symmetry

MDPI

Article

Shrinking Extragradient Method for Pseudomonotone Equilibrium Problems and Quasi-Nonexpansive Mappings

Manatchanok Khonchaliew [1], Ali Farajzadeh [2] and Narin Petrot [1,3,*]

[1] Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand; m.khonchaliew@gmail.com
[2] Department of Mathematics, Razi University, Kermanshah 67149, Iran; faraj1348@yahoo.com
[3] Centre of Excellence in Nonlinear Analysis and Optimization, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
* Correspondence: narinp@nu.ac.th

Received: 8 March 2019; Accepted: 29 March 2019; Published: 3 April 2019

Abstract: This paper presents two shrinking extragradient algorithms that can both find the solution sets of equilibrium problems for pseudomonotone bifunctions and find the sets of fixed points of quasi-nonexpansive mappings in a real Hilbert space. Under some constraint qualifications of the scalar sequences, these two new algorithms show strong convergence. Some numerical experiments are presented to demonstrate the new algorithms. Finally, the two introduced algorithms are compared with a standard, well-known algorithm.

Keywords: equilibrium problem; pseudomonotone bifunction; quasi-nonexpansive mapping; shrinking method

---

1. Introduction

The equilibrium problem started to gain interest after the publication of a paper by Blum and Oettli [1], which discussed the problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f : C \times C \to (-\infty, +\infty)$ is a bifunction. This well-known equilibrium model (1) has been used for studying a variety of mathematical models for physics, chemistry, engineering, and economics. In addition, the equilibrium problem (1) can be applied to many mathematical problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, and fixed point problems, see [1–4], and the references therein.

In order to solve the equilibrium problem (1), when f is a monotone bifunction, approximate solutions are frequently based on the proximal point method. That is, given x_k, at each step, the next iterate x_{k+1} can be found by solving the following regularized equilibrium problem:
find $x \in C$ such that

$$f(x, y) + \frac{1}{r_k}\langle y - x, x - x_k\rangle \geq 0, \forall y \in C, \tag{2}$$

where $\{r_k\} \subset (0, \infty)$. Note that the existence of each x_k is guaranteed, on condition that the subproblem (2) is a strongly monotone problem (see [5,6]). However, if f is a pseudomonotone bifunction (a property which is weaker than a monotone) the strong monotone-ness of the problem (2) cannot be guaranteed. Therefore, the sequence $\{x_k\}$ may not be well-defined. To overcome this drawback, Tran et al. [7] proposed the following extragradient method for solving the equilibrium problem, when the considered bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k = argmin\{\rho f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = argmin\{\rho f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \end{cases} \tag{3}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. Tran et al. guaranteed that the sequence $\{x_k\}$ generated by (3) converges weakly to a solution of the equilibrium problem (1).

On the other hand, for a nonempty closed convex subset C of H and a mapping $T : C \to C$, the fixed point problem is a problem of finding a point $x \in C$ such that $Tx = x$. This fixed point problem has many important applications, such as optimization problems, variational inequality problems, minimax problems, and saddle point problems, see [8–11], and the references therein. The set of fixed points of a mapping T will be represented by $Fix(T)$.

An iteration method for finding fixed points of the mapping T was proposed by Mann [12] as follows:

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T x_k, \end{cases} \tag{4}$$

where $\{\alpha_k\} \subset (0, 1)$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$. If T is a nonexpansive mapping and has a fixed point, then the sequence $\{x_k\}$ generated by (4) converges weakly to a fixed point of T. In addition, in 1994, Park and Jeong [13] showed that if T is a quasi-nonexpansive mapping with $I - T$ demiclosed at 0, then the sequence which is generated by the Mann iteration method converges weakly to a fixed point of T.

Furthermore, in order to obtain a strong convergence result for the Mann iteration method, Nakajo and Takahashi [14] proposed the following hybrid method:

$$\begin{cases} x_0 \in C, \\ y_k = \alpha_k x_k + (1 - \alpha_k)T x_k, \\ C_k = \{x \in C : \|y_k - x\| \leq \|x_k - x\|\}, \\ Q_k = \{x \in C : \langle x_0 - x_k, x - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases} \tag{5}$$

where $\{\alpha_k\} \subset [0, 1]$ such that $\alpha_k \leq 1 - \overline{\alpha}$ for some $\overline{\alpha} \in (0, 1]$, and $P_{C_k \cap Q_k}$ is the metric projection onto $C_k \cap Q_k$. Nakajo and Takahashi proved that if T is a nonexpansive mapping, then the sequence $\{x_k\}$ generated by (5) converges strongly to $P_{Fix(T)}(x_0)$.

In addition, in 1974, Ishikawa [15] proposed the following method for finding fixed points of a Lipschitz pseudocontractive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = (1 - \alpha_k)x_k + \alpha_k T x_k, \\ x_{k+1} = (1 - \beta_k)x_k + \beta_k T y_k, \end{cases} \tag{6}$$

where $0 \leq \beta_k \leq \alpha_k \leq 1$, $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k \beta_k = \infty$. If C is a convex compact subset of H, then the sequence $\{x_k\}$ generated by (6) converges strongly to fixed points of T. It has been previously shown that the Mann iteration method is generally not applicable for finding fixed points of a Lipschitz pseudocontractive mapping in a Hilbert space. For example, see [16].

In 2008, by using Ishikawa's iteration concept, Takahashi et al. [17] proposed the following hybrid method, called the shrinking projection method, which is different from Nakajo and Takahashi's method [14]:

$$\begin{cases} u_0 \in H, C_1 = C, \\ x_1 = P_{C_1}(u_0), \\ y_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ z_k = \beta_k x_k + (1-\beta_k)Ty_k, \\ C_{k+1} = \{x \in C_k : \|z_k - x\| \le \|x_k - x\|\}, \\ x_{k+1} = P_{C_{k+1}}(x_0), \end{cases} \tag{7}$$

where $\{\alpha_k\} \subset [\underline{\alpha}, \overline{\alpha}]$ with $0 < \underline{\alpha} \le \overline{\alpha} < 1$, and $\{\beta_k\} \subset [0, 1-\overline{\beta}]$ for some $\overline{\beta} \in (0,1)$. Takahashi et al. proved that if T is a nonexpansive mapping, then the sequence $\{x_k\}$ generated by (7) converges strongly to $P_{Fix(T)}(x_0)$.

In recent years, many algorithms have been proposed for finding a common element of the set of solutions of the equilibrium problem and the set of solutions of the fixed point problem. See, for instance, [8,11,18–23] and the references therein. In 2016, by using both hybrid and extragradient methods together in combination with Ishikawa's iteration concept, Dinh and Kim [24] proposed the following iteration method for finding a common element of fixed points of a symmetric generalized hybrid mapping T and the set of solutions of the equilibrium problem, when a bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k = argmin\{\rho_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = argmin\{\rho_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ t_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ u_k = \beta_k t_k + (1-\beta_k)Tz_k, \\ C_k = \{x \in H : \|x - u_k\| \le \|x - x_k\|\}, \\ Q_k = \{x \in H : \langle x - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k \cap C}(x_0), \end{cases} \tag{8}$$

where $\{\rho_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \le \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1-\overline{\beta}]$ for some $\overline{\beta} \in (0,1)$. Dinh and Kim proved that the sequence $\{x_k\}$ generated by (8) converges strongly to $P_{EP(f,C)\cap Fix(T)}(x_0)$, where $EP(f,C)$ is the solution set of the equilibrium problem.

Now, let us consider the problem of finding a common solution of a finite family of equilibrium problems (CSEP). Let C be a nonempty closed convex subset of H and let $f_i : C \times C \to (-\infty, +\infty)$, $i = 1, \ldots, N$, be bifunctions satisfying $f_i(x,x) = 0$ for each $x \in C$. The problem CSEP is to find $x^* \in C$ such that

$$f_i(x^*, y) \ge 0, \forall y \in C, i = 1, \ldots, N. \tag{9}$$

The solution set of the problem CSEP will be denoted by $\cap_{i=1}^N EP(f_i, C)$. It is worth pointing out that the problem CSEP is a generalization of many mathematical models, such as common solutions to variational inequality problems, convex feasibility problems and common fixed point problems. See [1,25–27] for more details. In 2016, Hieu et al. [28] considered the following problem:

$$\begin{cases} \text{find a point } x^* \in C \quad \text{such that} \quad T_j x^* = x^*, j = 1, \ldots, M, \\ \text{and} \quad f_i(x^*, y) \ge 0, \forall y \in C, i = 1, \ldots, N, \end{cases} \tag{10}$$

where C is a nonempty closed convex subset of H, $T_j : C \to C$, $j = 1, \ldots, M$, are mappings, and $f_i : C \times C \to (-\infty, +\infty)$, $i = 1, \ldots, N$, are bifunctions satisfying $f_i(x, x) = 0$ for each $x \in C$. From now on, the solution set of problem (10) will be denoted by S. That is:

$$S := (\cap_{j=1}^{M} Fix(T_j)) \cap (\cap_{i=1}^{N} EP(f_i, C)).$$

By using both hybrid and extragradient methods together in combination with Mann's iteration concept and parallel splitting-up techniques (see [25,29]), they proposed the following algorithm for finding the solution set of problem (10), when mappings are nonexpansive, and bifunctions are pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k^i = argmin\{\rho f_i(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \ldots, N, \\ z_k^i = argmin\{\rho f_i(y_k^i, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \ldots, N, \\ \bar{z}_k = argmax\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}, \\ u_k^j = \alpha_k x_k + (1 - \alpha_k) T_j \bar{z}_k, j = 1, 2, \ldots, M, \\ \bar{u}_k = argmax\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}, \\ C_k = \{x \in C : \|x - \bar{u}_k\| \leq \|x - x_k\|\}, \\ Q_k = \{x \in C : \langle x - x_k, x_0 - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases} \tag{11}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, and $\{\alpha_k\} \subset (0, 1)$ such that $\limsup_{k\to\infty} \alpha_k < 1$. Hieu et al. proved that the sequence $\{x_k\}$ generated by (PHMEM) converges strongly to $P_S(x_0)$. The algorithm (11) is called PHMEM method.

The current study will continue developing methods for finding the solution set of problem (10). Roughly speaking, some new iterative algorithms will be introduced for finding the solution set of problem (10). Some numerical examples will be considered and the introduced methods will be discussed and compared with the PHMEM algorithm.

This paper is organized as follows: In Section 2, some relevant definitions and properties will be reviewed for use in subsequent sections. Section 3 will present two shrinking extragradient algorithms and prove their convergence. Finally, in Section 4, the performance of the introduced algorithms will be compared to the performance of the PHMEM algorithm and discussed.

2. Preliminaries

This section will present some definitions and properties that will be used subsequently. First, let H be a real Hilbert space induced by the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The symbols $\to$ and $\rightharpoonup$ will be used here to denote the strong convergence and the weak convergence in H, respectively.

Now, recalled here are definitions of nonlinear mappings related to this work.

Definition 1 ([30,31])**.** *Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be:*

(i) pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator on H.

(ii) Lipschitzian if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$, then T is said to be nonexpansive.

(iii) quasi-nonexpansive if $Fix(T)$ is nonempty and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in Fix(T).$$

(iv) $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid if there exists $\alpha, \beta, \gamma, \delta \in (-\infty, +\infty)$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|y - Tx\|^2) + \gamma\|x - y\|^2$$

$$+\delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0, \forall x, y \in C.$$

Definition 2. *(see [32]) Let C be a nonempty closed convex subset of H and $T : C \to H$ be a mapping. The mapping T is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\} \subset C$ with $x_k \rightharpoonup x^* \in C$ and $Tx_k \to y$ imply $Tx^* = y$.*

Note that the class of pseudocontractive mappings includes the class of nonexpansive mappings. In addition, a nonexpansive mapping with at least one fixed point is a quasi-nonexpansive mapping, but the converse is not true. For example, see [33]. Moreover, if a $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping satisfies (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta > 0$ and (3) $\delta \geq 0$ then T is quasi-nonexpansive and $I - T$ demiclosed at 0 (see [34,35]). Moreover, $Fix(T)$ is closed and convex when T is a quasi-nonexpansive mapping (see [36]).

Next, we recall definitions and facts for considering the equilibruim problems.

Definition 3 ([1,4,37])**.** *Let C be a nonempty closed convex subset of H and $f : C \times C \to (-\infty, +\infty)$ be a bifunction. The bifunction f is said to be:*

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \forall x, y \in C;$$

(ii) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\forall x, y \in C, f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

(iv) Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \forall x, y, z \in C.$$

Remark 1. *From Definition 3, we observe that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). However, if f is pseudomonotone, f might not be monotone on C. For example, see [38].*

For a nonempty closed convex subset C of H and a bifunction $f : C \times C \to (-\infty, +\infty)$ satisfying $f(x, x) = 0$ for each $x \in C$. In this paper, we are concerned with the following assumptions:

$(A1)$ f is weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_k\}$, $\{y_k\}$ are two sequences in C converge weakly to x and y respectively, then $f(x_k, y_k)$ converges to $f(x, y)$;

$(A2)$ $f(x, \cdot)$ is convex and subdifferentiable on C for each fixed $x \in C$;

$(A3)$ f is psuedomonotone on C;

$(A4)$ f is Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

It is well-known that the solution set $EP(f, C)$ is closed and convex, when the bifunction f satisfies the assumptions $(A1) - (A3)$. See, for instance, [7,39,40].

The following facts are very important in order to obtain our main results.

Lemma 1 ([18]). *Let $f : C \times C \to (-\infty, +\infty)$ be satisfied $(A2)-(A4)$. If $EP(f, C)$ is nonempty set and $0 < \rho_0 < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. Let $x_0 \in C$. If y_0 and z_0 are defined by*

$$\begin{cases} y_0 = \arg\min\{\rho_0 f(x_0, w) + \frac{1}{2}\|w - x_0\|^2 : w \in C\}, \\ z_0 = \arg\min\{\rho_0 f(y_0, w) + \frac{1}{2}\|w - x_0\|^2 : w \in C\}, \end{cases}$$

then,

(i) $\rho_0\, [f(x_0, w) - f(x_0, y_0)] \geq \langle y_0 - x_0, y_0 - w\rangle$, *for all* $w \in C$;

(ii) $\|z_0 - q\|^2 \leq \|x_0 - q\|^2 - (1 - 2\rho_0 L_1)\|x_0 - y_0\|^2 - (1 - 2\rho_0 L_2)\|y_0 - z_0\|^2$, *for all* $q \in EP(f, C)$.

This section will be closed by recalling the projection mapping and calculus concepts in Hilbert space.

Let C be a nonempty closed convex subset of H. For each $x \in H$, we denote the metric projection of x onto C by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|y - x\|, \forall y \in C.$$

The following facts will also be used in this paper.

Lemma 2. *(see, for instance, [41,42]) Let C be a nonempty closed convex subset of H. Then*

(i) $P_C(x)$ *is singleton and well-defined for each* $x \in H$;
(ii) $z = P_C(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;
(iii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \forall x, y \in C$.

For a nonempty closed convex subset C of H and a convex function $g : C \to \mathbb{R}$, the subdifferential of g at $z \in C$ is defined by

$$\partial g(z) = \{w \in C : g(y) - g(z) \geq \langle w, y - z\rangle, \forall y \in C\}.$$

The function g is said to be subdifferentiable at z if $\partial g(z) \neq \emptyset$.

3. Main Result

In this section, we propose two shrinking extragradient algorithms for finding a solution of problem (10), when each mapping T_j, $j = 1, 2, \ldots, M$, is quasi-nonexpansive with $I - T_j$ demiclosed at 0, and each bifunction f_i, $i = 1, 2, \ldots, N$, satisfies all the assumptions $(A1)-(A4)$. We start by observing that if each bifunction f_i, $i = 1, 2, \ldots, N$, is Lipshitz-type continuous on C with constants $L_1^i > 0$ and $L_2^i > 0$, then

$$\begin{aligned} f_i(x, y) + f_i(y, z) &\geq f_i(x, z) - L_1^i\|x - y\|^2 - L_2^i\|y - z\|^2 \\ &\geq f_i(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \end{aligned}$$

where $L_1 = max\{L_1^i : i = 1, 2, \ldots, N\}$ and $L_2 = max\{L_2^i : i = 1, 2, \ldots, N\}$. This means the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$. Of course, we will use this notation in this paper. Moreover, for each $N \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we denote $[k]_N$ for a modulo function at k with respect to N, that is

$$[k]_N = k(mod\ N) + 1.$$

Now, we propose a following cyclic algorithm.

CSEM Algorithm (Cyclic Shrinking Extragradient Method)

Initialization. Pick $x_0 \in C =: C_0$, choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\}$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$.

Step 1. Solve the strongly convex program

$$y_k = argmin\{\rho_k f_{[k]_N}(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 2. Solve the strongly convex program

$$z_k = argmin\{\rho_k f_{[k]_N}(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 3. Compute

$$t_k = \alpha_k x_k + (1 - \alpha_k) T_{[k]_M} x_k,$$
$$u_k = \beta_k t_k + (1 - \beta_k) T_{[k]_M} z_k.$$

Step 4. Construct closed convex subset of C:

$$C_{k+1} = \{x \in C_k : \|x - u_k\| \leq \|x - x_k\|\}.$$

Step 5. The next approximation x_{k+1} is defined as the projection of x_0 onto C_{k+1}, i.e.,

$$x_{k+1} = P_{C_{k+1}}(x_0).$$

Step 6. Put $k = k + 1$ and go to **Step 1**.

Before going to prove the strong convergence of CSEM Algorithm, we need the following lemma.

Lemma 3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm is well-defined.*

Proof. To prove the Lemma, it suffices to show that C_k is a nonempty closed and convex subset of H, for each $k \in \mathbb{N} \cup \{0\}$. Firstly, we will show the non-emptiness by showing that $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Obviously, $S \subset C_0$.

Now, let $q \in S$. Then, by Lemma 1 (ii), we have

$$\|z_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k L_1)\|x_k - y_k\|^2 - (1 - 2\rho_k L_2)\|y_k - z_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|z_k - q\| \leq \|x_k - q\|, \tag{12}$$

for each $k \in \mathbb{N} \cup \{0\}$. On the other hand, since $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$) and the definitions of t_k, u_k that

$$\begin{aligned} \|t_k - q\| &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|T_{[k]_M} x_k - q\| \\ &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \|u_k - q\| &\leq \beta_k\|t_k - q\| + (1 - \beta_k)\|T_{[k]_M} z_k - q\| \\ &\leq \beta_k\|t_k - q\| + (1 - \beta_k)\|z_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. The relations (12) and (13) imply that

$$\begin{aligned} \|u_k - q\| &\leq \beta_k \|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{14}$$

for each $k \in \mathbb{N} \cup \{0\}$. Now, suppose that $S \subset C_k$. Thus, by using (14), we see that $S \subset C_{k+1}$. So, by induction, we have $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Since S is a nonempty set, we obtain that C_k is a nonempty set, for each $k \in \mathbb{N} \cup \{0\}$.

Next, we show that C_k is a closed and convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Note that we already have that C_0 is a closed and convex subset. Now, suppose that C_k is a closed and convex subset, we will show that C_{k+1} is likewise. To do this, let us consider a set $B_k = \{x \in H : \|x - u_k\| \leq \|x - x_k\|\}$. We see that

$$B_k = \{x \in H : \langle x_k - u_k, x \rangle \leq \frac{1}{2}(\|x_k\|^2 - \|u_k\|^2)\}.$$

This means that B_k is a halfspace and $C_{k+1} = C_k \cap B_k$. Thus, C_{k+1} is a closed and convex subset. Thus, by induction, we can conclude that C_k is a closed and convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Consequently, we can guarantee that $\{x_k\}$ is well-defined. □

Theorem 1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of x_{k+1}, we observe that $x_{k+1} \in C_{k+1} \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k = P_{C_k}(x_0)$ and $x_{k+1} \in C_k$, we have

$$\|x_k - x_0\| \leq \|x_{k+1} - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. This means that $\{\|x_k - x_0\|\}$ is a nondecreasing sequence. Similarly, for each $q \in S \subset C_{k+1}$, we obtain that

$$\|x_{k+1} - x_0\| \leq \|q - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. By the above inequalities, we get

$$\|x_k - x_0\| \leq \|q - x_0\|, \tag{15}$$

for each $k \in \mathbb{N} \cup \{0\}$. So $\{\|x_k - x_0\|\}$ is a bounded sequence. Consequently, we can conclude that $\{\|x_k - x_0\|\}$ is a convergent sequence. Moreover, we see that $\{x_k\}$ is bounded. Thus, in view of (13) and (14), we get that $\{t_k\}$ and $\{u_k\}$ are also bounded. Suppose $k, j \in \mathbb{N} \cup \{0\}$ such that $k > j$. It follows that $x_k \in C_k \subset C_j$. Then, by Lemma 2 (iii), we have

$$\|P_{C_j}(x_k) - P_{C_j}(x_0)\|^2 \leq \|x_0 - x_k\|^2 - \|P_{C_j}(x_k) - x_k + x_0 - P_{C_j}(x_0)\|^2.$$

Consequently,

$$\|x_k - x_j\|^2 \leq \|x_0 - x_k\|^2 - \|x_j - x_0\|^2.$$

Thus, by using the existence of $\lim_{k\to\infty} \|x_k - x_0\|$, we get

$$\lim_{k,j\to\infty} \|x_k - x_j\| = 0.$$

That is $\{x_k\}$ is a Cauchy sequence in C. Since C is closed, there exists $p \in C$ such that

$$\lim_{k\to\infty} x_k = p. \tag{16}$$

By the definition of C_{k+1} and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - u_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|u_k - x_k\| &\leq \|u_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{17}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $x_{k+1} \to p$, as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0.$$

This together with (17) imply that

$$\lim_{k\to\infty} \|u_k - x_k\| = 0. \tag{18}$$

Since $\lim_{k\to\infty} \alpha_k = 1$ and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), it follows that

$$\begin{aligned} \lim_{k\to\infty} \|t_k - x_k\| &= \lim_{k\to\infty} \|\alpha_k x_k + (1-\alpha_k)T_{[k]_M} x_k - x_k\| \\ &= \lim_{k\to\infty} (1-\alpha_k)\|x_k - T_{[k]_M} x_k\| \\ &= 0. \end{aligned} \tag{19}$$

Consider,

$$\begin{aligned} \|u_k - q\|^2 &= \|\beta_k(t_k - q) + (1-\beta_k)(T_{[k]_M} z_k - q)\|^2 \\ &= \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2 - \beta_k(1-\beta_k)\|t_k - T_{[k]_M} z_k\|^2 \\ &\leq \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. By using (13) and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we obtain

$$\|u_k - q\|^2 \leq \beta_k\|x_k - q\|^2 + (1-\beta_k)\|z_k - q\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by Lemma 1 (ii), we have

$$\begin{aligned} \|u_k - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k L_1)\|x_k - y_k\|^2 - (1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\leq \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2], \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$(1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2] \leq \|x_k - u_k\|(\|x_k - q\| + \|u_k - q\|), \tag{20}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (18) and the choices of $\{\beta_k\}$, $\{\rho_k\}$, we have

$$\lim_{k\to\infty} \|x_k - y_k\| = 0, \tag{21}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{22}$$

These imply that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{23}$$

Then, by $\lim_{k\to\infty} x_k = p$, we also have

$$\lim_{k\to\infty} y_k = p, \tag{24}$$

and

$$\lim_{k\to\infty} z_k = p. \tag{25}$$

Next, we claim that $p \in S$. From the definition of u_k, we see that

$$\begin{aligned} (1-\beta_k)\|T_{[k]_M} z_k - z_k\| &= \|u_k - z_k - \beta_k(t_k - z_k)\| \\ &\leq \|u_k - z_k\| + \beta_k\|t_k - z_k\| \\ &\leq \|u_k - x_k\| + \beta_k\|t_k - x_k\| + (1+\beta_k)\|x_k - z_k\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using (18), (19) and (23), we have

$$\lim_{k\to\infty} \|T_{[k]_M} z_k - z_k\| = 0. \tag{26}$$

Furthermore, for each fixed $j \in \{1, 2, \ldots, M\}$, we observe that

$$[(j-1) + kM]_M = j,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, it follows from (26) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|T_{[(j-1)+kM]_M} z_{(j-1)+kM} - z_{(j-1)+kM}\| \\ &= \lim_{k\to\infty} \|T_j z_{(j-1)+kM} - z_{(j-1)+kM}\|, \end{aligned} \tag{27}$$

for each $j \in \{1, 2, \ldots, M\}$. Since $z_k \to p$, as $k \to \infty$, then for each $j \in \{1, 2, \ldots, M\}$, we get $z_{(j-1)+kM} \to p$, as $k \to \infty$. Combining with (27), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

Similarly, for each fixed $i \in \{1, 2, \ldots, N\}$, we note that

$$[(i-1) + kN]_N = i,$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $y_k \to p$, as $k \to \infty$, then for each $i \in \{1, 2, \ldots, N\}$, we have $x_{(i-1)+kN} \to p$ and $y_{(i-1)+kN} \to p$, as $k \to \infty$. By Lemma 1 (i), for each $i \in \{1, 2, \ldots, N\}$, we obtain

$$\begin{aligned} &\rho_{(i-1)+kN}[f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y) - f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y_{(i-1)+kN})] \\ &\geq \langle y_{(i-1)+kN} - x_{(i-1)+kN}, y_{(i-1)+kN} - y\rangle, \forall y \in C. \end{aligned}$$

It follows that, for each $i \in \{1, 2, \ldots, N\}$, we have

$$\begin{aligned} &f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y) - f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y_{(i-1)+kN}) \\ &\geq -\frac{1}{\rho_{(i-1)+kN}}\|y_{(i-1)+kN} - x_{(i-1)+kN}\|\|y_{(i-1)+kN} - y\|, \forall y \in C. \end{aligned}$$

By using (21) and weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$), we get that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Then, we had shown that $p \in S$.

Finally, we will show that $p = P_S(x_0)$. In fact, since $P_S(x_0) \in S$, it follows from (15) that

$$\|x_k - x_0\| \leq \|P_S(x_0) - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the continuity of norm and $\lim_{k\to\infty} x_k = p$, we see that

$$\|p - x_0\| = \lim_{k\to\infty} \|x_k - x_0\| \leq \|P_S(x_0) - x_0\|.$$

Thus, by the definition of $P_S(x_0)$ and $p \in S$, we obtain that $p = P_S(x_0)$. This completes the proof. □

Next, by replacing cyclic method by parallel method, we propose the following algorithm.

PSEM Algorithm (Parallel Shrinking Extragradient Method)

Initialization. Pick $x_0 \in C =: C_0$, choose parameters $\{\rho_k^i\}$ with $0 < \inf \rho_k^i \leq \sup \rho_k^i < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}, i = 1, 2, \ldots, N$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\}$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$.

Step 1. Solve N strongly convex programs

$$y_k^i = argmin\{\rho_k^i f_i(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 2. Solve N strongly convex programs

$$z_k^i = argmin\{\rho_k^i f_i(y_k^i, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 3. Find the farthest element from x_k among z_k^i, $i = 1, 2, \ldots, N$, i.e.,

$$\bar{z}_k = argmax\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}.$$

Step 4. Compute

$$t_k^j = \alpha_k x_k + (1 - \alpha_k) T_j x_k, j = 1, 2, \ldots, M,$$
$$u_k^j = \beta_k t_k^j + (1 - \beta_k) T_j \bar{z}_k, j = 1, 2, \ldots, M.$$

Step 5. Find the farthest element from x_k among u_k^j, $j = 1, 2, \ldots, M$, i.e.,

$$\bar{u}_k = argmax\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}.$$

Step 6. Construct closed convex subset of C:

$$C_{k+1} = \{x \in C_k : \|x - \bar{u}_k\| \leq \|x - x_k\|\}.$$

Step 7. The next approximation x_{k+1} is defined as the projection of x_0 onto C_{k+1}, i.e.,

$$x_{k+1} = P_{C_{k+1}}(x_0).$$

Step 8. Put $k = k + 1$ and go to **Step 1**.

Theorem 2. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PSEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of $\bar{z}_k$, we suppose that $i_k \in \{1, 2, \ldots, N\}$ such that $z_k^{i_k} = \bar{z}_k = argmax\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}$. Then, by Lemma 1 (ii), we have

$$\|\bar{z}_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k^{i_k} L_1)\|x_k - y_k^{i_k}\|^2 - (1 - 2\rho_k^{i_k} L_2)\|y_k^{i_k} - \bar{z}_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|\bar{z}_k - q\| \leq \|x_k - q\|, \tag{28}$$

for each $k \in \mathbb{N} \cup \{0\}$. On the other hand, by the definition of t_k^j and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we have

$$\begin{aligned} \|t_k^j - q\| &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|T_j x_k - q\| \\ &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{29}$$

for each $k \in \mathbb{N} \cup \{0\}$. Additionally, by the definition of $\bar{u}_k$, we suppose that $j_k \in \{1, 2, \ldots, M\}$ such that $u_k^{j_k} = \bar{u}_k = argmax\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}$. It follows from the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$) that

$$\begin{aligned} \|\bar{u}_k - q\| &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|T_{j_k} \bar{z}_k - q\| \\ &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|\bar{z}_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. The relations (28) and (29) imply that

$$\begin{aligned} \|\bar{u}_k - q\| &\leq \beta_k \|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{30}$$

for each $k \in \mathbb{N} \cup \{0\}$. Following the proof of Lemma 3 and Theorem 1, we can show that C_k is a closed convex subset of H and $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Moreover, we can check that the sequence $\{x_k\}$ is a convergent sequence, say

$$\lim_{k \to \infty} x_k = p, \tag{31}$$

for some $p \in C$.

By the definition of C_{k+1} and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - \bar{u}_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|\bar{u}_k - x_k\| &\leq \|\bar{u}_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{32}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $x_{k+1} \to p$, as $k \to \infty$, we obtain that

$$\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0.$$

This together with (32) implies that

$$\lim_{k \to \infty} \|\bar{u}_k - x_k\| = 0.$$

Then, by the definition of $\overline{u}_k$, we have

$$\lim_{k\to\infty} \|u_k^j - x_k\| = 0, \tag{33}$$

for each $j = 1, 2, \ldots, M$. Since $\lim_{k\to\infty} \alpha_k = 1$ and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), it follows that

$$\begin{aligned} \lim_{k\to\infty} \|t_k^j - x_k\| &= \lim_{k\to\infty} \|\alpha_k x_k + (1-\alpha_k) T_j x_k - x_k\| \\ &= \lim_{k\to\infty} (1-\alpha_k)\|x_k - T_j x_k\| \\ &= 0, \end{aligned} \tag{34}$$

for each $j = 1, 2, \ldots, M$. Beside, by the definition of u_k^j , for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned} \|u_k^j - q\|^2 &= \|\beta_k(t_k^j - q) + (1-\beta_k)(T_j\overline{z}_k - q)\|^2 \\ &= \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2 - \beta_k(1-\beta_k)\|t_k^j - T_j\overline{z}_k\|^2 \\ &\leq \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (29) and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we have

$$\|u_k^j - q\|^2 \leq \beta_k\|x_k - q\|^2 + (1-\beta_k)\|\overline{z}_k - q\|^2,$$

for $k \in \mathbb{N} \cup \{0\}$. So, by Lemma 1 (ii), for each $j = 1, 2, \ldots, M$, we get that

$$\begin{aligned} \|u_k^j - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 - (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\ &= \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2], \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $j = 1, 2, \ldots, M$, we have

$$\begin{aligned} &(1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\ \leq\ &\|x_k - q\|^2 - \|u_k^j - q\|^2 \\ =\ &\|x_k - u_k^j\|(\|x_k - q\| + \|u_k^j - q\|), \end{aligned} \tag{35}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (33) and the choices of $\{\beta_k\}$, $\{\rho_k^i\}$, we see that

$$\lim_{k\to\infty} \|x_k - y_k^{i_k}\| = 0, \tag{36}$$

and

$$\lim_{k\to\infty} \|y_k^{i_k} - \overline{z}_k\| = 0. \tag{37}$$

These imply that

$$\lim_{k\to\infty} \|x_k - \overline{z}_k\| = 0. \tag{38}$$

Then, by the definition of $\overline{z}_k$, we have

$$\lim_{k\to\infty} \|x_k - z_k^i\| = 0, \tag{39}$$

for each $i = 1, 2, \ldots, N$. Moreover, by Lemma 1 (ii), for each $i = 1, 2, \ldots, N$, we get that

$$\|z_k^i - q\|^2 \leq \|x_k - q\|^2 - (1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 - (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}(1 - 2\rho_k^i L_1)\|x_k - y_k^i\|^2 + (1 - 2\rho_k^i L_2)\|y_k^i - z_k^i\|^2 &\leq \|x_k - q\|^2 - \|z_k^i - q\|^2 \\ &= \|x_k - z_k^i\|(\|x_k - q\| + \|z_k^i - q\|),\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Combining with (39) implies that

$$\lim_{k\to\infty} \|x_k - y_k^i\| = 0, \tag{40}$$

and

$$\lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \tag{41}$$

for each $i = 1, 2, \ldots, N$. Thus, by using (38), (40) and $\lim_{k\to\infty} x_k = p$, we have

$$\lim_{k\to\infty} \bar{z}_k = p, \tag{42}$$

and

$$\lim_{k\to\infty} y_k^i = p, \tag{43}$$

for each $i = 1, 2, \ldots, N$.

Next, we claim that $p \in S$. From the definition of u_k^j, for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned}(1 - \beta_k)\|T_j\bar{z}_k - \bar{z}_k\| &= \|u_k^j - \bar{z}_k - \beta_k(t_k^j - \bar{z}_k)\| \\ &\leq \|u_k^j - \bar{z}_k\| + \beta_k\|t_k^j - \bar{z}_k\| \\ &\leq \|u_k^j - x_k\| + \beta_k\|t_k^j - x_k\| + (1 + \beta_k)\|x_k - \bar{z}_k\|,\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, in view of (33), (34), and (38), we get that

$$\lim_{k\to\infty} \|T_j\bar{z}_k - \bar{z}_k\| = 0, \tag{44}$$

for each $j = 1, 2, \ldots, M$. Combining with (42), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

On the other hand, by Lemma 1 (i), for each $i = 1, 2, \ldots, N$, we see that

$$\rho_k^i[f_i(x_k, y) - f_i(x_k, y_k^i)] \geq \langle y_k^i - x_k, y_k^i - y\rangle, \forall y \in C.$$

It follows that, for each $i = 1, 2, \ldots, N$, we get

$$f_i(x_k, y) - f_i(x_k, y_k^i) \geq -\frac{1}{\rho_k^i}\|y_k^i - x_k\|\|y_k^i - y\|, \forall y \in C.$$

By using (31), (40), (43) and weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$), we have

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Thus, we can conclude that $p \in S$. The rest of the proof is similar to the arguments in the proof of Theorem 1, and it leads to the conclusion that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. □

Remark 2. *We note that for the PSEM algorithm we solve y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions and compute t_k^j, u_k^j, $j = 1, 2, \ldots, M$, by using M mappings. The farthest elements from x_k among all z_k^i and u_k^j are chosen for the next step calculation. However, we solve only y_k, z_k, by using a bifunction and compute only t_k, u_k, by using a mapping for the CSEM algorithm. After that, we construct closed convex subset C_{k+1}, and the approximation x_{k+1} is the projection of x_0 onto C_{k+1} for both algorithms. We claim that the numbers of iterations of the PSEM algorithm should be less than the CSEM algorithm. However, the computational times of the CSEM algorithm should be less than the PSEM algorithm for sufficiently large N, M. On the other hand, for the PHMEM algorithm they solved y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions, and computed u_k^j, $j = 1, 2, \ldots, M$, by using M mappings. The farthest elements from x_k among all z_k^i and u_k^j are chosen similar to the PSEM algorithm. However, they constructed two closed convex subsets C_k, Q_k, and the approximation x_{k+1} is the projection of x_0 onto $C_k \cap Q_k$, which is difficult to compute. We will focus on these observations in the next section.*

4. A Numerical Experiment

This section will compare the two introduced algorithms, CSEM and PSEM, with the PHMEM algorithm, which was presented in [28]. The following setting is taken from Hieu et al. [28]. Let $H = \mathbb{R}$ be a Hilbert space with the standard inner product $\langle x, y \rangle = xy$ and the norm $\|x\| = |x|$, for each $x, y \in H$. To be considered here are the nonexpansive self-mappings T_j, $j = 1, 2, \ldots, M$, and the bifunctions f_i, $i = 1, 2, \ldots, N$, which are given on $C = [0, 1]$ by

$$T_j(x) = \frac{x^j \sin^{j-1}(x)}{2j-1}, \quad j = 1, 2, \ldots, M,$$

and

$$f_i(x, y) = B_i(x)(y - x), \quad i = 1, 2, \ldots, N,$$

where $B_i(x) = 0$ if $0 \le x \le \xi_i$, and $B_i(x) = e^{x-\xi_i} + \sin(x - \xi_i) - 1$ if $\xi_i < x \le 1$. Moreover, $0 < \xi_1 < \xi_2 < \ldots < \xi_N < 1$. Then, the bifunctions f_i, $i = 1, 2, \ldots, N$, satisfy conditions $(A1)-(A4)$ (see [28]). Indeed, the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipshitz-type continuous with constants $L_1 = L_2 = 2$. Note that the solution set S is nonempty because $0 \in S$.

The following numerical experiment is considered with these parameters: $\rho_k = \frac{1}{5}$, $\xi_{[k]_N} = \frac{[k]_N}{N+1}$ for the CSEM algorithm; $\rho_k^i = \frac{1}{5}$, $\xi_i = \frac{i}{N+1}$, $i = 1, 2, \ldots, N$ for the PSEM algorithm, when $N = 1000$ and $M = 2000$. The following six cases of the parameters α_k and β_k are considered:

Case 1. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = \dfrac{1}{k+2}$.

Case 2. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 3. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = 0.99 - \dfrac{1}{k+2}$.

Case 4. $\alpha_k = 1$, $\beta_k = \dfrac{1}{k+2}$.

Case 5. $\alpha_k = 1$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 6. $\alpha_k = 1$, $\beta_k = 0.99 - \dfrac{1}{k+2}$.

The experiment was written in Matlab R2015b and performed on a PC desktop with Intel(R) Core(TM) i3-3240 CPU @ 3.40GHz 3.40GHz and RAM 4.00 GB. The function $fmincon$ in Matlab Optimization Toolbox was used to solve vectors y_k, z_k for the CSEM algorithm; y_k^i, z_k^i, $i = 1, 2, \ldots, N$, for the PSEM algorithm. The set C_{k+1} was computed by using the function *solve* in Matlab Symbolic

Math Toolbox. One can see that the set C_{k+1} is the interval $[a, b]$, where $a, b \in [0, 1]$, $a \leq b$. Consequently, the metric projection of a point x_0 onto the set C_{k+1} was computed by using this form

$$P_{C_{k+1}}(x_0) = \max\{\min\{x_0, b\}, a\},$$

see [41]. The CSEM and PSEM algorithms were tested along with the PHMEM algorithm by using the stopping criteria $|x_{k+1} - x_k| < 10^{-4}$ and the results below were presented as averages calculated from four starting points: x_0 at 0.01, 0.25, 0.75 and 1.

Table 1 shows that the parameter $\beta_k = \dfrac{1}{k+2}$ yields faster computational times and fewer computational iterations than other cases. Compare cases 1–3 with each other and cases 4–6 with each other. Meanwhile, the parameter $\alpha_k = 1$, in which the Ishikawa iteration reduces to the Mann iteration, yields slower computational times and more computational iterations than the other case. Compare cases 1 with 4, 2 with 5, and 3 with 6. Moreover, the computational times of the CSEM algorithm are faster than other algorithms, while the computational iterations of the PSEM algorithm are fewer than or equal to other algorithms. Finally, we see that both computational times and iterations of the CSEM and PSEM algorithms are better than or equal to those of the PHMEM algorithm.

Table 1. Numerical results for six different cases of parameters α_k and β_k.

	Average Times (sec)			Average Iterations		
Cases	**CSEM**	**PSEM**	**PHMEM**	**CSEM**	**PSEM**	**PHMEM**
1	4.905197	165.099794	173.347257	14.25	13.75	14.25
2	7.326055	287.918141	345.025914	25.25	24.25	28.25
3	20.371064	834.001035	2004.693844	91.25	74.25	177
4	5.079676	173.091716	173.347257	14.75	14.25	14.25
5	8.016109	342.870819	345.025914	28.75	28.25	28.25
6	42.035240	1986.147273	2004.693844	200	177	177

Remark 3. *Let us consider the case of parameters $\alpha_k = 1$ and $\beta_k = 0$, in which the Ishikawa iteration will be reduced to the Picard iteration. We notice that the convergence of PHMEM algorithm cannot be guaranteed in this setting. The computational results of the CSEM and PSEM algorithms are shown as follows.*

From Table 2, we see that both computational times and iterations are better than all those cases presented in Table 1. However, it should be warned that the Picard iteration method may not always converge to a fixed point of a nonexpansive mapping in general. For example, see [43].

Table 2. Numerical results for parameters $\alpha_k = 1$ and $\beta_k = 0$.

Average Times (sec)		Average Iterations	
CSEM	**PSEM**	**CSEM**	**PSEM**
4.657696	137.200812	12.50	11.50

5. Conclusions

We introduce the methods for finding a common element of the set of fixed points of a finite family for quasi-nonexpansive mappings and the solution set of equilibrium problems of a finite family for pseudomonotone bifunctions in a real Hilbert space. In fact, we consider both extragradient and shrinking projection methods together in combination with Ishikawa's iteration concept for introducing a sequence which is strongly convergent to a common solution of the considered problems. Some numerical experiments are also provided and discussed. For the future research direction, the convergence analysis of the proposed algorithms and some practical applications should be considered and implemented.

Author Contributions: Conceptualization, M.K., A.F. and N.P.; methodology, M.K., A.F. and N.P.; formal analysis, M.K., A.F. and N.P.; investigation, M.K., A.F. and N.P.; writing—original draft preparation, M.K., A.F. and N.P.; writing—review and editing, M.K., A.F. and N.P.; funding acquisition, N.P.

Funding: This research is patially supported by Faculty of Science, Naresuan University.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Blum, E.; Oettli, W. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1994**, *63*, 127–149.
2. Bigi, G.; Castellani, M.; Pappalardo, M.; Passacantando, M. Existence and solution methods for equilibria. *Eur. J. Oper. Res.* **2013**, *227*, 1–11. [CrossRef]
3. Daniele, P.; Giannessi, F.; Maugeri, A. *Equilibrium Problems and Variational Models*; Kluwer: Dordrecht, The Netherlands, 2003.
4. Muu, L.D.; Oettli, W. Convergence of an adaptive penalty scheme for finding constrained equilibria. *Nonlinear Anal. TMA* **1992**, *18*, 1159–1166. [CrossRef]
5. Combettes, P.L.; Hirstoaga, A. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.* **2005**, *6*, 117–136.
6. Moudafi, A. Proximal point algorithm extended to equilibrium problems. *J. Nat. Geom.* **1999**, *15*, 91–100.
7. Tran, D.Q.; Dung, L.M.; Nguyen, V.H. Extragradient algorithms extended to equilibrium problems. *Optimization* **2008**, *57*, 749–776. [CrossRef]
8. Ahn, P.N. A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems. *Bull. Malays. Math. Sci. Soc.* **2013**, *36*, 107–116.
9. Ansari, Q.H.; Nimana, N.; Petrot, N. Split hierarchical variational inequality problems and related problems. *Fixed Point Theory Appl.* **2014**, *2014*, 208. [CrossRef]
10. Iiduka, H. Convex optimization over fixed point sets of quasi-nonexpansive and nonexpansive mappings in utility-based bandwidth allocation problems with operational constraints. *J. Comput. Appl. Math.* **2015**, *282*, 225–236. [CrossRef]
11. Moradlou, F.; Alizadeh, S. Strong convergence theorem by a new iterative method for equilibrium problems and symmetric generalized hybrid mappings. *Mediterr. J. Math.* **2016**, *13*, 379–390. [CrossRef]
12. Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
13. Park, J.Y.; Jeong, J.U. Weak convergence to a fixed point of the sequence of Mann type iterates. *J. Math. Anal. Appl.* **1994**, *184*, 75–81. [CrossRef]
14. Nakajo, K.; Takahashi, W. Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *J. Math. Anal. Appl.* **2003**, *279*, 372–379. [CrossRef]
15. Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **1974**, *40*, 147–150. [CrossRef]
16. Chidume, C.E.; Mutangadura, S.A. An example of the Mann iteration method for Lipschitz pseudocontractions. *Proc. Am. Math. Soc.* **2001**, *129*, 2359–2363. [CrossRef]
17. Takahashi, W.; Takeuchi, Y.; Kubota, R. Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* **2008**, *341*, 276–286. [CrossRef]
18. Ahn, P.N. A hybrid extragradient method extended to fixed point problems and equilibrium problems. *Optimization* **2013**, *62*, 271–283.
19. Anh, P.N.; Muu, L.D. A hybrid subgradient algorithm for nonexpansive mappings and equilibrium problems. *Optim. Lett.* **2014**, *8*, 727–738. [CrossRef]
20. Ceng, L.C.; Al-Homidan, S.; Ansari, Q.H.; Yao, J.C. An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings. *J. Comput. Appl. Math.* **2009**, *223*, 967–974. [CrossRef]
21. Maingé, P.E. A hybrid extragradient viscosity methods for monotone operators and fixed point problems. *SIAM J. Control. Optim.* **2008**, *47*, 1499–1515. [CrossRef]
22. Plubtieng, S.; Kumam, P. Weak convergence theorem for monotone mappings and a countable family of nonexpansive semigroups. *J. Comput. Appl. Math.* **2009**, *224*, 614–621. [CrossRef]
23. Vuong, P.T.; Strodiot, J.J.; Nguyen, V.H. On extragradient-viscosity methods for solving equilibrium and fixed point problems in a Hilbert space. *Optimization* **2015**, *64*, 429–451. [CrossRef]

24. Dinh, B.V.; Kim, D.S. Extragradient algorithms for equilibrium problems and symmetric generalized hybrid mappings. *Optim. Lett.* **2016**, *11*, 537–553. [CrossRef]
25. Anh, P.K.; Hieu, D.V. Parallel and sequential hybrid methods for a finite family of asymptotically quasi ϕ-nonexpansive mappings. *J. Appl. Math. Comput.* **2015**, *48*, 241–263. [CrossRef]
26. Censor, Y.; Chen, W.; Combettes, P.L.; Davidi, R.; Herman, G.T. On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints. *Comput. Optim. Appl.* **2012**, *51*, 1065–1088. [CrossRef]
27. Censor, Y.; Gibali, A.; Reich, S.; Sabach, S. Common solutions to variational inequalities. *Set-Valued Var. Anal.* **2012**, *20*, 229–247. [CrossRef]
28. Hieu, D.V.; Muu, L.D.; Anh, P.K. Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings. *Numer. Algor.* **2016**, *73*, 197–217. [CrossRef]
29. Anh, P.K.; Chung, C.V. Parallel hybrid methods for a finite family of relatively nonexpansive mappings. *Numer. Funct. Anal. Optim.* **2014**, *35*, 649–664. [CrossRef]
30. Browder, F.E.; Petryshyn, W.V. Construction of fixed points of nonlinear mappings in Hilbert spaces. *J. Math. Anal. Appl.* **1967**, *20*, 197–228. [CrossRef]
31. Takahashi, W.; Wong, N.C.; Yao, J.C. Fixed point theorems for new generalized hybrid mappings in Hilbert spaces and applications. *Taiwan J. Math.* **2013**, *17*, 1597–1611. [CrossRef]
32. Browder, F.E. Semicontractive and semiaccretive nonlinear mappings in Banach spaces. *Bull. Am. Math. Soc.* **1968**, *74*, 660–665. [CrossRef]
33. Dotson, W.G., Jr. Fixed points of quasi-nonexpansive mappings. *J. Aust. Math. Soc.* **1972**, *13*, 167–170. [CrossRef]
34. Hojo, M.; Suzuki, T.; Takahashi, W. Fixed point theorems and convergence theorems for generalized hybrid non-self mappings in Hilbert spaces. *J. Nonlinear Convex Anal.* **2013**, *14*, 363–376.
35. Kawasaki, T.; Takahashi,W. Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces. *J. Nonlinear Convex Anal.* **2013**, *14*, 71–87.
36. Itoh, S.; Takahashi, W. The common fixed point theory of single-valued mappings and multi-valued mappings. *Pac. J. Math.* **1978**, *79*, 493–508. [CrossRef]
37. Mastroeni, G. On auxiliary principle for equilibrium problems. In *Equilibrium Problems and Variational Models*; Daniele, P., Giannessi, F., Maugeri, A., Eds.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2003; pp. 289–298.
38. Karamardian, S.; Schaible, S.; Crouzeix, J.P. Characterizations of generalized monotone maps. *J. Optim. Theory Appl.* **1993**, *76*, 399–413. [CrossRef]
39. Bianchi, M.; Schaible, S. Generalized monotone bifunctions and equilibrium problems. *J. Optim. Theory Appl.* **1996**, *90*, 31–43. [CrossRef]
40. Quoc, T.D.; Anh, P.N.; Muu, L.D. Dual extragradient algorithms extended to equilibrium problems. *J. Glob. Optim.* **2012**, *52*, 139–159. [CrossRef]
41. Andrzej, C. *Iterative Methods for Fixed Point Problems in Hilbert Spaces*; Springer: Berlin/Heidelberg, Germany, 2012.
42. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Marcel Dekker: New York, NY, USA, 1984.
43. Krasnoselski, M.A. Two observations about the method of succesive approximations. *Uspehi Math. Nauk* **1955**, *10*, 123–127.