PC_{1-q}[t_0,T]}.
\end{aligned} \tag{18}$$

Next let $t \in (s_0, t_1]$. Then

$$\begin{aligned}
\sup_{t\in(s_0,t_1]} ||T x_1(t) - T x_2(t)|| &= \sup_{t\in(s_0,t_1]} ||\phi_1(t, x_1(s_0-0)) - \phi_1(t, x_2(s_0-0))|| \\
&\le l_1 ||x_1 - x_2||_{PC_{1-q}[t_0,T]} \le K ||x_1 - x_2||_{PC_{1-q}[t_0,T]}.
\end{aligned} \tag{19}$$

Continuing this procedure. For example let $t \in (t_k, s_k]$. Then we get

$$\begin{aligned}
&\sup_{t\in(t_k,s_k]} ||(t-t_0)^{1-q} T x_1(t) - (t-t_0)^{1-q} T x_2(t)|| \\
&\le ||(t_k-t_0)^{1-q} \Big(\phi_k(t_k, x_1(t_k), x_1(s_{k-1}-0)) - \phi_k(t_k, x_2(t_k), x_2(s_{k-1}-0))\Big)|| \\
&\quad ||\frac{1}{\Gamma(q)} \int_{t_0}^{t_k} \Big(\Big(\frac{t_k-t_0}{t_k-s}\Big)^{1-q} - \Big(\frac{t-t_0}{t-s}\Big)^{1-q} \Big) (f(s, x_1(s)) - f(s, x_2(s))) ds|| \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} \Big(\frac{t-t_0}{t-s}\Big)^{1-q} ||f(s, x_1(s)) - f(s, x_2(s))|| ds \\
&\le l_k (t_k-t_0)^{1-q} ||x_1(t_k) - x_2(t_k)|| \\
&+ l_k \Big(\frac{t_k-t_0}{s_{k-1}-t_0}\Big)^{1-q} (s_{k-1}-t_0)^{1-q} ||x_1(s_{k-1}-0) - x_2(s_{k-1}-0)|| \\
&\quad + \frac{L}{\Gamma(q)} \int_{t_0}^{t_k} \Big| \Big(\frac{t_k-t_0}{t_k-s}\Big)^{1-q} - \Big(\frac{t-t_0}{t-s}\Big)^{1-q} \Big| ||x_1(s)) - x_2(s)|| ds \\
&\quad + \frac{L}{\Gamma(q)} \int_{t_k}^{t} \Big(\frac{t-t_0}{(t-s)(t_k-t_0)}\Big)^{1-q} ||(s-t_0)^{1-q} x_1(s) - x_2(s))|| ds \\
&\le K ||x_1 - x_2||_{PC_{1-q}[t_0,T]}
\end{aligned} \tag{20}$$

where the inequalities

$$\begin{aligned}
&||x_1(t_k) - x_2(t_k)|| \leq \sup_{t\in(s_{k-1},t_k]} ||x_1(t) - x_2(t)|| \leq ||x_1 - x_2||_{PC_{1-q}[t_0,T]},\\
&||(s_{k-1} - t_0)^{1-q}(x_1(s_{k-1} - 0) - x_2(s_{k-1} - 0))||\\
&\quad \leq \sup_{t\in(t_{k-1},s_{k-1}]} ||(t - t_0)^{1-q}(x_1(t) - x_2(t))|| \leq ||x_1 - x_2||_{PC_{1-q}[t_0,T]},\\
&\int_{s_i}^{t_{i+1}} (t-s)^{q-1} ds = \frac{(t-s_i)^q - (t-t_{i+1})^q}{q},\\
&\int_{s_i}^{t_{i+1}} (t_k-s)^{q-1} ds = \frac{(t_k-s_i)^q - (t_k-t_{i+1})^q}{q},\\
&\int_{t_i}^{s_i} (t-s)^{q-1} ds = \frac{(t-t_i)^q - (t-s_i)^q}{q},\\
&\int_{t_i}^{s_i} (t_k-s)^{q-1} ds = \frac{(t_k-t_i)^q - (t_k-s_i)^q}{q},\\
&\int_{t_k}^{t} (t-s)^{q-1} ds = \frac{(t-t_k)^q}{q},
\end{aligned}$$

$$\int_{t_k}^{t} \Big(\frac{t-t_0}{(t-s)(t_k-t_0)}\Big)^{1-q} ds = \Big(\frac{t-t_0}{t_k-t_0}\Big)^{1-q}\frac{(t-t_k)^q}{q} \leq \Big(\frac{s_k-t_0}{t_k-t_0}\Big)^{1-q}\frac{(s_k-t_k)^q}{q}$$

and

$$\begin{aligned}
&\int_{t_0}^{t_k} \Bigg(\Big(\frac{t_k-t_0}{t_k-s}\Big)^{1-q} - \Big(\frac{t-t_0}{t-s}\Big)^{1-q}\Bigg) ||x_1(s) - x_2(s)|| ds\\
&\leq \sum_{i=0,1,\dots,k} \int_{t_i}^{s_i} \Bigg(\Big(\frac{t_k-t_0}{(t_k-s)(t_i-t_0)}\Big)^{1-q} - \Big(\frac{t-t_0}{(t-s)(t_i-t_0)}\Big)^{1-q}\Bigg)\times\\
&\times ||(s-t_0)^{1-q}(x_1(s) - x_2(s))|| ds\\
&+ \sum_{i=0,1,\dots,k-1} \int_{s_i}^{t_{i+1}} \Bigg(\Big(\frac{t_k-t_0}{t_k-s}\Big)^{1-q} - \Big(\frac{t-t_0}{t-s}\Big)^{1-q}\Bigg) ||x_1(s) - x_2(s))|| ds\\
&\leq \Bigg(\frac{(t_k-t_0)^{1-q}}{q}\Big(\sum_{i=0,1,\dots,k}\frac{(t_k-t_i)^q - (t_k-s_i)^q}{(t_i-t_0)^{1-q}} + \sum_{i=0,1,\dots,k-1}\Big((t_k-s_i)^q - (t_k-t_{i+1})^q\Big)\\
&\;- \Big(\frac{(t-t_0)^{1-q}}{q}\sum_{i=0,1,\dots,k}\frac{(t-t_i)^q - (t-s_i)^q}{(t_i-t_0)^{1-q}} + \sum_{i=0,1,\dots,k-1}\Big((t-s_i)^q - (t-t_{i+1})^q\Big)\Bigg)\times\\
&\times ||x_1 - x_2||_{PC_{1-q}[t_0,T]} \leq P_k ||x_1 - x_2||_{PC_{1-q}[t_0,T]}
\end{aligned}$$

are used.

From the inequalities (18)–(20) it follows that $||Tx_1 - Tx_2||_{PC_{1-q}[t_0,T]} \leq K||x_1 - x_2||_{PC_{1-q}[t_0,T]}$ which proves the Theorem.

3.1.2. Weighted Initial Value Problem

Consider the NIRLFrDE (8) with impulsive conditions (9) and the weighted Cauchy type condition

$$\lim_{t\to t_0}\Big((t-t_0)^{1-q}x(t)\Big) = x_0 \in \mathbb{R}. \tag{21}$$

Then the following auxiliary results hold.

Lemma 7. *Let $h : [t_0, T] \to \mathbb{R} : \ h \in C_{1-q}[t_0, T]$, for all $k = 1, 2, \ldots, p$ the functions $\psi_k \in C([t_k, s_k] \times \mathbb{R})$ and $x : [0, T] \to \mathbb{R}$ be a function defined by*

$$x(t) = \begin{cases} x_0(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s) ds, \quad t \in (t_0, s_0] \\ \psi_k(t, x(s_{k-1}-0)), \ \ t \in (s_{k-1}, t_k], \ k = 1, 2, 3, \ldots, p, \\ \left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \psi_k(t_k, x(s_{k-1}-0)) \\ \quad -\left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} h(s) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s) ds, \ \ t \in (t_k, s_k], \ k = 1, 2, \ldots, p. \end{cases} \tag{22}$$

Then:

(i) *the function x satisfies the linear problem*

$$\begin{aligned} &{}_{t_0}^{RL}D_t^q x(t) = h(t) \text{ for } t \in (t_k, s_k], \ k = 0, 1, \ldots, p, \\ &x(t) = \psi_k(t, x(s_{k-1}-0)) \ \ \text{for } t \in (s_{k-1}, t_k], \ k = 1, 2, \ldots, p, \\ &\lim_{t \to t_0} \Big((t-t_0)^{1-q} x(t) \Big) = x_0. \end{aligned} \tag{23}$$

(ii) *The function x is continuous at t_i, $i = 1, 2, \ldots, p$.*

The proof follows from Lemma 5 and Remark 1 so we omit it.

Lemma 8. *Let $h \in C_{1-q}[t_0, T]$. If $x \in \mathcal{PC}_q[t_0, T]$ and satisfies (23), then x satisfies (22).*

Now, we introduce the concept of a mild solution for IVP for NIRLFrDE (8), (9), (21).

Definition 3. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (8), (9), (21) if it satisfies the following Volterra integral–algebraic equation (compare with (16)):*

$$x(t) = \begin{cases} x_0(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \ \ t \in (s_{k-1}, t_k], \ k = 1, 2, 3, \ldots, p, \\ \left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \phi_k(t_k, x(t_k), x(s_{k-1}-0)) \\ \quad -\left(\frac{t_k - t_0}{t - t_0}\right)^{1-q} \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} f(s, x(s)) ds \\ \quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \ \ t \in (t_k, s_k], \ k = 1, 2, \ldots, p. \end{cases} \tag{24}$$

Remark 6. *The existence and uniqueness Theorem 1 is true for the mild solution to the weighted form of IVP for NIRLFrDE (8), (9), (21) defined by (24).*

3.2. Changed Lower Bounds of the RL Fractional Derivative at the Impulsive Points

Here our function $f(t, x)$ is defined only for $t \in \cup_{k=0}^{p} [t_k, s_k]$ and $x \in \mathbb{R}^n$.

In this case, we will consider the RL fractional derivative with a changeable lower bound at any point of jump, i.e., instead of ${}_{t_0}^{RL}D_t^q$ for all $t > t_0$ we consider ${}_{t_k}^{RL}D_t^q$ on $(t_k, s_k]$, $k = 0, 1, \ldots$. It does not seem to be possible to consider the usual Cauchy conditions at the point t_k to the RLFrDE ${}_{t_k}^{RL}D_t^q x(t) = f(t, x(t))$, because the solution to this problem, in general, has a singularity at t_k and therefore it is not bounded and continuous at the point t_k (see Section 3.4.2 [14]). There are two types of initial conditions to RL fractional differential equations: the integral form (2) and the weighted form (3). Following this idea we will define four different types of initial value problems for NIRLFrDE.

3.2.1. Integral form of the Initial Conditions and Impulses

Consider the IVP for NIRLFrDE

$$ {}_{t_k}^{RL}D_t^q x(t) = f(t,x) \text{ for } t \in (t_k, s_k],\ k = 0,1,\dots,p, \tag{25} $$

with impulsive conditions in the form of RL integrals

$$ \begin{aligned} &x(t) = \phi_k(t, x(t), x(s_{k-1}-0)) \quad \text{for } t \in (s_{k-1}, t_k),\ k = 1,2,\dots,p, \\ &{}_{t_k}I_t^{1-q} x(t)|_{t=t_k} = \phi_k(t_k, x(t_k), x(s_{k-1}-0)),\ k = 1,2,\dots,p, \end{aligned} \tag{26} $$

and initial conditions

$$ {}_{t_0}I_t^{1-q} x(t)|_{t=t_0} = x_0, \tag{27} $$

where $x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^p [t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_{k-1}, t_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 1,2,\dots,p)$.

Lemma 9. *Let $h : \cup_{k=0}^p [t_k, s_k] \to \mathbb{R} :\ h \in \cup_{k=0}^p C_{1-q}[t_k, s_k]$, for all $k = 1,2,\dots,p$ the functions $\psi_k \in C([t_k, s_k] \times \mathbb{R})$ and $x : [t_0, T] \to \mathbb{R}$ be a function defined by*

$$ x(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} h(s) ds, \quad t \in (t_0, s_0] \\ \psi_k(t, x(s_{k-1}-0)),\ \ t \in (s_{k-1}, t_k],\ k = 1,2,3,\dots,p \\ \frac{\psi_k(t_k, x(s_{k-1}-0))}{\Gamma(q)}(t-t_k)^{q-1} \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} h(s) ds,\ \ t \in (t_k, s_k],\ k = 1,2,\dots,p. \end{cases} \tag{28} $$

Then:

(i) the function x satisfies the linear problem

$$ \begin{aligned} &{}_{t_k}^{RL}D_t^q x(t) = h(t) \text{ for } t \in (t_k, s_k],\ k = 0,1,\dots,p, \\ &x(t) = \psi_k(t, x(s_{k-1}-0)) \quad \text{for } t \in (s_{k-1}, t_k),\ k = 1,2,\dots,p, \\ &{}_{t_k}I_t^{1-q} x(t)|_{t=t_k} = \psi_k(t_k, x(s_{k-1}-0)),\ k = 1,2,\dots,p \\ &{}_{t_0}I_t^{1-q} x(t)|_{t=t_0} = x_0. \end{aligned} \tag{29} $$

(ii) The function x satisfies $\lim_{t\to t_k+} \left({}_{t_k}I_t^{1-q} x(t) \right) = \lim_{t\to t_k-} x(t)$ for $k = 1,2,\dots,p$.

Proof. (i). Let $t \in (t_k, s_k)$, $k = 1,2,\dots,p$. According to Lemma 4i and Proposition 1 the derivative ${}_{t_k}^{RL}D_t^q x(t)$ exists. Applying the operator ${}_{t_k}^{RL}D_t^q$ to both sides of the last equality of (28), and useing Corollary 1 and Lemma 4i, we obtain

$$ \begin{aligned} {}_{t_k}^{RL}D_t^q x(t) = &\frac{\psi_k(t_k, x(s_{k-1}-0))}{\Gamma(q)}\, {}_{t_k}^{RL}D_t^q (t-t_k)^{q-1} \\ &+ \frac{1}{\Gamma(q)}\, {}_{t_k}^{RL}D_t^q \int_{t_k}^t (t-s)^{q-1} h(s) ds = h(t). \end{aligned} \tag{30} $$

Therefore the function $x(t)$ satisfies the first equation in (29). A similar argument is needed for $t \in (t_0, s_0]$.

Let $t \in (t_k, s_k),\ k = 1,2,\dots,p$ and apply the operator ${}_{t_k}I_t^{1-q}x(t)$ to both sides of the last equality of (28), use Corollary 1 and the equality ${}_{t_k}I_t^{1-q}\ {}_{t_k}I_t^{q} = {}_{t_k}I_t^{1}$ (see p. 10, (1.10) [16]), to get

$$\begin{aligned} {}_{t_k}I_t^{1-q}x(t) &= \frac{\psi_k(t_k, x(s_{k-1}-0))}{\Gamma(q)}\ {}_{t_k}I_t^{1-q}(t-t_k)^{q-1} + {}_{t_k}I_t^{1-q}\ {}_{t_k}I_t^{q}h(t) \\ &= \psi_k(t_k, x(s_{k-1}-0)) + \frac{1}{\Gamma(1)}\int_{t_k}^{t} h(s)ds. \end{aligned} \tag{31}$$

Therefore, $\lim_{t\to t_k}\left({}_{t_k}I_t^{1-q}x(t)\right) = \psi_k(t_k, x(s_{k-1}-0))$ and the function $x(t)$ satisfies the impulsive condition in (29).

Apply the operator ${}_{t_0}I_t^{1-q}x(t)$ to the first equation of (28), use Corollary 1 and the equality ${}_{t_k}I_t^{1-q}\ {}_{t_k}I_t^{q} = {}_{t_0}I_t^{1}$ (see p. 10, (1.10) [16]) and we get

$${}_{t_k}I_t^{1-q}x(t) = \frac{x_0}{\Gamma(q)}\ {}_{t_k}I_t^{1-q}(t-t_k)^{q-1} + {}_{t_k}I_t^{1-q}\ {}_{t_0}I_t^{q}h(t) = x_0 + \frac{1}{\Gamma(1)}\int_{t_0}^{t} h(s)ds.$$

Therefore, $\lim_{t\to t_0}\ {}_{t_0}I_t^{1-q}x(t) = x_0$ and the function $x(t)$ satisfies the initial condition in (29).

(ii) Let $k = 1,2,\dots,p$. According to (31) we obtain

$$\lim_{t\to t_k+} x(t) = \lim_{t\to t_k+}\left({}_{t_k}I_t^{1-q}x(t)\right) = \psi_k(t_k, x(s_{k-1}-0) = \lim_{t\to t_k-}\psi_k(t, x(s_{k-1}-0)) = \lim_{t\to t_k-} x(t).$$

□

Lemma 10. *Let $h \in \cup_{k=0}^{p} C_{1-q}[t_k, s_k]$. If $x \in \mathcal{PC}_q[t_0, T]$ and satisfies (29), then x satisfies (28).*

Proof. Let $t \in (t_0, s_0]$ and ${}_{t_0}^{RL}D_t^q x(t) = h(t)$. Then according to Lemma 4ii we get

$${}_{t_0}I_t^{q}h(t) = {}_{t_0}I_t^{q}\left({}_{t_0}^{RL}D_t^q x(t)\right) = x(t) - \frac{{}_{t_0}I_t^{1-q}x(t)|_{t=t_0}}{\Gamma(q)}(t-t_0)^{q-1},$$

i.e.,

$$x(t) = {}_{t_0}I_t^{q}h(t) + \frac{{}_{t_0}I_t^{1-q}x(t)|_{t=t_0}}{\Gamma(q)}(t-t_0)^{q-1} = \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + {}_{t_0}I_t^{q}h(t),$$

i.e., the function $x(t)$ satisfies the first equation of (28).

Let $t \in (t_1, s_1]$ and ${}_{t_1}^{RL}D_t^q x(t) = h(t)$. According to Lemma 2 and (6) with $a = t_k,\ b = s_k,\ B = \psi_1(t_1, x(s_0-0))$ we get

$$x(t) = \frac{\psi_1(t_1, x(s_0-0))}{\Gamma(q)}(t-t_1)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_1}^{t}\frac{f(s, x(s))}{(t-s)^{1-q}}ds, \tag{32}$$

i.e., the last equality in (28) is satisfied for $k = 1$.

Similarly, we can prove that x satisfies (28) for the other subintervals. □

Now, we define a mild solution for IVP for NIRLFrDE (25)–(27) following the idea with the presence of impulses in differential equations ([17]) and Lemma 2 with $t_0 = t_k, k = 1,2,\dots$ and $b = \phi_k(t_k, x(t_k), x(s_{k-1}-0))$.

Definition 4. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (25)–(27) if it satisfies the following Volterra integral–algebraic equations:*

$$x(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \ \ t \in (s_{k-1}, t_k], \ k = 1,2,3,\ldots,p, \\ \frac{\phi_k(t_k, x(t_k), x(s_{k-1}-0))}{\Gamma(q)}(t-t_k)^{q-1} \\ \quad + \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1} f(s,x(s))ds, \ \ t \in (t_k, s_k], \ k = 1,2,\ldots,p. \end{cases} \tag{33}$$

We give conditions for the existence of a mild solution to the integral type IVP for NIRLFrDE (25)–(27).

Theorem 2. *(Existence and uniqueness). Let the following assumptions be satisfied:*

1. *The function $f(t,x) : \cup_{k=0}^p [t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t,x) \in \cup_{k=0}^p C_{1-q}[t_k, s_k]$ for any $x \in \mathbb{R}^n$ and the inequality $||f(t,x) - f(t,y)|| \le L||x-y||$ holds for all $t \in [t_k, s_k], k = 0,1,\ldots,p$ and $x, y \in \mathbb{R}^n$, where $L > 0$.*
2. *For all $k = 1,2,\ldots,p$ the functions $\phi_k(t,x,y) \in C[s_{k-1}, t_k]$ for any $x, y \in \mathbb{R}^n$ and the inequality $||\phi_k(t,x_1,y_1) - \phi_k(t,x_2,y_2)|| \le l_k(||x_1-x_2|| + ||y_1-y_2||)$ holds for $t \in [s_{k-1}, t_k]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ where $l_k > 0$, $k = 1,2,\ldots,p$.*
3. *The inequality $K < 1$ holds where*

$$K = \max\Big\{ \max_{k=1,2,\ldots p} l_k, \frac{2^{1-2q} L\sqrt{\pi}(s_0-t_0)^q}{\Gamma(0.5+q)}, \\ \frac{l_k}{\Gamma(q)} + \frac{l_k}{\Gamma(q)(s_{k-1}-t_{k-1})^{1-q}} + \frac{L 2^{1-2q}\sqrt{\pi}(t-t_k)^q}{\Gamma(0.5+q)} \Big\}.$$

Then there exists a unique mild solution to the integral form of IVP for NIRLFrDE (8)–(10) in the space $\mathcal{PC}_{1-q}[t_0, T]$.

We will apply the Banach contraction principle. For any function $x \in \mathcal{PC}_{1-q}[t_0, T]$ we define the operator

$$Tx(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + {}_{t_0}I_t^q f(t, x(t)), \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \ \ t \in (s_{k-1}, t_k], \ k = 1,2,3,\ldots,p, \\ \frac{\phi_k(t_k, x(t_k), x(s_{k-1}-0))}{\Gamma(q)}(t-t_k)^{q-1} + {}_{t_k}I_t^q f(t, x(t)), \\ \quad t \in (t_k, s_k], \ k = 1,2,\ldots,p. \end{cases} \tag{34}$$

From condition 1 it follows that the operator T is well defined.
Step 1. We prove that $Tx(t) \in \mathcal{PC}_{1-q}[t_0, T]$ for $x \in \mathcal{PC}_{1-q}[t_0, T]$.
From (34) it follows that $Tx(t) \in \cup_{k=0}^p C(t_k, s_k) \bigcup \cup_{k=0}^p C(s_k, t_{k+1})$.
Let $t \in (t_0, s_0]$. Then from condition 1 we get

$$(t-t_0)^{1-q} Tx(t) = \frac{x_0}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^t \Big(\frac{t-t_0}{t-s}\Big)^{1-q} f(s,x(s))ds \in C(t_0, s_0].$$

Let $t \in (t_k, s_k]$, $k = 1,2,\ldots,p$. Then from (34) and condition 1 we get

$$(t-t_k)^{1-q} Tx(t) \in C(t_k, s_k).$$

From (34), condition 1, Corollary 1 and ${}_{t_k}I_t^{1-q}\ {}_{t_k}I_t^{q} = {}_{t_k}I_t^{1}$ we get

$$\begin{aligned}{}_{t_k}I_t^{1-q}Tx(t) &= \frac{\phi_k(t_k, x(t_k), x(s_{k-1}-0))}{\Gamma(q)}\ {}_{t_k}I_t^{1-q}(t-t_k)^{q-1} \\ &\quad + {}_{t_k}I_t^{1-q}\Big({}_{t_k}I^{q} f(t, x(t))\Big) \\ &= \phi_k(t_k, x(t_k), x(s_{k-1}-0) + {}_{t_k}I_t^{1} f(t, x(t)).\end{aligned}$$

Therefore,

$$\lim_{t\to t_k+}\Big({}_{t_k}I_t^{1-q}x(t)\Big) = \lim_{t\to t_k-}\phi_k(t, x(t), x(s_{k-1}-0) = \lim_{t\to t_k-} x(t).$$

Step 2. The operator T is a contraction in $\mathcal{PC}_{1-q}[t_0, T]$.
Let $x_1, x_2 \in \mathcal{PC}_{1-q}[t_0, T]$. First let $t \in (t_0, s_0]$. Then similar to (18) we get

$$\begin{aligned}&\sup_{t\in[t_0,s_0]} ||(t-t_0)^{1-q}Tx_1(t) - (t-t_0)^{1-q}Tx_2(t)|| \\ &\le ||x_1 - x_2||_{\mathcal{PC}_{1-q}[t_0,T]}\frac{2^{1-2q}L\sqrt{\pi}(t-t_0)^q}{\Gamma(0.5+q)} \le K||x_1-x_2||_{\mathcal{PC}_{1-q}[t_0,T]}\end{aligned} \tag{35}$$

where the equality

$$\int_a^t \Big(\frac{1}{(t-s)(s-a)}\Big)^{1-q} ds = \frac{2^{1-2q}\sqrt{\pi}(t-a)^{2q-1}\Gamma(q)}{\Gamma(0.5+q)} \tag{36}$$

is applied. Next let $t \in (s_0, t_1]$. As in (19) we get

$$\sup_{t\in(s_0,t_1]} ||Tx_1(t) - Tx_2(t)|| \le l_1||x_1-x_2||_{\mathcal{PC}_{1-q}[t_0,T]} \le K||x_1-x_2||_{\mathcal{PC}_{1-q}[t_0,T]}. \tag{37}$$

Continuing this procedure. For example let $t \in (t_k, s_k]$. Then from (34) we get

$$\begin{aligned}&\sup_{t\in(t_k,s_k]} ||(t-t_k)^{1-q}Tx_1(t) - (t-t_k)^{1-q}Tx_2(t)|| \\ &\le ||\frac{\phi_k(t_k, x_1(t_k), x_1(s_{k-1}-0)) - \phi_k(t_k, x_2(t_k), x_2(s_{k-1}-0))}{\Gamma(q)}|| \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_k}^t \Big(\frac{t-t_k}{t-s}\Big)^{1-q}||f(s, x_1(s)) - f(s, x_2(s))||ds \\ &\le \frac{l_k}{\Gamma(q)}||x_1(t_k) - x_2(t_k)|| \\ &+ \frac{l_k}{\Gamma(q)(s_{k-1}-t_{k-1})^{1-q}}(s_{k-1}-t_{k-1})^{1-q}||x_1(s_{k-1}-0) - x_2(s_{k-1}-0)|| \\ &\quad + \frac{L}{\Gamma(q)}\int_{t_k}^t \Big(\frac{t-t_k}{(t-s)(s-t_k)}\Big)^{1-q}||(s-t_k)^{1-q}(x_1(s) - x_2(s))||ds \\ &\le \Big(\frac{l_k}{\Gamma(q)} + \frac{l_k}{\Gamma(q)(s_{k-1}-t_{k-1})^{1-q}} + \frac{L2^{1-2q}\sqrt{\pi}(t-t_k)^q}{\Gamma(0.5+q)}\Big)||x_1-x_2||_{\mathcal{PC}_{1-q}[t_0,T]} \\ &\le K||x_1-x_2||_{\mathcal{PC}_{1-q}[t_0,T]}\end{aligned} \tag{38}$$

where the inequality (36) with $a = t_k$ and

$$||x_1(t_k) - x_2(t_k)|| \leq \sup_{t\in(s_{k-1},t_k]} ||x_1(t) - x_2(t)|| \leq ||x_1 - x_2||_{\mathcal{PC}_{1-q}[t_0,T]},$$
$$||(s_{k-1} - t_0)^{1-q}(x_1(s_{k-1} - 0) - x_2(s_{k-1} - 0))||$$
$$\leq \sup_{t\in(t_{k-1},s_{k-1}]} ||(t - t_0)^{1-q}(x_1(t) - x_2(t))|| \leq ||x_1 - x_2||_{\mathcal{PC}_{1-q}[t_0,T]}$$

are used.

From the inequalities (35), (37), (38) it follows that $||Tx_1 - Tx_2||_{PC_{1-q}[t_0,T]} \leq K||x_1 - x_2||_{PC_{1-q}[t_0,T]}$ which proves the Theorem.

3.2.2. Weighted Form of the Initial Conditions and Impulses

Consider the NIRLFrDE (25) with impulsive conditions in weighted form

$$\begin{aligned} & x(t) = \phi_k(t, x(t), x(s_{k-1} - 0)) \quad \text{for } t \in (s_{k-1}, t_k),\ k = 1, 2, \ldots, p, \\ & \lim_{t\to t_k} \Big((t - t_k)^{1-q} x(t)\Big) = \phi_k(t_k, x(t_k), x(s_{k-1} - 0)),\ k = 1, 2, \ldots, p \end{aligned} \tag{39}$$

and initial conditions in the weighted form:

$$\lim_{t\to t_0} \Big((t - t_0)^{1-q} x(t)\Big) = x_0. \tag{40}$$

Applying Lemmas 2, 9 and 10 and (6) we could define a mild solution of IVP for NIRLFrDE (25), (39), (40):

Definition 5. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (25), (39), (40) if it satisfies the following Volterra integral–algebraic equations:*

$$x(t) = \begin{cases} x_0(t - t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s))ds, & t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1} - 0)),\ t \in (s_{k-1}, t_k],\ k = 1, 2, 3, \ldots, p, \\ \phi_k(t_k, x(t_k), x(s_{k-1} - 0))(t - t_k)^{q-1} \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s))ds,\ t \in (t_k, s_k],\ k = 1, 2, \ldots, p. \end{cases} \tag{41}$$

Note Theorem 2 for existence and uniqueness could be easily converted to the mild solution of the IVP for NIRLFrDE (25), (39), (40) given by (41).

3.2.3. Mixed Forms of the Initial Conditions and Impulses

We have the following cases:

Case 1. Consider the IVP for NIRLFrDE (25) with mixed impulsive and initial conditions of the form:

$$\begin{aligned} & x(t) = \phi_k(t, x(t), x(s_{k-1} - 0)) \quad \text{for } t \in (s_{k-1}, t_k),\ k = 1, 2, \ldots, p, \\ & {}_{t_k}I_t^{1-q} x(t)|_{t=t_k} = \phi_k(t_k, x(t_k), x(s_{k-1} - 0)),\ k = 1, 2, \ldots, p \end{aligned} \tag{42}$$

and initial conditions in the weighted form:

$$\lim_{t\to t_0} \Big((t - t_0)^{1-q} x(t)\Big) = x_0, \tag{43}$$

Then the mild solution to the IVP (25), (42), (43) will be:

Definition 6. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (25), (42), (43) if it satisfies the following Volterra integral–algebraic equations:*

$$x(t) = \begin{cases} x_0(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \; t \in (s_{k-1}, t_k], \; k = 1, 2, 3, \dots, p, \\ \frac{\phi_k(t_k, x(t_k), x(s_{k-1}-0))}{\Gamma(q)} (t-t_k)^{q-1} \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s)) ds, \; t \in (t_k, s_k], \; k = 1, 2, \dots, p. \end{cases} \tag{44}$$

Case 2. Consider the IVP for NIRLFrDE (25) with impulsive conditions in the weighted form:

$$\begin{aligned} & x(t) = \phi_k(t, x(t), x(s_{k-1}-0)) \quad \text{for } t \in (s_{k-1}, t_k), \; k = 1, 2, \dots, p, \\ & \lim_{t \to t_k} \Big((t-t_k)^{1-q} x(t) \Big) = \phi_k(t_k, x(t_k), x(s_{k-1}-0)), \; k = 1, 2, \dots, p \end{aligned} \tag{45}$$

and initial conditions in integral form:

$$_{t_0}I_t^{1-q} x(t)|_{t=t_0} = x_0. \tag{46}$$

Then the mild solution to the IVP (25), (45), (46) will be:

Definition 7. *A function $x : [t_0, T] \to \mathbb{R}^n$ is called a mild solution of the IVP for NIRLFrDE (25), (45), (46) if it satisfies the following Volterra integral–algebraic equations:*

$$x(t) = \begin{cases} \frac{x_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \quad t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_{k-1}-0)), \; t \in (s_{k-1}, t_k], \; k = 1, 2, 3, \dots, p, \\ \phi_k(t_k, x(t_k), x(s_{k-1}-0))(t-t_k)^{q-1} \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s)) ds, \; t \in (t_k, s_k], \; k = 1, 2, \dots, p. \end{cases} \tag{47}$$

Note Theorem 2 could be easily converted to mild solutions of the IVP for NIRLFrDE with mixed initial and impulsive conditions.

3.3. Examples

To illustrate the application of the above formulas for the mild solutions of non-instantaneous impulsive RL fractional differential equations and the types of impulsive functions we consider the following examples in the scalar case.

Example 1. Let $t_k = 2k, s_k = 2k+1, \; k = 0, 1, 2, \dots, p, \; T = 2p+1, \; p$ is a natural number, and $f(t, x) = \frac{1}{t-1.5}$ for $t \in [0, T]$.

Consider the IVP for the scalar NIRLFrDE (8)–(10) with $t_0 = 0$. The function $f(t, x)$ is defined on the whole interval $[0, T]$. However (16) cannot be applied because the integral $\int_0^t (t-s)^{q-1} \frac{1}{s-1.5} ds$ is not convergent for $t > 1.5$.

Now, consider the IVP for the scalar NIRLFrDE (25), (27). The application of formula (33) for the solution causes no problem since we use the integral $\int_{2k}^t (t-s)^{q-1} \frac{1}{s-1.5} ds$ for $t \in (2k, 2k+1], \; k = 0, 1, 2, \dots, p$ which is convergent. Example 2. Consider the partial case of (8)–(10) with $n = 1, t_0 = 0, q = 0.8$ and $f(t, x) \equiv 1$ for $t \in [0, T]$.

We will consider several types of impulsive functions $\phi_k(t, x, y)$.

Case 1. Let $\phi_k(t, x, y) = g_k(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \dots, p)$.

Case 1.1. Consider (8)–(10) and (16) to obtain the solution:

$$x(t) = \begin{cases} x_0 \frac{t^{-0.2}}{\Gamma(0.8)} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (0, s_0] \\ g_k(t) & t \in (s_{k-1}, t_k],\ k = 1,2,3,\ldots,p \\ \left(\frac{t_k}{t}\right)^{0.2} g_k(t_k) - \frac{1.25t_k}{\Gamma(0.8)t^{0.2}} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (t_k, s_k],\ k = 1,2,\ldots,p. \end{cases}$$

The solution depends on the initial value x_0 only on the interval $(0, s_0]$. Therefore, two solutions with different initial values $x_0 \neq \tilde{x_0}$ will coincide for all $t > s_0$.

Case 1.2. Consider (25), (27) and apply (33) to obtain the solution:

$$x(t) = \begin{cases} x_0 \frac{t^{-0.2}}{\Gamma(0.8)} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (0, s_0] \\ g_k(t) & t \in (s_{k-1}, t_k],\ k = 1,2,3,\ldots,p \\ \frac{g_k(t_k)}{\Gamma(0.8)}(t - t_k)^{-0.2} + \frac{1.25(t-t_k)^{0.8}}{\Gamma(0.8)} & t \in (t_k, s_k],\ \ k = 1,2,\ldots,p. \end{cases}$$

Similarly to Case 1.1 we obtain that two solutions with different initial values $x_0 \neq \tilde{x_0}$ coincide for all $t > s_0$.

Case 2. Let $\phi_k(t, x, y) = a_k x + b_k$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$ where a_k, b_k are constants.

- If $a_k = 1,\ b_k = 0$, then the impulsive condition (9) is reduced to the condition $x(t) = x(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$ and obviously the IVP for NIRLFrDE (8)–(10), respectively (25), (27), will have an infinite number of solutions.
- If $a_k = 1,\ b_k \neq 0$, then the impulsive condition (9) is reduced to the condition $x(t) = x(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$ which has no solution and the IVP for NIRLFrDE (8)–(10), respectively (25), (27), will have no solution.
- If $a_k \neq 1,\ b_k = 0$, then the impulsive condition (9) is reduced to the condition $x(t) = ax(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$ which has only the zero solution, and therefore any solution of IVP for NIRLFrDE (8)–(10), respectively (25), (27) will be zero on $(s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$. In this case from (16) we obtain the solution for (8)–(10):

$$x(t) = \begin{cases} x_0 \frac{t^{-0.2}}{\Gamma(0.8)} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (0, s_0] \\ 0 & t \in (s_{k-1}, t_k],\ k = 1,2,3,\ldots,p, \\ -\frac{1.25t_k}{\Gamma(0.8)t^{0.2}} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (t_k, s_k],\ k = 1,2,\ldots,p. \end{cases}$$

and from (33) we obtain the solution for (25), (27):

$$x(t) = \begin{cases} x_0 \frac{t^{-0.2}}{\Gamma(0.8)} + \frac{1.25t^{0.8}}{\Gamma(0.8)} & t \in (0, s_0] \\ 0 & t \in (s_{k-1}, t_k],\ k = 1,2,3,\ldots,p, \\ \frac{1.25(t-t_k)^{0.8}}{\Gamma(0.8)} & t \in (t_k, s_k],\ \ k = 1,2,\ldots,p. \end{cases}$$

- If $a_k \neq 1,\ b_k \neq 0$, then the impulsive condition (9) is reduced to the condition $x(t) = ax(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,\ldots,p)$ which has an unique solution and we can talk about uniqueness of the solution IVP for NIRLFrDE (8)–(10), respectively (25), (27).

Case 3. Let $\phi_k(t, x, y) = \arctan(x) + \cos(x) + y$ for $t \in [s_{k-1}, t_k]$, $(k = 1,2,3,\ldots,p)$. Then the impulsive condition (9) is reduced to the algebraic equation $x = \arctan(x) + \cos(x) + y$ which could have more than one solution (for example if $y = 1$, then there are 5 constant solutions), i.e., we do not have uniqueness for the IVP for NIRLFrDE (8)–(10), respectively (25), (27).

Example 3. Consider the partial case of (8)–(10) with $t_0 = 0$, $q = 0.5$, $f(t,x) = ax$ for $t \in [0,T], x \in \mathbb{R}$, $a = const \neq 0$ and $\phi_k(t,x,y) = b_k y$, $k = 1,2,\ldots,p$. In this case we can consider both IVP (8), (10) and IVP (25), (27).

Case 1. When we apply (16) we obtain the solution of (8), (10):

$$x(t) = \begin{cases} \frac{x_0}{\sqrt{t}} E_{0.5,0.5}(-a\sqrt{t}) & \text{for } t \in [0,s_0], \\ b_k x(s_{k-1}) & \text{for } t \in (s_{k-1},t_k], \\ \sqrt{\frac{t_k}{t}}\Big(b_k x(s_{k-1}) - \frac{a}{\Gamma(0.5)}\int_0^{t_k}\frac{x(s)}{\sqrt{t_k-s}}ds\Big) + \frac{a}{\Gamma(0.5)}\int_0^t \frac{x(s)}{\sqrt{t-s}}ds & \text{for } t \in (t_k,s_k]. \end{cases}$$

Case 2. When we apply (33) and Proposition 1 we obtain the solution of (25), (27):

$$x(t) = \begin{cases} x_0 \frac{E_{0.5,0.5}(-a\sqrt{t})}{\sqrt{t}} & \text{for } t \in [0,s_0], \\ x_0 \prod_{j=1}^k \frac{b_j E_{0.5,0.5}(-a\sqrt{s_{j-1}-t_{j-1}})}{\sqrt{s_{j-1}-t_{j-1}}} & \text{for } t \in (s_{k-1},t_k], \\ x_0 \frac{E_{0.5,0.5}(-a\sqrt{t-t_k})}{\sqrt{t-t_k}} \prod_{j=1}^k \Big(\frac{b_j E_{0.5,0.5}(-a\sqrt{s_{j-1}-t_{j-1}})}{\sqrt{s_{j-1}-t_{j-1}}}\Big) & \text{for } t \in [t_k,s_k], \end{cases}$$

4. Brief Overview of RL Fractional Equations with Instantaneous Impulses

From Remark 4 note the case of instantaneous impulses is a special case of non-instantaneous impulses.

Several authors set up and studied various types of instantaneous impulsive differential equations for RL fractional differential equations and here we will give a brief overview of the types of instantaneous impulsive conditions in RL fractional differential equations. We emphasize only the case of the fractional order $q \in (0,1)$. Also, we will assume that the points $\{t_k\}$ are the points of impulses.

In [18] Kosmatov studied the RL fractional differential equations of fractional order q with the following impulsive conditions:

$$ {}_0^{RL}D_t^{\beta} x(t)|_{t=t_i+0} - {}_0^{RL}D_t^{q} x(t)|_{t=t_i-0} = J_i(x(t_i)), \;\; i = 1,2,\ldots,p, \tag{48}$$

where $\beta \in (0,q)$. As is shown in [12] the impulse functions ${}_0^{RL}D_t^{\beta} x(t)$ are singular at $t = t_i$, i=1,2,..., p, for $\beta \in (0,q)$ and the impulsive conditions (48) are unsuitable (see Remarks 4.1 [12]). Note the existence of a mild solution for Riemann–Liouville fractional differential equations with fractional order q and impulsive conditions of the form (48) is studied in [10]. In [19] Zhao studied a nonlinear Riemann–Liouville fractional differential equation with instantaneous impulsive conditions of the type

$$\Delta x(t_i) = J_i(x(t_i)), \;\; i = 1,2,\ldots,p, \tag{49}$$

where $\Delta x(t_i) = x(t_i+0) - x(t_i-0)$. Note similar impulsive conditions are used in [20,21]. As is shown in [12] the integral representation of the solution with impulsive condition (49) is not correct (see Remark 4.3 [12]). In [8] the initial value problems of nonlinear impulsive RL fractional differential equations are studied with instantaneous impulsive conditions of the type

$$\lim_{t\to t_i+0}(t-t_i)^{1-q}x(t) = J_i(t_i, x(t_i), {}_0^{RL}D_t^{\beta} x(t)|_{t=t_i-0}), \;\; i = 1,2,\ldots,p, \tag{50}$$

with $\beta \in (0,q)$. Similar impulsive conditions are studied in [11,22] but written in the form:

$$\Delta I_{t_i}^{1-q} x(t_i) = J_i(t_i, x(t_i-)), \;\; i = 1,2,\ldots,p, \tag{51}$$

where $\Delta I_{t_i}^{1-q}x(t_i) = I_{t_i+}^{1-q}x(t_i+) - I_{t_i-}^{1-q}x(t_i-) = \lim_{t\to t_i+0}(t-t_i)^{1-q}x(t) - \lim_{t\to t_i-}(t-t_i)^{1-q}x(t)$. In [23] the instantaneous impulsive conditions are given by

$$ {}_{0+}^{RL}I_t^{1-q}x(t_i+0) - {}_{0+}^{RL}I_t^{1-q}x(t_i-0) = J_i(x(t_i)), \; i = 1,2,\dots,p. \tag{52} $$

The integral representation of the solution of instantaneous impulsive RL fractional derivative of order $q \in (0,1)$ is given in [8] in the case when the impulsive conditions are

$$ \lim_{t\to t_i+0}(t-t_i)^{1+\beta-q}x(t) = x_i, \; i = 1,2,\dots,p, $$

and the initial condition is

$$ {}_{0}^{RL}I_t^{1-q}x(t)|_{t=0} = x_0 $$

where $\beta \in (0,q)$.

Author Contributions: Conceptualization, R.A., S.H. and D.O.; methodology, R.A., S.H. and D.O.; validation, R.A., S.H. and D.O.; formal analysis, R.A., S.H. and D.O.; writing—original draft preparation, R.A., S.H. and D.O.; writing—review and editing, R.A., S.H. and D.O.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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