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Non-Unique Fixed Point Theorems in Modular Metric Spaces

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Abstract: This paper is devoted to the study of Ćirić-type non-unique fixed point results in modular metric spaces. We obtain various theorems about a fixed point and periodic points for a self-map on modular spaces which are not necessarily continuous and satisfy certain contractive conditions. Our results extend the results of Ćirić, Pachpatte, and Achari in modular metric spaces.

Keywords: f-orbitally ω-complete; strong Ćirić type ω-contraction; strong Pachpatte type ω-contraction

MSC: 47H09; 54H25

1. Introduction

Metric fixed point theory was initiated by the renowned theorem of Banach [1], known as the Banach Contraction Mapping Principle. He stated that every contraction in a complete metric space has a unique fixed point. Following this pioneering work, many authors have generalized this elegant result by refining the contraction condition and/or by changing the metric space to more refined abstract spaces (see, e.g., [2–5] and the related references therein). In 1974, Ćirić [6] studied non-unique fixed point results in metric spaces. He obtained various theorems about a fixed point and periodic points for a self-map f on a metric space M which is not necessarily continuous and satisfies the condition

$$\min\{d(fx,fy),d(x,fx),d(y,fy)\}-\min\{d(x,fy),d(y,fx)\}\le kd(x,y),$$

where $x,y\in M$ and $k\in(0,1)$. Later on, Pachpatte [7] proved that an orbitally continuous self-map f on an f-orbitally complete metric space M satisfying the condition

$$\min\left\{[d(fx,fy)]^2,d(x,y)d(fx,fy),[d(y,fy)]^2\right\}\\ -\min\{d(x,fx)d(y,fy),d(x,fy)d(y,fx)\}\le kd(x,fx)d(y,fy),$$

where $x,y\in M$ and $k\in(0,1)$, has a fixed point. Achari [8] established some fixed point theorems when the self-mapping f on a metric space (M,d) satisfies the inequality

$$\frac{\min\{d(fx,fy)d(x,y),d(x,fx)d(y,fy)\}-\min\{d(x,fx)d(x,fy),d(y,fy)d(y,fx)\}}{\min\{d(x,fx),d(y,fy)\}}\\ \le kd(x,y),$$

where $x, y \in M$, $k \in (0,1)$, and $d(x, fx) \neq 0, d(y, fy) \neq 0$. Inspired by this pioneering work, many researchers have studied non-unique fixed point results for different types of contractions on metric spaces (see [9–18]), as well as in many other abstract spaces (see [19–26]).

On the other hand, Nakno [27] initiated the theory of modular spaces, which was re-defined and extended by Musielak and Orlicz [28–30]. In 2008, Chistyakov [31] gave the concept of a modular metric space generated by an F-modular and the advanced theory of modular spaces. As a generalization of metric spaces, Chistyakov [32,33]) introduced and studied modular metric spaces on an arbitrary set and, in [34], proved fixed point results for contractive maps in modular spaces. The existence of fixed point theorems in modular spaces has received a great deal of attention from researchers, recently (see [35–38] and references therein).

Inspired by the works of Chistyakov and Ćirić, in this paper, we study non-unique fixed points and periodic points in modular metric spaces. Our results extend the results of Ćirić, Pachpatte, and Achari in modular metric spaces.

2. Preliminaries

In this section, we recollect some basic notions and results about modular metric spaces, which will be used later. Throughout the article, we assume that M is a nonempty set, λ is a non-negative real number (i.e., $\lambda \in (0, \infty)$), and $\omega : (0, \infty) \times M \times M \to [0, \infty]$ is a function (that will also be written as $\omega(\lambda, x, y) = \omega_\lambda(x, y)$ for all $\lambda > 0$ and $x, y \in M$) such that $\omega = \{\omega_\lambda\}_{\lambda>0}$ with $\omega_\lambda : M \times M \to [0, \infty]$.

Definition 1. *[32] A map $\omega : (0, \infty) \times M \times M \to [0, \infty]$ is called a (metric) modular on M if it satisfies the following conditions:*

(i) $\omega_\lambda(x, y) = 0$ *if and only if* $x = y$*;*
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$*; and*
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$,

for all $\lambda, \mu > 0$ *and* $x, y, z \in M$.

If, in lieu of (i), ω *satisfies only*

(i_p) $\omega_\lambda(x, x) = 0$ *for all* $x \in M$ *and* $\lambda > 0$,

then ω *is called a pseudomodular on* M*. Furthermore,* ω *is called a strict modular on* M *if it satisfies* (i_p) *and*

(i_s) *given* $x, y \in M$*, if there exists a non-negative real number* λ*, possibly depending on* x *and* y*, such that* $\omega_\lambda(x, y) = 0$*, then* $x = y$*.*

A modular (strict modular, pseudomodular) is called a convex modular if, in place of (iii)*, it satisfies*

(iv) $\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$

for all $\lambda, \mu > 0$ *and* $x, y, z \in M$.

It was shown, in [32], that if ω is a convex modular then, for all $0 < \lambda \leq \mu$ and $x, y \in M$, one has

$$\omega_\mu(x, y) \leq \frac{\lambda}{\mu}\omega_\lambda(x, y) \leq \omega_\lambda(x, y). \tag{1}$$

By using condition (iii) of Definition 1, one can show that a modular (pseudomodular) ω satisfies

$$\omega_{\mu_2}(x, y) \leq \omega_{\mu_1}(x, y) \tag{2}$$

for $\mu_1 < \mu_2$ and for all $x, y \in M$.

Definition 2. *[32] Let* ω *be a pseudomodular on* M *and* $x \in M$*. Then, the sets*

$$M_\omega = M_\omega(x) = \{y \in M : \omega_\lambda(x,y) \to 0 \text{ as } \lambda \to \infty\}$$
$$M^*_\omega = M^*_\omega(x) = \{y \in M : \text{ there exists } \lambda = \lambda(y) > 0 \text{ such that } \omega_\lambda(x,y) < \infty\},$$

are called modular metric spaces (around x).

It was shown, in [32], that, in general, M_ω is contained in M^*_ω. According to ([32], Theorem 2.6), if ω is a modular metric on M, then the modular space M_ω can be equipped with a non-trivial metric generated by ω, given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\},$$

for all $x, y \in M_\omega$. If ω is a convex modular on M, then it follows, from ([32], Section 3.5 and Theorem 3.6), that $M_\omega = M^*_\omega$ holds and they are equipped with the metric d^*_ω, given by

$$d^*_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\}.$$

Definition 3. *[32,33] Let M_ω and M^*_ω be modular metric spaces.*

(i) A sequence $\{x_n\}$ in M^*_ω (or M_ω) is called ω-convergent to $x \in M$ if and only if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$. Then, x is said to be the modular limit of $\{x_n\}$

(ii) A sequence $\{x_n\}$ in M^*_ω (or M_ω) is called ω-Cauchy if $\lim_{n,m\to\infty} \omega_\lambda(x_n, x_m) = 0$, for some $\lambda > 0$.

(iii) A subset X of M^*_ω (or M_ω) is called $\omega-$complete if every ω-Cauchy sequence in X is ω-convergent to $x \in X$.

By using the properties of modular metrics and the definition of convergence, one can easily prove that if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$, then $\lim_{n\to\infty} \omega_\mu(x_n, x) = 0$ for all $\mu > \lambda > 0$. It was also shown, in [33], that if ω is pseudomodular on M, then the modular metric M^*_ω and M_ω are closed with respect to ω-convergence.

Definition 4. *[34] A pseudomodular ω on M is said to satisfy the Δ_2-condition (on M^*_ω) if the following condition holds: Given a sequence $\{x_n\} \subset M^*_\omega$ and $x \in M^*_\omega$, if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, then $\lim_{n\to\infty} \omega_{\frac{\lambda}{2}}(x_n, x) = 0$.*

Now, we state the definitions of modular contractive mappings and a fixed point theorem for such mappings (given in [34]).

Definition 5. *Let ω be a modular metric on M.*

(i) A map $f : M^*_\omega \to M^*_\omega$ is said to be ω-contractive if there exists $k \in (0,1)$ and $\lambda_0 = \lambda_0(k) > 0$ such that

$$\omega_{k\lambda}(fx, fy) \le \omega_\lambda(x,y),$$

for all $0 < \lambda < \lambda_0$ and $x, y \in M^*_\omega$.

(ii) A map $f : M^*_\omega \to M^*_\omega$ is said to be strong ω-contractive if there exists $k \in (0,1)$ and $\lambda_0 = \lambda_0(k) > 0$ such that

$$\omega_{k\lambda}(fx, fy) \le k\omega_\lambda(x,y),$$

for all $0 < \lambda < \lambda_0$ and $x, y \in M^*_\omega$.

Theorem 1. *Let ω be a strict convex metric modular on M and $f : M^*_\omega \to M^*_\omega$ be a ω-contractive (or strong ω-contractive) mapping on a complete modular metric space M^*_ω induced by ω. If, for every $\lambda > 0$, there exists an $x = x(\lambda) \in M^*_\omega$ such that $\omega_\lambda(x, fx) < \infty$, then f has a fixed point in M^*_ω. Moreover, if $\omega_\lambda(x,y) < \infty$ for all $x, y \in M^*_\omega$ and every $\lambda > 0$, then f has a unique fixed point in M^*_ω.*

3. Extension of Non-Unique Fixed Point of Ćirić on Modular Metric Spaces

Let M_ω^* and M_ω be modular metric spaces and $f : M_\omega^* \to M_\omega^*$ (or $f : M_\omega \to M_\omega$) be a self-map. Let $x \in M_\omega^*$ (or M_ω). We call $\mathcal{O}(x) = \{f^n x : n = 0, 1, 2, 3, \ldots\}$ the orbit of x, and f is called orbitally continuous if $\lim_i f^{n_i} x = z$ implies $\lim_i f f^{n_i} x = fz$ for each $x \in M_\omega^*$ (or M_ω). The space M_ω^* (or M_ω) is f-orbitally ω-complete if every ω-Cauchy sequence of the form $\{f^{n_i} x\}_{i=1}^\infty$, $x \in M_\omega^*$, (or M_ω), converges in M_ω^* (or M_ω).

Definition 6. *Let ω be a metric modular on M. A mapping $f : M_\omega^* \to M_\omega^*$ is called a strong Ćirić-type ω-contraction if there exists $k \in (0, 1)$ and $\lambda_0 = \lambda_0(k)$, such that*

$$\min\{\omega_{k\lambda}(fx, fy), \omega_{k\lambda}(x, fx), \omega_{k\lambda}(y, fy)\} - \min\{\omega_{k\lambda}(x, fy), \omega_{k\lambda}(y, fx)\} \le k\omega_\lambda(x, y) \tag{3}$$

holds for all $0 < \lambda < \lambda_0$ and $x, y \in M_\omega^*$.

Theorem 2. *Let ω be a convex modular on M. Suppose $f : M_\omega^* \to M_\omega^*$ is an orbitally continuous mapping on a f-orbitally ω-complete modular space M_ω^* and f is a strong Ćirić-type ω-contraction. Assume that, for every $\lambda > 0$, there exists an $x \in M_\omega^*$ such that $\omega_\lambda(x, fx) = C < \infty$. Then, for each $x \in M_\omega^*$, the sequence $\{f^n x\}_{n=1}^\infty$ converges to a fixed point of f.*

Proof. Let $x \in M_\omega^*$ be arbitrary such that $\omega_\lambda(x, fx) = C < \infty$. Define the iterative sequence $\{x_n\}$ by

$$x_0 = x, x_1 = fx_0 = fx, x_2 = fx_1 = f^2 x, \cdots, x_n = fx_{n-1} = f^n x.$$

We shall show that $\{x_n\}$ is an ω-Cauchy sequence. As $\omega_\lambda(x_{j-1}, x_j) = 0$ for some $j \in \mathbb{N}$ immediately implies that $\{x_n\}$ is an ω-Cauchy sequence, we assume that $\omega_\lambda(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. By inequality (3) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned}
&\min\{\omega_{k\lambda}(fx_{n-1}, fx_n), \omega_{k\lambda}(x_{n-1}, fx_{n-1}), \omega_{k\lambda}(x_n, fx_n)\}\\
&- \min\{\omega_{k\lambda}(x_{n-1}, fx_n), \omega_{k\lambda}(x_n, fx_{n-1})\}\\
&= \min\{\omega_{k\lambda}(x_n, x_{n+1}), \omega_{k\lambda}(x_{n-1}, x_n)\} \le k\omega_\lambda(x_{n-1}, x_n).
\end{aligned}$$

From the fact $0 < k\lambda < \lambda$, we then have $\omega_\lambda(x_{n-1}, x_n) \le \omega_{k\lambda}(x_{n-1}, x_n)$. As $\omega_{k\lambda}(x_{n-1}, x_n) \le k\omega_\lambda(x_{n-1}, x_n) \le k\omega_{k\lambda}(x_{n-1}, x_n)$ is not possible (as $k < 1$), we have

$$\omega_{k\lambda}(x_n, x_{n+1}) \le k\omega_\lambda(x_{n-1}, x_n), \tag{4}$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. As $0 < k^n\lambda < \lambda < \lambda_0$, by (4), we obtain

$$\omega_{k^n\lambda}(x_n, x_{n+1}) = \omega_{kk^{n-1}\lambda}(x_n, x_{n+1}) \le k\omega_{k^{n-1}\lambda}(x_{n-1}, x_n),$$

or, inductively,

$$\omega_{k^n\lambda}(x_n, x_{n+1}) \le k^n\omega_\lambda(x, fx) = k^n C,$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. By letting $n \to \infty$, we get

$$\lim_{n\to\infty} \omega_{k^n\lambda}(x_n, x_{n+1}) = 0, \tag{5}$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. By setting $\lambda_1 = (1 - k)\lambda_0 < \lambda_0$, we obtain

$$\omega_{k^n\lambda_1}(x_n, x_{n+1}) \le k^n\omega_{\lambda_1}(x, fx) = k^n C,$$

for all $n \in \mathbb{N}$. By letting $n \to \infty$, we get

$$\lim_{n\to\infty} \omega_{k^n\lambda_1}(x_n, x_{n+1}) = 0, \tag{6}$$

for all $0 < \lambda_1 < \lambda_0$, As ω is convex, for any $m, n \in \mathbb{N}$ such that $m < n$, we get

$$\omega_{\lambda^*}(x_n, x_m) \le \sum_{j=m}^{n-1} \frac{\lambda_j}{\lambda^*} \omega_{\lambda_j}(x_j, x_{j+1}), \tag{7}$$

where

$$\lambda^* = \sum_{j=m}^{n-1} \lambda_j.$$

Now, putting $\lambda_j = k^j\lambda_1$, $j = m, m+1, \cdots, n-1$, in (7), we have

$$\omega_{\lambda^*}(x_n, x_m) \le \sum_{j=m}^{n-1} \frac{k^j\lambda_1}{\lambda^*} \omega_{k^j\lambda_1}(x_j, x_{j+1}), \tag{8}$$

where

$$\lambda^* = \sum_{j=m}^{n-1} k^j\lambda_1 = k^m\lambda_1 \frac{1-k^{n-m}}{1-k} = k^m(1-k^{n-m})\lambda_0 < \lambda_0.$$

Taking into account $0 < k^j\lambda_1 < \lambda_1 < \lambda_0$ for $j = m, m+1, \cdots, n-1$ and (6), we get

$$\lim_{n,m\to\infty} \omega_{\lambda^*}(x_n, x_m) = 0, \tag{9}$$

for all $0 < \lambda^* < \lambda_0$. From the fact that $0 < \lambda^* < \lambda_0$, we then have

$$\omega_{\lambda_0}(x_n, x_m) \le \frac{\lambda^*}{\lambda_0} \omega_{\lambda^*}(x_n, x_m) = k^m(1-k^m)\omega_{\lambda^*}(x_n, x_m) \le k^m \omega_{\lambda^*}(x_n, x_m). \tag{10}$$

Now, from (9) we have

$$\lim_{n,m\to\infty} \omega_{\lambda_0}(x_n, x_m) = 0. \tag{11}$$

This shows that $\{x_n\}$ is a ω-Cauchy sequence in M^*_ω. By the f-orbitally ω-completeness of M^*_ω, there exists some z in M^*_ω such that $\lim_n f^n x = z$. The orbital continuity of f implies

$$fz = \lim_n ff^n x = z,$$

which shows that z is a fixed point of f. $\square$

Theorem 3. *Let ω be a convex modular on M. Suppose $f : M^*_\omega \to M^*_\omega$ is an orbitally continuous mapping on a f-orbitally ω-complete modular space M^*_ω and let $\varepsilon > 0$. Suppose that there exists $k \in (0,1)$, $\lambda_0 = \lambda_0(k)$ and $x \in M^*_\omega$ such that $\omega_\lambda(x, f^q x) < \varepsilon$, for some $q \in \mathbb{N}$ and for all $\lambda < \lambda_0$. If*

$$0 < \omega_\lambda(x, y) < \varepsilon \textit{ implies } \min\{\omega_{k\lambda}(fx, fy), \omega_{k\lambda}(x, fx), \omega_{k\lambda}(y, fy)\} \le k\omega_\lambda(x, y) \tag{12}$$

holds, for all $0 < \lambda < \lambda_0$ and $x, y \in M^*_\omega$, then f has a periodic point.

Proof. Let $Q = \{q : \omega_\lambda(x, f^q x) < \varepsilon$ for some $x \in M_\omega^*$ and for all $\lambda < \lambda_0\}$ be the subset of $\mathbb{N}$ which is non-empty, due to the assumption of the Theorem. Let $x \in M_\omega^*$ such that $\omega_\lambda(x, f^m x) < \varepsilon$, where $m = \min Q$.

If $m = 1$, by using (12) with x and fx, we get

$$\min\left\{\omega_{k\lambda}(fx, f^2x), \omega_{k\lambda}(x, fx), \omega_{k\lambda}(fx, f^2x)\right\} \le k\omega_\lambda(x, fx).$$

By the fact that $k\lambda < \lambda$, we have

$$\omega_\lambda(x, fx) \le \omega_{k\lambda}(x, fx).$$

As $\omega_{k\lambda}(x, fx) \le \omega_\lambda(x, fx) \le k\omega_{k\lambda}(x, fx)$ is impossible (as $k < 1$), we have

$$\omega_{k\lambda}(fx, f^2x) \le k\omega_\lambda(x, fx) < k\varepsilon,$$

for all $0 < \lambda < \lambda_0$. Proceeding as in Theorem 2, we obtain that $fz = z$ for some $z \in M_\omega^*$.

Now, take $m \ge 2$; that is,

$$\omega_\lambda(y, fy) \ge \varepsilon, \tag{13}$$

for all $y \in M_\omega^*$ and $\lambda < \lambda_0$. Then, from $0 < \omega_\lambda(x, f^m x) < \varepsilon$ and by (22), we get

$$\min\left\{\omega_{k\lambda}(fx, f^{m+1}x), \omega_{k\lambda}(x, fx), \omega_{k\lambda}(f^m x, f^{m+1}x)\right\} \le k\omega_\lambda(x, f^m x).$$

By the fact that $k\lambda < \lambda < \lambda_0$ and (13), thus $\omega_{k\lambda}(x, fx) \ge \varepsilon$ and $\omega_{k\lambda}(f^m x, f^{m+1}x) = \omega_{k\lambda}(f^m x, ff^m x) \ge \varepsilon$, and we get

$$\omega_{k\lambda}(fx, f^{m+1}x) \le k\omega_\lambda(x, f^m x) < k\varepsilon,$$

for all $x \in M_\omega^*$ and $\lambda < \lambda_0$. Similarly,

$$\omega_{k^2\lambda}(f^2x, f^{m+2}x) \le k^2\omega_\lambda(x, f^m x) < k^2\varepsilon.$$

Continuing in this manner, we get

$$\omega_{k^n\lambda}(f^n x, f^{m+n}x) \le k^n\omega_\lambda(x, f^m x) < k^n\varepsilon,$$

for all $n \in \mathbb{N}$. Therefore, for the sequence

$$x_0 = x, x_1 = f^m x_0, x_2 = f^m x_1, \cdots x_n = f^m x_{n-1},$$

we have that

$$\omega_{k^{nm}\lambda}(x_n, x_{n+1}) = \omega_{k^{nm}\lambda}(f^{nm}x, f^{m+nm}x) \le k^{nm}\omega_\lambda(x, f^m x) < k^{nm}\varepsilon.$$

Then, following the same method as in Theorem 2, we conclude that $\{x_n\}$ is a Cauchy sequence. As $\{x_n\} \subseteq \{f^n x\}$ and M_ω^* is f-orbitally ω-complete, there exists some $z \in M_\omega^*$ such that

$$z = \lim_n x_n = \lim_n f^{nm}.$$

Taking into account that if f is orbital continuous, then f^r is also orbital continuous for all $r \in \mathbb{N}$, we have

$$f^m z = \lim_n x_n = \lim_n f^m f^{nm} == \lim_n f^{(n+1)m} = z,$$

which shows that z is a periodic point of f. $\square$

Theorem 4. *Let ω be a modular on M and $f : M^*_\omega \to M^*_\omega$ be an orbitally continuous mapping on a modular space M^*_ω. Suppose that, whenever $x \neq y$, f satisfies the following*

$$\min\{\omega_\lambda(fx, fy), \omega_\lambda(x, fx), \omega_\lambda(y, fy)\} - \min\{\omega_\lambda(x, fy), \omega_\lambda(y, fx)\} < \omega_\lambda(x, y), \tag{14}$$

for all $\lambda > 0$ and $x, y \in M^*_\omega$. If, for some $x \in M^*_\omega$, the sequence $\{f^n x\}_{n=1}^\infty$ has a limit point $z \in M^*_\omega$, then z is a fixed point of f.

Proof. If, for some $m \in \mathbb{N}$, $\omega_\lambda(x_{m-1}, x_m) = 0$, then $x_n = x_m = z$ for $n \geq m$, and the assertion holds. Suppose, then, that $\omega_\lambda(x_{m-1}, x_m) \neq 0$ for all $m \in \mathbb{N}$. Let $\lim\limits_i x_{n_i} = z$. Then, by (14), for $x_{n-1}, x_n \in M^*_\omega$. Then,

$$\begin{aligned}
&\min\{\omega_\lambda(fx_{n-1}, fx_n), \omega_\lambda(x_{n-1}, fx_{n-1}), \omega_\lambda(x_n, fx_n)\} - \\
&\min\{\omega_\lambda(x_{n-1}, fx_n), \omega_\lambda(x_n, fx_{n-1})\} = \min\{\omega_\lambda(x_n, x_{n+1}), \omega_\lambda(x_{n-1}, x_n)\} \\
&< \omega_\lambda(x_{n-1}, x_n).
\end{aligned}$$

As $\omega_\lambda(x_{n-1}, x_n) < \omega_\lambda(x_{n-1}, x_n)$ is impossible, we have $\omega_\lambda(x_n, x_{n+1}) < \omega_\lambda(x_{n-1}, x_n)$ for all $\lambda > 0$. Therefore, $\{\omega_\lambda(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is a decreasing, and hence convergent, sequence of real numbers. As $\lim\limits_i \omega_\lambda(x_{n_i}, x_{n_i+1}) = \omega_\lambda(z, fz)$ and $\{\omega_\lambda(x_{n_i}, x_{n_i+1})\} \subseteq \{\omega_\lambda(x_n, x_{n+1})\}$, it follows that

$$\lim_n \omega_\lambda(x_n, x_{n+1}) = \omega_\lambda(z, fz). \tag{15}$$

Furthermore, as $\lim\limits_i x_{n_i+1} = fz$, $\lim\limits_i x_{n_i+2} = f^2 z$, and $\{\omega_\lambda(x_{n_i+1}, x_{n_i+2})\} \subseteq \{\omega_\lambda(x_n, x_{n+1})\}$, by (15), we have

$$\lim_n \omega_\lambda(fz, f^2 z) = \omega_\lambda(z, fz). \tag{16}$$

If $\omega_\lambda(z, fz) > 0$, then (14) implies that $\omega_\lambda(fz, f^2 z) < \omega_\lambda(z, fz)$, a contradiction. Hence, $\omega_\lambda(z, fz) = 0$, i.e., $fz = z$. This completes the proof of the Theorem. $\square$

Theorem 5. *Let ω be a modular on M satisfying the Δ_2-condition on M^*_ω. Suppose that $f : M^*_\omega \to M^*_\omega$ is an orbitally continuous mapping on a modular space M^*_ω and $\varepsilon > 0$. Suppose that f satisfies the following*

$$0 < \omega_\lambda(x, y) < \varepsilon \text{ implies } \min\{\omega_\lambda(fx, fy), \omega_\lambda(x, fx), \omega_\lambda(y, fy)\} < \omega_\lambda(x, y), \tag{17}$$

for all $\lambda > 0$ and $x, y \in M^*_\omega$. If, for some $x \in M^*_\omega$, the sequence $\{f^n x\}_{n=1}^\infty$ has a limit point $z \in M^*_\omega$, then z is the periodic point of f.

Proof. Let $\lim\limits_i x_{n_i} = z$, then there exists $r \in \mathbb{N}$ such that $i > r$ implies $\omega_{\frac{\lambda}{2}}(x_{n_i}, z) < \frac{\varepsilon}{2}$. Hence,

$$\omega_\lambda(x_{n_i}, x_{n_i+1}) \leq \omega_{\frac{\lambda}{2}}(x_{n_i}, z) + \omega_{\frac{\lambda}{2}}(x_{n_i+1}, z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the set

$$S = \{s \in \mathbb{N} : \omega_\lambda(x_p, x_{p+s}) < \varepsilon \text{ for some } p \in \mathbb{N}\}$$

is non-empty. Put $m = \min S$. If $\omega_\lambda(x_s, x_{s+m}) = 0$ for some $s \in \mathbb{N}$, then $z = x_s = f^m z$, and the assertion holds. Now, assume that $\omega_\lambda(x_s, x_{s+m}) > 0$ for every $s \in \mathbb{N}$ and $\lambda > 0$. Let $q \in \mathbb{N}$ such that $\omega_\lambda(x_q, x_{q+m}) < \varepsilon$.

If $m = 1$, then, by (17) (as in the proof of the Theorem 4), $\{\omega_\lambda(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a decreasing sequence for $n \geq q$, which implies that $fz = z$.

So, suppose that $m \geq 2$; that is, that

$$\omega_\lambda(x_n, x_{n+1}) \geq \varepsilon, \tag{18}$$

for all $n \in \mathbb{N}$ and $\lambda > 0$. As f is orbital continuous, $\lim_i x_{n_i+s} = f^s z$. By (18),

$$\omega_\lambda(f^s z, f^{s+1}) = \lim_i (x_{n_i+s}, x_{n_i+s+1}) \geq \varepsilon, \tag{19}$$

for all $s \in \mathbb{N}$. By (17) and the assumption $0 < \omega_\lambda(x_q, x_{q+m}) < \varepsilon$, we have

$$\min\{\omega_\lambda(fx_q, fx_{q+m}), \omega_\lambda(x_q, fx_q), \omega_\lambda(x_{q+m}, fx_{q+m})\} < \omega_\lambda(x_q, x_{q+m}),$$

or

$$\min\{\omega_\lambda(x_{q+1}, x_{q+m+1}), \omega_\lambda(x_q, x_{q+1}), \omega_\lambda(x_{q+m}, x_{q+m+1})\} < \omega_\lambda(x_q, x_{q+m}).$$

Hence, by (18), we get

$$\omega_\lambda(x_{q+1}, x_{q+m+1}) < \omega_\lambda(x_q, x_{q+m}) < \varepsilon.$$

In a similar way, we get

$$\varepsilon > \omega_\lambda(x_q, x_{q+m}) > \omega_\lambda(x_{q+1}, x_{q+m+1}) > \omega_\lambda(x_{q+2}, x_{q+m+2}) > \cdots, \tag{20}$$

which shows that $\{\omega_\lambda(x_n, x_{n+m}) : n \geq q \text{ and } \lambda > 0\}$ is decreasing and, hence, is a convergent sequence of real numbers. As the subsequences $\{\omega_\lambda(x_{n_i}, x_{n_i+m})\}_{i\in\mathbb{N}}$ and $\{\omega_\lambda(x_{n_i+1}, x_{n_i+m+1})\}_{i\in\mathbb{N}}$ converge to $\omega_\lambda(z, f^m z)$ and $\omega_\lambda(z, f^{m+1} z)$, respectively, then, by the orbital continuity of f and as $\lim_i f^{n_i} = z$, we have

$$\omega_\lambda(fz, f^{m+1} z) = \omega_\lambda(z, f^m z) = \lim_n \omega_\lambda(x_n, x_{m+n}). \tag{21}$$

By (20) and (21), we get $\omega_\lambda(z, f^m z) < \varepsilon$. If $\omega_\lambda(z, f^m z) > 0$, then, from (17), we obtain

$$\min\left\{\omega_\lambda(fz, f^{m+1} z), \omega_\lambda(z, fz), \omega_\lambda(f^m z, f^{m+1} z)\right\} < \omega_\lambda(z, f^m z) < \varepsilon.$$

By (19),

$$\omega_\lambda(fz, f^{m+1} z) < \omega_\lambda(z, f^m z),$$

which is a contradiction. Hence, $\omega_\lambda(z, f^m z) = 0$, which implies that z is the periodic point of f. □

4. Extension of Non-Unique Fixed Point of Pachpatte on Modular Metric Spaces

In this section, non-unique fixed point theorems for Pachpatte-type contractions are proved in the setting of modular metric spaces. We start this section with the following definition.

Definition 7. *Let ω be a metric modular on M. A mapping $f : M^*_\omega \to M^*_\omega$ is called a strong Pachpatte-type ω-contraction if there exists $k \in (0,1)$ and $\lambda_0 = \lambda_0(k)$, such that*

$$\begin{aligned}\min&\left\{[\omega_{k\lambda}(fx, fy)]^2, \omega_{k\lambda}(x,y)\omega_{k\lambda}(fx, fy), [\omega_{k\lambda}(y, fy)]^2\right\}\\ &- \min\{\omega_{k\lambda}(x, fx), \omega_{k\lambda}(y, fy), \omega_{k\lambda}(x, fy)\omega_{k\lambda}(y, fx)\} \leq k\omega_\lambda(x, fx)\omega_\lambda(y, fy)\end{aligned} \tag{22}$$

holds, for all $\lambda_0 > \lambda > 0$, and $x, y \in M_\omega^*$.

Theorem 6. *Let ω be a convex modular on M. Suppose $f : M_\omega^* \to M_\omega^*$ is an orbitally continuous mapping on a f-orbitally ω-complete modular space M_ω^* and f is a strong Pachpatte-type ω-contraction. Assume, for every $\lambda > 0$, there exists an $x \in M_\omega^*$ such that $\omega_\lambda(x, fx) = C < \infty$. Then, for each $x \in M_\omega^*$, the sequence $\{f^n x\}_{n=1}^\infty$ converges to a fixed point of f.*

Proof. Let $x \in M_\omega^*$ be arbitrary, such that $\omega_\lambda(x, fx) = C < \infty$. Define the iterative sequence $\{x_n\}$ by

$$x_0 = x, x_1 = fx_0 = fx, x_2 = fx_1 = f^2x, \cdots, x_n = fx_{n-1} = f^n x.$$

We shall show that $\{x_n\}$ is an ω-Cauchy sequence. As $\omega_\lambda(x_{j-1}, x_j) = 0$ for some $j \in \mathbb{N}$ immediately implies that $\{x_n\}$ is ω-Cauchy sequence, we assume that $\omega_\lambda(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. By (22) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} &\min\left\{[\omega_{k\lambda}(x_n, x_{n+1})]^2, \omega_{k\lambda}(x_{n-1}, x_n)\omega_{k\lambda}(x_n, x_{n+1}), [\omega_{k\lambda}(x_n, x_{n+1})]^2\right\} \\ &\quad - \min\{\omega_{k\lambda}(x_{n-1}, x_n), \omega_{k\lambda}(x_n, x_{n+1}), \omega_{k\lambda}(x_{n-1}, x_{n+1})\omega_{k\lambda}(x_n, x_n)\} \\ &= \min\left\{[\omega_{k\lambda}(x_n, x_{n+1})]^2, \omega_{k\lambda}(x_{n-1}, x_n)\omega_{k\lambda}(x_n, x_{n+1})\right\} \le k\omega_\lambda(x_{n-1}, x_n)\omega_\lambda(x_n, x_{n+1}). \end{aligned}$$

From the fact that $0 < k\lambda < \lambda$, we have

$$\omega_\lambda(x_{n-1}, x_n)\omega_\lambda(x_n, x_{n+1}) \le \omega_{k\lambda}(x_n, x_{n+1})\omega_{k\lambda}(x_{n-1}, x_n).$$

As

$$\omega_{k\lambda}(x_{n-1}, x_n)\omega_{k\lambda}(x_n, x_{n+1}) \le k\omega_\lambda(x_{n-1}, x_n)\omega_\lambda(x_n, x_{n+1}) \le k\omega_{k\lambda}(x_{n-1}, x_n)\omega_{k\lambda}(x_n, x_{n+1})$$

is impossible (as $k < 1$), we have

$$[\omega_{k\lambda}(x_n, x_{n+1})]^2 \le k\omega_\lambda(x_{n-1}, x_n)\omega_\lambda(x_n, x_{n+1}) \le k\omega_\lambda(x_{n-1}, x_n)\omega_{k\lambda}(x_n, x_{n+1}),$$

or

$$\omega_{k\lambda}(x_n, x_{n+1}) \le k\omega_\lambda(x_{n-1}, x_n), \tag{23}$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. As $0 < k^n\lambda < \lambda < \lambda_0$, by (23), we obtain

$$\omega_{k^n\lambda}(x_n, x_{n+1}) = \omega_{kk^{n-1}\lambda}(x_n, x_{n+1}) \le k\omega_{k^{n-1}\lambda}(x_{n-1}, x_n),$$

or, inductively,

$$\omega_{k^n\lambda}(x_n, x_{n+1}) \le k^n\omega_\lambda(x, fx) = k^n C,$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. By letting $n \to \infty$, we get

$$\lim_{n\to\infty} \omega_{k^n\lambda}(x_n, x_{n+1}) = 0, \tag{24}$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. Following the same procedure as in the proof of Theorem 2, we conclude that $\{x_n\}$ is an ω-Cauchy sequence in M_ω^*. By the f-orbitally ω-completeness of M_ω^*, there is some z in M_ω^* such that $\lim_n f^n x = z$. The orbital continuity of f implies that

$$fz = \lim_n ff^n x = z,$$

which shows that z is a fixed point of f. □

Theorem 7. *Let ω be a modular on M and $f : M^*_\omega \to M^*_\omega$ be an orbitally continuous mapping on a modular space M^*_ω. Suppose that, whenever $x \neq y$, f satisfies the following*

$$\begin{aligned} &\min\left\{[\omega_\lambda(fx,fy)]^2, \omega_\lambda(x,y)\omega_\lambda(fx,fy), [\omega_\lambda(y,fy)]^2\right\} \\ &\quad - \min\{\omega_\lambda(x,fx), \omega_\lambda(y,fy), \omega_\lambda(x,fy)\omega_\lambda(y,fx)\} < \omega_\lambda(x,fx)\omega_\lambda(y,fy), \end{aligned} \tag{25}$$

for all $\lambda > 0$ and $x, y \in M^*_\omega$. If, for some $x \in M^*_\omega$, the sequence $\{f^n x\}_{n=1}^\infty$ has a limit point $z \in M^*_\omega$, then z is a fixed point of f.

Proof. If for some $m \in \mathbb{N}$, $\omega_\lambda(x_{m-1}, x_m) = 0$, then $x_n = x_m = z$ for $n \geq m$, and the assertion holds. Suppose, then, that $\omega_\lambda(x_{m-1}, x_m) \neq 0$ for all $m \in \mathbb{N}$. Let $\lim\limits_i x_{n_i} = z$. Then, by (25), for $x_{n-1}, x_n \in M^*_\omega$, we have

$$\begin{aligned} &\min\left\{[\omega_\lambda(x_n,x_{n+1})]^2, \omega_\lambda(x_{n-1},x_n)\omega_\lambda(x_n,x_{n+1}), [\omega_\lambda(x_n,x_{n+1})]^2\right\} \\ &\quad - \min\{\omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_n,x_{n+1}), \omega_\lambda(x_{n-1},x_{n+1})\omega_\lambda(x_n,x_n)\} \\ &= \min\left\{[\omega_\lambda(x_n,x_{n+1})]^2, \omega_\lambda(x_{n-1},x_n)\omega_\lambda(x_n,x_{n+1})\right\} < \omega_\lambda(x_{n-1},x_n)\omega_\lambda(x_n,x_{n+1}). \end{aligned}$$

As $\omega_\lambda(x_{n-1},x_n)\omega_\lambda(x_n,x_{n+1}) < \omega_\lambda(x_{n-1},x_n)\omega_\lambda(x_n,x_{n+1})$ is impossible, we have $\omega_\lambda(x_n,x_{n+1}) < \omega_\lambda(x_{n-1},x_n)$ for all $\lambda > 0$. Therefore, $\{\omega_\lambda(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is a decreasing, and hence convergent, sequence of real numbers. As $\lim\limits_i \omega_\lambda(x_{n_i},x_{n_i+1}) = \omega_\lambda(z,fz)$ and $\{\omega_\lambda(x_{n_i},x_{n_i+1})\} \subseteq \{\omega_\lambda(x_n,x_{n+1})\}$, it follows that

$$\lim_n \omega_\lambda(x_n,x_{n+1}) = \omega_\lambda(z,fz). \tag{26}$$

Furthermore, as $\lim\limits_i x_{n_i+1} = fz$, $\lim\limits_i x_{n_i+2} = f^2 z$ and $\{\omega_\lambda(x_{n_i+1},x_{n_i+2})\} \subseteq \{\omega_\lambda(x_n,x_{n+1})\}$, by (26), we have

$$\omega_\lambda(fz, f^2z) = \omega_\lambda(z,fz). \tag{27}$$

If $\omega_\lambda(z,fz)\omega_\lambda(fz,f^2z) > 0$, then (25) implies $\omega_\lambda(fz,f^2z) < \omega_\lambda(z,fz)$, a contradiction. Hence, $\omega_\lambda(z,fz)\omega_\lambda(fz,f^2z) = 0$. From (27), we have $\omega_\lambda(z,fz) = 0$; that is, $fz = z$. This completes the proof of the Theorem. □

Remark 1. *The conclusion of Theorem 6 remains true if we replace condition (22) by*

$$\begin{aligned} \min\left\{[\omega_{k\lambda}(fx,fy)]^2, \omega_{k\lambda}(x,y)\omega_{k\lambda}(fx,fy), [\omega_{k\lambda}(y,fy)]^2\right\} - \min\{\omega_{k\lambda}(x,fy), \omega_{k\lambda}(y,fx)\} \\ \leq k\omega_\lambda(x,fx)\omega_\lambda(y,fy), \end{aligned}$$

and, similarly, the conclusion of Theorem 7 remains true if we replace condition (25) by

$$\begin{aligned} \min\left\{[\omega_\lambda(fx,fy)]^2, \omega_\lambda(x,y)\omega_\lambda(fx,fy), [\omega_\lambda(y,fy)]^2\right\} - \min\{\omega_\lambda(x,fy), \omega_\lambda(y,fx)\} \\ < \omega_\lambda(x,fx)\omega_\lambda(y,fy). \end{aligned}$$

5. Extension of Non-Unique Fixed Point of Achari on Modular Metric Spaces

In this section, non-unique fixed point theorems for Achari-type contractions are proved in the setting of modular metric spaces. We start this section with the following definition.

Definition 8. *Let ω be a metric modular on M. A mapping $f : M_\omega^* \to M_\omega^*$ is called a strong Achari-type ω-contraction if there exists $k \in (0,1)$ and $\lambda_0 = \lambda_0(k)$, such that*

$$\frac{A(x,y) - B(x,y)}{C(x,y)} \leq k\omega_\lambda(x,y) \tag{28}$$

holds for all $\lambda_0 > \lambda > 0$, and $x, y \in M_\omega^*$, where

$$\begin{aligned} A(x,y) &= \min\{\omega_{k\lambda}(fx,fy)\omega_{k\lambda}(x,y), \omega_{k\lambda}(x,fx)\omega_{k\lambda}(y,fy)\}, \\ B(x,y) &= \min\{\omega_{k\lambda}(x,fx)\omega_{k\lambda}(x,fy), \omega_{k\lambda}(y,fy)\omega_{k\lambda}(y,fx)\}, \end{aligned}$$

and

$$C(x,y) = \min\{\omega_{k\lambda}(x,fx), \omega_{k\lambda}(y,fy)\},$$

such that $\omega_{k\lambda}(x,fx) \neq 0, \omega_{k\lambda}(y,fy) \neq 0$.

Theorem 8. *Let ω be a convex modular on M. Suppose $f : M_\omega^* \to M_\omega^*$ is an orbitally continuous mapping on a f-orbitally ω-complete modular space M_ω^* and f is a strong Achari-type ω-contraction. Assume that, for every $\lambda > 0$, there exists an $x \in M_\omega^*$ such that $\omega_\lambda(x,fx) = C < \infty$. Then, for each $x \in M_\omega^*$, the sequence $\{f^n x\}_{n=1}^\infty$ converges to a fixed point of f.*

Proof. Let $x \in M_\omega^*$ be arbitrary, such that $\omega_\lambda(x,fx) = C < \infty$. Define the iterative sequence $\{x_n\}$ by

$$x_0 = x, x_1 = fx_0 = fx, x_2 = fx_1 = f^2x, \cdots, x_n = fx_{n-1} = f^n x.$$

We shall show that $\{x_n\}$ is an ω-Cauchy sequence. As $\omega_\lambda(x_{j-1}, x_j) = 0$ for some $j \in \mathbb{N}$ immediately implies that $\{x_n\}$ is an ω-Cauchy sequence, we assume that $\omega_\lambda(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$ and $\lambda > 0$. By inequality (28) with $x = x_0$ and $y = x_1$, we get

$$\frac{\omega_{k\lambda}(x_1,x_2)\omega_{k\lambda}(x_0,x_1)}{\min\{\omega_{k\lambda}(x_1,x_2), \omega_{k\lambda}(x_0,x_1)\}} \leq k\omega_\lambda(x_0,x_1).$$

From the fact $0 < k\lambda < \lambda$, we have $\omega_\lambda(x_0,x_1) \leq \omega_{k\lambda}(x_0,x_1)$. As $\omega_{k\lambda}(x_0,x_1) \leq k\omega_\lambda(x_0,x_1) \leq k\omega_{k\lambda}(x_0,x_1)$ is not possible (as $k < 1$), we have

$$\omega_{k\lambda}(x_1,x_2) \leq k\omega_\lambda(x_0,x_1), \tag{29}$$

for all $0 < \lambda < \lambda_0$. As $0 < k^n\lambda < \lambda < \lambda_0$, proceeding in the same manner, we obtain

$$\omega_{k^n\lambda}(x_n,x_{n+1}) = \omega_{kk^{n-1}\lambda}(x_n,x_{n+1}) \leq k\omega_{k^{n-1}\lambda}(x_{n-1},x_n) \cdots \leq k^n\omega_\lambda(x_0,x_1) = k^n C,$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. By letting $n \to \infty$, we get

$$\lim_{n\to\infty} \omega_{k^n\lambda}(x_n,x_{n+1}) = 0, \tag{30}$$

for all $n \in \mathbb{N}$ and $0 < \lambda < \lambda_0$. By setting $\lambda_1 = (1-k)\lambda_0 < \lambda_0$, we obtain

$$\omega_{k^n\lambda_1}(x_n,x_{n+1}) \leq k^n\omega_{\lambda_1}(x,fx) = k^n C,$$

for all $n \in \mathbb{N}$. By letting $n \to \infty$, we get

$$\lim_{n\to\infty} \omega_{k^n\lambda_1}(x_n,x_{n+1}) = 0, \tag{31}$$

for all $0 < \lambda_1 < \lambda_0$. Following the same procedure as in the proof of Theorem 2, we conclude that $\{x_n\}$ is an ω-Cauchy sequence in M^*_ω. By the f-orbitally ω-completeness of M^*_ω, there is some z in M^*_ω such that $\lim_n f^n x = z$. The orbital continuity of f implies that

$$fz = \lim_n ff^n x = z,$$

which shows that z is a fixed point of f. $\square$

Theorem 9. *Let ω be a modular on M and $f : M^*_\omega \to M^*_\omega$ be an orbitally continuous mapping on a modular space M^*_ω. Suppose that whenever $x \neq y$, f satisfies*

$$\frac{P(x,y) - Q(x,y)}{R(x,y)} < \omega_\lambda(x,y), \tag{32}$$

for all $\lambda > 0$, and $x, y \in M^*_\omega$; where

$$P(x,y) = \min\{\omega_\lambda(fx,fy)\omega_\lambda(x,y), \omega_\lambda(x,fx)\omega_\lambda(y,fy)\},$$
$$Q(x,y) = \min\{\omega_\lambda(x,fx)\omega_\lambda(x,fy), \omega_\lambda(y,fy)\omega_\lambda(y,fx)\},$$

and

$$R(x,y) = \min\{\omega_\lambda(x,fx), \omega_\lambda(y,fy)\},$$

such that $\omega_\lambda(x,fx) \neq 0, \omega_\lambda(y,fy) \neq 0$. If, for some $x \in M^*_\omega$, the sequence $\{f^n x\}_{n=1}^\infty$ has a limit point $z \in M^*_\omega$, then z is a fixed point of f.

Proof. If, for some $m \in \mathbb{N}$, $\omega_\lambda(x_{m-1}, x_m) = 0$, then $x_n = x_m = z$ for $n \geq m$, and the assertion holds. Suppose, then, that $\omega_\lambda(x_{m-1}, x_m) \neq 0$ for all $m \in \mathbb{N}$. Let $\lim_i x_{n_i} = z$. Then, by (32), for $x_{n-1}, x_n \in M^*_\omega$, we have

$$\frac{\min\{\omega_\lambda(x_n,x_{n+1})\omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_n,x_{n+1})\omega_\lambda(x_{n-1},x_n)\} - 0}{\min\{\omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_n,x_{n+1})\}} < \omega_\lambda(x_{n-1},x_n),$$

or

$$\frac{\omega_\lambda(x_n,x_{n+1})\omega_\lambda(x_{n-1},x_n)}{\min\{\omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_n,x_{n+1})\}} < \omega_\lambda(x_{n-1},x_n).$$

If $\min\{\omega_\lambda(x_{n-1},x_n), \omega_\lambda(x_n,x_{n+1})\} = \omega_\lambda(x_n,x_{n+1})$, then $\omega_\lambda(x_{n-1},x_n) < \omega_\lambda(x_{n-1},x_n)$ is impossible. Hence, we have $\omega_\lambda(x_n,x_{n+1}) < \omega_\lambda(x_{n-1},x_n)$ for all $\lambda > 0$ and $n \in \mathbb{N}$. Therefore, $\{\omega_\lambda(x_{n+1},x_n)\}_{n\in\mathbb{N}}$ is a decreasing, and hence convergent, sequence of real numbers. As $\lim_i \omega_\lambda(x_{n_i}, x_{n_i+1}) = \omega_\lambda(z,fz)$ and $\{\omega_\lambda(x_{n_i}, x_{n_i+1})\} \subseteq \{\omega_\lambda(x_n,x_{n+1})\}$, it follows that

$$\lim_n \omega_\lambda(x_n,x_{n+1}) = \omega_\lambda(z,fz). \tag{33}$$

Furthermore, as $\lim_i x_{n_i+1} = fz$, $\lim_i x_{n_i+2} = f^2 z$, and $\{\omega_\lambda(x_{n_i+1}, x_{n_i+2})\} \subseteq \{\omega_\lambda(x_n,x_{n+1})\}$, by (33), we have

$$\omega_\lambda(fz, f^2 z) = \omega_\lambda(z,fz). \tag{34}$$

If $\omega_\lambda(z,fz) > 0$, then (32) implies $\omega_\lambda(fz,f^2 z) < \omega_\lambda(z,fz)$, a contradiction. Hence, $\omega_\lambda(z,fz) = 0$; that is, $fz = z$. This completes the proof of the Theorem. $\square$

Remark 2. *The conclusion of Theorem 8 remains true if we replace condition (28) by*

$$\frac{A(x,y) - \min\{\omega_{k\lambda}(x,fy), \omega_{k\lambda}(y,fx)\}}{C(x,y)} \leq k\omega_{\lambda}(x,y),$$

and, similarly, the conclusion of Theorem 9 remains true if we replace condition (32) by

$$\frac{P(x,y) - \min\{\omega_{\lambda}(x,fy), \omega_{\lambda}(y,fx)\}}{R(x,y)} < \omega_{\lambda}(x,y).$$

6. Conclusions

Several generalizations of the concept of metric spaces have been introduced. Among them, modular metric spaces [31], partial metric spaces [39], extended b-metric spaces [40], and cone metric spaces [41] have been studied by the several researchers recently. Non-unique fixed points of Ćirić-type were investigated in extended b-metric spaces [19], partial metric spaces [23], and cone metric spaces [25]. This approach can be applied in several abstract spaces and has various applications in (fractional) differential equations and integral equations. Inspired by this work, we studied non-unique fixed points of Ćirić-type in modular metric spaces. We obtained various theorems about fixed points and periodic points for self-maps on modular spaces which are not necessarily continuous and satisfy certain contractive conditions. Our results unify and extend some existing results in the literature. The study of non-unique fixed points in the current context would be an interesting topic for future study.

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