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On Generating Functions for Boole Type Polynomials and Numbers of Higher Order and Their Applications

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Abstract: The purpose of this manuscript is to study and investigate generating functions for Boole type polynomials and numbers of higher order. With the help of these generating functions, many properties of Boole type polynomials and numbers are presented. By applications of partial derivative and functional equations for these functions, derivative formulas, recurrence relations and finite combinatorial sums involving the Apostol-Euler polynomials, the Stirling numbers and the Daehee numbers are given.

Keywords: Apostol-Euler polynomials and numbers; Peters polynomials and numbers; Boole polynomials and numbers; Stirling numbers; Daehee numbers; generating functions; Changheenumbers and polynomials

MSC: 05A10; 05A15; 11B68; 11B73; 11B83; 26C05

1. Introduction

In literature, there are various different and useful manuscripts related to not only Boole type polynomials and numbers, but also the Peters type polynomials and numbers. Some of those have been recently given by Boas [1], Jordan [2], Kim et al. [3–10], Kucukoglu et al. [11], Kruchinin [12], Roman [13], Simsek [14–20], Simsek and So [21], and also Srivastava et al. [22,23]. By using generating function method, we give many important and fundamental properties of Boole type polynomials and numbers of higher order. We need the following notations:

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ demonstrate respectively sets of integer numbers, real numbers, complex numbers, and p-adic integers. Marking that for $n = 0$, $0^n = 1$ and for $n \in \mathbb{N}$, $0^n = 0$.

$$(x)_v = x(x-1)\cdots(x-v+1),$$

$(x)_0 = 1$ and

$$\binom{x}{v} = \frac{x(x-1)\cdots(x-v+1)}{v!},$$

where $v \in \mathbb{N}_0$ (*cf.* [1–32]).

The definition of the Apostol-Euler polynomials of order v, shown by $\mathcal{E}_n(x, \lambda)$, is given below.

$$F_{\mathcal{E}}(t, x; \lambda, v) = \left(\frac{2}{\lambda e^t + 1}\right)^v e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(v)}(x, \lambda) \frac{t^n}{n!} \tag{1}$$

(*cf.* [25,28,29,31,32]; and the references cited therein).

Setting $v = 1$ in (1), we have the Apostol-Euler polynomials

$$\mathcal{E}_n(x,\lambda) = \mathcal{E}_n^{(1)}(x,\lambda).$$

When $x = 0$, we also have the Apostol-Euler numbers

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0,\lambda).$$

When $\lambda = 1$, the above equation reduces to the well-known the Euler numbers

$$E_n = \mathcal{E}_n(1)$$

(*cf*. [1–32]).

The definition of the Stirling numbers of the first kind, shown by $S_1(n,k)$, is given below.

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{2}$$

If $k > n$, then

$$S_1(n,k) = 0$$

(*cf*. [2–30,32]).

The definition of the Stirling numbers of the second kind, shown by $S_2(n,k)$, is given below.

$$F_S(t,k) = \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{3}$$

If $k > n$, then

$$S_2(n,k) = 0$$

(*cf*. [2–30,32]).

The Peters polynomials are one of the members of the Sheffer polynomials, which are a very broad family of polynomial sequences. The definition of the Peters polynomials, shown by $s_n(x;\lambda,\mu)$, is given below.

$$F_P(t,x;\lambda,\mu) = \frac{(1+t)^x}{\left(1+(1+t)^{\lambda}\right)^{\mu}} = \sum_{n=0}^{\infty} s_n(x;\lambda,\mu)\frac{t^n}{n!}, \tag{4}$$

where $x, t \in \mathbb{C}$ (*cf*. [1,2,13]).

Remark 1. *Recently, there have been various studies and papers about the Peters (type) polynomials. For example, see for detail, Boas [1], Jordan [2], Kim et al. [3–10], Kucukoglu et al. [11], Kruchinin [12], Roman [13], Simsek [14–20], Simsek and So [21], and also Srivastava et al. [22,23].*
We now present some appropriate values of the $s_n(x;\lambda,\mu)$.
When $x = 0$, *we have the Peters numbers:*

$$s_n(\lambda,\mu) = s_n(0;\lambda,\mu)$$

(cf. [15,22]). When $\mu = 1$, *we have the Boole polynomials:*

$$\xi(x,\lambda) = s_n(x;\lambda,1)$$

(cf. [2,13]). If $\lambda = \mu = 1$, *we get the Changhee polynomials*

$$Ch_n(x) = 2s_n(x;1,1)$$

(*cf.* [3,4]).

The definition of the Daehee numbers, shown by D_n, is given below.

$$F_D(t,k) = \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{5}$$

(*cf.* [5]). For further information and generalization see also (*cf.* [11,15–18,23]). Recently, the first author defined the following combinatorial numbers and polynomials, respectively:

$$F(t,\lambda) = \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!}$$

and

$$F(t,x,\lambda) = (1+\lambda t)^x F(t,\lambda) = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!}$$

(*cf.* [18]).

Let d be an odd integer and χ be a Dirichlet character. That is $\chi(x+d) = \chi(x)$. The first author [18] [Equation-(2.3)] defined the following interesting p-adic integral representation and equation:

$$\int_{\mathbb{X}} \lambda^x (1+\lambda t)^x \chi(x) d\mu_{-q}(x) = \frac{1+q}{(\lambda q)^d (1+\lambda t)^d + 1} \sum_{j=0}^{d-1} (-1)^j \chi(j)(\lambda q)^j (1+\lambda t)^j,$$

where p is a fixed prime and d is a fixed positive integer with $(p,d) = 1$, hence

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \\ \mathbb{X}_1 &= \mathbb{Z}_p, \end{aligned}$$

$$\mu_q(x) = \mu_q(x + p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]},$$

where $q \in \mathbb{Z}_p$ with $\mid 1-q \mid_p < 1$ (see for detail, [26]) and

$$[x]_q = \frac{1-q^x}{1-q}.$$

Here q is an indeterminate. If $q \in \mathbb{C}$, we assume that $|q| < 1$. It is well-known that

$$\lim_{q \to 1} [x]_q = x$$

(see, for detail, [18] [Equation-(2.3)]).

By using the previous equation, we have

$$\frac{(1+q)\sum_{j=0}^{d-1}(-1)^j \chi(j)(\lambda q)^j (1+\lambda t)^{j+z}}{(\lambda q)^d (1+\lambda t)^d + 1} = \sum_{n=0}^{\infty} \mathfrak{Ch}_{n,\chi}(z;\lambda,q)\frac{t^n}{n!},$$

where the polynomials $\mathfrak{Ch}_{n,\chi}(z;\lambda,q)$, which are the so-called generalized Apostol-Changhee polynomials, are given by

$$\mathfrak{Ch}_{n,\chi}(z;\lambda,q) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} (z)_{n-j} \mathfrak{Ch}_{j,\chi}(\lambda,q)$$

where the numbers $\mathfrak{Ch}_{n,\chi}(\lambda, q)$, which are the so-called generalized Apostol-Changhee numbers, are given by

$$\mathfrak{Ch}_{n,\chi}(\lambda, q) = \int\limits_{\mathbb{X}} \lambda^{x+n}(x)_n \chi(x) d\mu_{-q}(x) \tag{6}$$

(see, for detail, [18] [Equations-(2.4) and (2.5)]).

In light of the previous equations, the authors [21] defined the following special polynomials $y_{7,n}(x; \lambda, q, d)$:

$$K_d(t, x; \lambda, q) = \frac{(1+q)(1+\lambda t)^x}{(\lambda q)^d (1+\lambda t)^d + 1} = \sum_{n=0}^{\infty} y_{7,n}(x; \lambda, q, d) \frac{t^n}{n!}. \tag{7}$$

When $x = 0$, we have the special combinatorial numbers:

$$y_{7,n}(\lambda, q, d) = y_{7,n}(0; \lambda, q, d).$$

Substituting $d = 1$ into (7), we also have the special combinatorial polynomials:

$$y_{7,n}(x; \lambda, q) = y_{7,n}(x; \lambda, q, 1)$$

(*cf.* [19,21]).

Here, brief information about notations and index for the above special combinatorial numbers and polynomials is given as follows:

The first author has recently defined many different Peters and Boole type combinatorial numbers and polynomials. He gave some notations for these numbers and polynomials. For instance, in order to distinguish them from each other, these polynomials are labeled by the following symbols:

$y_{j,n}(x; \lambda, q)$, $j = 1, 2, ..., 7$, and also $Y_n(x; \lambda)$. Therefore, the number 7 is only used for index representation for these polynomials (*cf.* [16–21]).

Results of this paper are briefly summarized below.

Some fundamental properties of Boole type numbers of higher order and Boole type polynomials of higher order. We derive some fundamental properties of these numbers, and polynomials are given in Section 2.

PDEs and functional equations related to generating functions for Boole type polynomials of higher order, the Daehee numbers and logarithm function are given. Using these equations, derivative formulae and recurrence relations are given in Section 3.

2. Generating Function for the Polynomials $y_{7,n}(x; \lambda, q, d)$ of Order v and the Numbers $y_{7,n}(\lambda, q, d)$ of Order v

In this section, we define the generalization of the numbers $y_{7,n}(\lambda, q, d)$ as follows:

$$\mathcal{F}_v(t; \lambda, q, d) = \left(\frac{1+q}{(\lambda q)^d (1+\lambda t)^d + 1}\right)^v = \sum_{n=0}^{\infty} y_{7,n}^{(v)}(\lambda, q, d) \frac{t^n}{n!}. \tag{8}$$

We also define the generalization of the polynomials $y_{7,n}(x; \lambda, q, d)$ as follows:

$$\mathcal{G}_v(t, x; \lambda, q, d) = (1+\lambda t)^x \mathcal{F}_v(t; \lambda, q, d) = \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x; \lambda, q, d) \frac{t^n}{n!}. \tag{9}$$

We investigate some properties of the polynomials $y_{7,n}(x; \lambda, q, d)$ and the numbers $y_{7,n}(\lambda, q, d)$. We give identities and formulas involving these numbers and polynomials, the Apostol-Euler numbers, and the Stirling numbers.

By (8) and (9), we have

$$y_{7,n}^{(v)}(\lambda, q, d) = y_{7,n}^{(v)}(0; \lambda, q, d),$$

and

$$y_{7,n}(x;\lambda,q,d)=y^{(1)}_{7,n}(x;\lambda,q,d).$$

In order to give a computation formula for the numbers $y^{(v)}_{7,n}(\lambda,q,d)$, we set

$$\left(\frac{1+q}{(\lambda q)^d(1+\lambda t)^d+1}\right)^v=\left(\frac{1+q}{2}\right)^v\left(\frac{2}{(\lambda q)^d e^{d\log(1+\lambda t)}+1}\right)^v.$$

Combining the above equation with (1) and (2), we get

$$\begin{aligned}\sum_{m=0}^{\infty}y^{(v)}_{7,m}(\lambda,q,d)\frac{t^m}{m!}&=\left(\frac{1+q}{2}\right)^v\sum_{n=0}^{\infty}\mathcal{E}^{(v)}_n(\lambda^d q^d)\frac{d^n(\log(1+\lambda t))^n}{n!}\\&=\left(\frac{1+q}{2}\right)^v\sum_{m=0}^{\infty}\sum_{n=0}^{m}d^n\lambda^m\mathcal{E}^{(v)}_n(\lambda^d q^d)S_1(m,n)\frac{t^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, a computation formula for the numbers $y^{(v)}_{7,m}(\lambda,q,d)$ is given by the following theorem:

Theorem 1. *Let $q>0$, $v,d,m\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Then we have*

$$y^{(v)}_{7,m}(\lambda,q,d)=\left(\frac{1+q}{2}\right)^v\lambda^m\sum_{n=0}^{m}d^n\mathcal{E}^{(v)}_n(\lambda^d q^d)S_1(m,n).$$

Using (8), we obtain

$$(1+q)^v=\sum_{j=0}^{v}\binom{v}{j}(\lambda q)^{dj}(1+\lambda t)^{dj}\sum_{n=0}^{\infty}y^{(v)}_{7,n}(\lambda,q,d)\frac{t^n}{n!}.$$

From the previous equation, we have

$$(1+q)^v=\sum_{n=0}^{\infty}\sum_{j=0}^{v}\binom{v}{j}(\lambda q)^{dj}\binom{dj}{n}\lambda^n t^n\sum_{n=0}^{\infty}y^{(v)}_{7,n}(\lambda,q,d)\frac{t^n}{n!}.$$

Hence

$$(1+q)^v=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{v}\binom{n}{k}\binom{v}{j}(\lambda q)^{dj}(dj)_k\lambda^k y^{(v)}_{7,n-k}(\lambda,q,d)\frac{t^n}{n!}.$$

Making some straightforward calculations in the previous equation, a recurrence relation for $y^{(v)}_{7,n}(\lambda,q,d)$ is obtained. This relation is given by the following theorem:

Theorem 2. *Let*

$$y^{(v)}_{7,0}(\lambda,q,d)=\left(\frac{1+q}{(\lambda q)^d+1}\right)^v.$$

For $n\in\mathbb{N}$, we have

$$\sum_{k=0}^{n}\sum_{j=0}^{v}\binom{n}{k}\binom{v}{j}(\lambda q)^{dj}(dj)_k\lambda^k y^{(v)}_{7,n-k}(\lambda,q,d)=0.$$

With the help of Equation (8), setting the following equation:

$$\mathcal{F}_{v_1+v_2}(t;\lambda,q,d)=\mathcal{F}_{v_1}(t;\lambda,q,d)\mathcal{F}_{v_2}(t;\lambda,q,d).$$

Making some calculations in the previous equation, another recurrence relation for $y_{7,n}^{(v)}(\lambda,q,d)$ is also obtained. This relation is given by the following theorem:

Theorem 3. *Let $q>0$, $v_1,v_2,d,n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Then we have*

$$y_{7,n}^{(v_1+v_2)}(\lambda,q,d)=\sum_{j=0}^{n}\binom{n}{j}y_{7,j}^{(v_1)}(\lambda,q,d)y_{7,n-j}^{(v_2)}(\lambda,q,d). \tag{10}$$

Setting $v_1=v_2=1$ in (10), we compute the following few values of the numbers $y_{7,n}^{(2)}(\lambda,q,d)$:

$$y_{7,0}^{(2)}(\lambda,q,d)=\left(\frac{1+q}{(\lambda q)^d+1}\right)^2,$$

$$y_{7,1}^{(2)}(\lambda,q,d)=-\frac{2d\lambda(\lambda q)^d(1+q)^2}{((\lambda q)^d+1)^3}$$

and

$$\begin{aligned} y_{7,2}^{(2)}(\lambda,q,d) &= \frac{8(d\lambda)^2(\lambda q)^{2d}(1+q)^2}{((\lambda q)^d+1)^4} \\ &\quad -\frac{(\lambda^2(\lambda q)^d((d)_2+(2d)_2(\lambda q)^d)(1+q)^2}{((\lambda q)^d+1)^4}. \end{aligned}$$

A relation between the numbers $y_{7,n}^{(v)}(\lambda,q,d)$ and the polynomials $y_{7,n}^{(v)}(x;\lambda,q,d)$ is given by the following theorem.

Theorem 4. *Let $q>0$, $v,d,n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Then we have*

$$y_{7,n}^{(v)}(x;\lambda,q,d)=\sum_{j=0}^{n}\binom{n}{j}(x)_j\lambda^j y_{7,n}^{(v)}(\lambda,q,d).$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} &= (1+\lambda t)^x\sum_{n=0}^{\infty}y_{7,n}^{(v)}(\lambda,q,d)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\binom{x}{n}(\lambda t)^n\sum_{n=0}^{\infty}y_{7,n}^{(v)}(\lambda,q,d)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(x)_j\lambda^j y_{7,n}^{(v)}(\lambda,q,d)\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the derived result. □

Using definition of the numbers $y_{7,n}^{(v)}(x;\lambda,q,d)$, we have

$$(1+\lambda t)^x=\left(\frac{1+q}{(\lambda q)^d(1+\lambda t)^d+1}\right)^{-v}\sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.$$

After elementary calculation, we obtain

$$\sum_{n=0}^{\infty}(x)_n\lambda^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}y_{7,n}^{(-v)}(x;\lambda,q,d)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.$$

Making some straightforward calculations in the previous equation, and after that comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the following theorem:

Theorem 5. *Let $q > 0$, $v, d, n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then we have*

$$(x)_n = \frac{1}{\lambda^n} \sum_{j=0}^{n} \binom{n}{j} y_{7,n-j}^{(-v)}(x; \lambda, q, d) y_{7,j}^{(v)}(x; \lambda, q, d).$$

Substituting $\lambda t = e^z - 1$ into (9), we have

$$\left(\frac{1+q}{2}\right)^v \left(\frac{2}{(\lambda q)^d e^{zd} + 1}\right)^v e^{zx} = \sum_{m=0}^{\infty} y_{7,m}^{(v)}(x; \lambda, q, d) \frac{(e^z - 1)^m}{\lambda^m m!}.$$

Combining the previous equation with (1) and (3), we obtain

$$\left(\frac{1+q}{2}\right)^v \sum_{n=0}^{\infty} d^n \mathcal{E}_n^{(v)}\left(\frac{x}{d}; \lambda^d q^d\right) \frac{z^n}{n!} = \sum_{m=0}^{\infty} \frac{y_{7,m}^{(v)}(x; \lambda, q, d)}{\lambda^m} \sum_{n=0}^{\infty} S_2(n, m) \frac{z^n}{n!}.$$

Since $S_2(n, m) = 0$ for $m > n$, we have

$$\left(\frac{1+q}{2}\right)^v \sum_{n=0}^{\infty} d^n \mathcal{E}_n^{(v)}\left(\frac{x}{d}; \lambda^d q^d\right) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{y_{7,m}^{(v)}(x; \lambda, q, d)}{\lambda^m} S_2(n, m)\right) \frac{z^n}{n!}.$$

Comparing the coefficients of $\frac{z^n}{n!}$ on the both sides of the above equation, we derive the following theorem:

Theorem 6. *Let $q > 0$, $v, d, n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then we have*

$$\left(\frac{1+q}{2}\right)^v \mathcal{E}_n^{(v)}\left(\frac{x}{d}; \lambda^d q^d\right) = \frac{1}{d^n} \left(\sum_{m=0}^{n} \frac{y_{7,m}^{(v)}(x; \lambda, q, d)}{\lambda^m} S_2(n, m)\right).$$

Setting

$$F_{v+k}(t; \lambda, q, d) = F_v(t; \lambda, q, d) F_k(t; \lambda, q, d).$$

Using the previous equation, we derive the following theorem:

Theorem 7. *Let $q > 0$, $v, d, n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Then we have*

$$y_{7,n}^{(v)}(x + y; \lambda, q, d) = \sum_{j=0}^{n} \binom{n}{j} (x + y)_j y_{7,n-j}^{(v)}(\lambda, q, d). \tag{11}$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x + y; \lambda, q, d) \frac{t^n}{n!} &= (1 + \lambda t)^{x+y} \left(\frac{1+q}{(\lambda q)^d (1 + \lambda t)^d + 1}\right)^v \\ &= \sum_{n=0}^{\infty} (y)_n \lambda^n \frac{t^n}{n!} \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x; \lambda, q, d) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the derived result. □

Combining the following the Chu-Vandermonde identity with (11)

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n-m}{k-j}=\binom{n}{k}. \tag{12}$$

we have

$$y_{7,n}^{(v)}(x+y;\lambda,q,d)=\sum_{j=0}^{n}\binom{n}{j}(x)_{n-j}y_{7,n}^{(v)}(x;\lambda,q,d)$$

and

$$y_{7,n}^{(v)}(x+y;\lambda,q,d)=\sum_{j=0}^{n}\binom{n}{j}(y)_{n-j}y_{7,n}^{(v)}(\lambda,q,d).$$

Combining (12) with (11), we arrive at the following corollary:

Corollary 1. *Let $q>0$, $v,d,n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Then we have*

$$y_{7,n}^{(v)}(x+y;\lambda,q,d)=\sum_{j=0}^{n}\binom{n}{j}\sum_{k=0}^{j}(x)_k(y)_{j-k}y_{7,n-j}^{(v)}(\lambda,q,d).$$

Kucukoglu [27] defined the following generating functions:

$$\mathcal{F}_d(t;\lambda,q,v)=\left(\frac{\log(1+\lambda t)}{(\lambda q)^d(1+\lambda t)^d-1}\right)^v=\sum_{n=0}^{\infty}I_{n,d}^{(v)}(\lambda,q)\frac{t^n}{n!} \tag{13}$$

and

$$\mathcal{G}_d(t,x;\lambda,q,v)=(1+\lambda t)^x\mathcal{F}_d(t;\lambda,q,v)=\sum_{n=0}^{\infty}I_{n,d}^{(v)}(x;\lambda,q)\frac{t^n}{n!}. \tag{14}$$

Combining (14) with (9), we have

$$\mathcal{G}_v(t,x;\lambda,q,d)\mathcal{G}_d(t,x;\lambda,q,v)=(1+\lambda t)^{2x}\frac{(1+q)^v(\log(1+\lambda t))^v}{((\lambda q)^{2d}(1+\lambda t)^{2d}-1)^v}.$$

From the above equation, we get

$$\mathcal{G}_v(t,x;\lambda,q,d)\mathcal{G}_d(t,x;\lambda,q,v)=(1+q)^v\mathcal{G}_{2d}(t,2x;\lambda,q,v).$$

From the equality in (14) with $2d$ and $2x$ instead of d respectively x, we arrive at the following one:

$$\sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}\sum_{n=0}^{\infty}I_{n,d}^{(v)}(x;\lambda,q)\frac{t^n}{n!}=(1+q)^v\sum_{n=0}^{\infty}I_{n,2d}^{(v)}(2x;\lambda,q)\frac{t^n}{n!}.$$

Using the Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we have the following theorem:

Theorem 8. *Let $q>0$, $v,d,n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Then we have*

$$I_{n,2d}^{(v)}(t,2x;\lambda,q)=\frac{1}{(1+q)^v}\sum_{j=0}^{n}\binom{n}{j}I_{j,d}^{(v)}(x;\lambda,q)y_{7,n-j}^{(v)}(x;\lambda,q,d). \tag{15}$$

Remark 2. *When $v=1$, (15) reduce to*

$$I_{n,2d}(t,2x;\lambda,q)=\frac{1}{1+q}\sum_{j=0}^{n}\binom{n}{j}I_{j,d}(x;\lambda,q)y_{7,n-j}(x;\lambda,q,d)$$

(*cf*. [21]).

3. Partial Derivative Equations and Their Applications

In this section, we deal with some partial derivative equations and functional equations involving generating functions for the polynomials $y_{7,n}^{(v)}(x;\lambda,q,d)$, the Daehee numbers and logarithm function. By using these equations, we derive derivative formulas for the polynomials $y_{7,n}^{(v)}(x;\lambda,q,d)$, and some identities including these polynomials, recurrence relations of these polynomials, the Daehee numbers and finite combinatorial sums.

3.1. Partial Derivative Equations and Derivative Formulas

Differentiating both side of (9) with respect to x, we get the following partial differential equations:

$$\frac{\partial}{\partial x}\left\{\mathcal{G}_v(t,x;\lambda,q,d)\right\}=\log(1+\lambda t)\mathcal{G}_v(t,x;\lambda,q,d) \tag{16}$$

and

$$\frac{\partial}{\partial x}\left\{\mathcal{G}_v(t,x;\lambda,q,d)\right\}=\lambda tF_D(\lambda t)\mathcal{G}_v(t,x;\lambda,q,d). \tag{17}$$

By using the above derivative equations, here we derive two derivative formulas for the polynomials $y_{7,n}^{(v)}(x;\lambda,q,d)$. Using these formulas, we derive a combinatorial sums including these polynomials and the Daehee numbers.

Combining (9) with (16), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left\{y_{7,n}^{(v)}(x;\lambda,q,d)\right\}\frac{t^n}{n!}=\sum_{n=1}^{\infty}(-1)^{n-1}\lambda^n\frac{t^n}{n}\sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.$$

After some elementary calculations from the above equation, we arrive at the following theorem:

Theorem 9. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{\partial}{\partial x}\left\{y_{7,n}^{(v)}(x;\lambda,q,d)\right\}=\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}j!\lambda^{j+1}y_{7,n-1-j}^{(v)}(x;\lambda,q,d). \tag{18}$$

Combining (9) with (17), we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left\{y_{7,n}^{(v)}(x;\lambda,q,d)\right\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\lambda^{n+1}D_n\frac{t^{n+1}}{n!}\sum_{n=0}^{\infty}y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}\left\{y_{7,n}^{(v)}(x;\lambda,q,d)\right\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}n\sum_{j=0}^{n-1}\binom{n-1}{j}D_j\lambda^{j+1}y_{7,n-1-j}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 10. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{\partial}{\partial x}\left\{y_{7,n}^{(v)}(x;\lambda,q,d)\right\}=n\sum_{j=0}^{n-1}\binom{n-1}{j}D_j\lambda^{j+1}y_{7,n-1-j}^{(v)}(x;\lambda,q,d). \tag{19}$$

Using (5), the following well-known explicit formula for the Daehee numbers is given by

$$D_j = (-1)^j \frac{j!}{j+1}$$

(*cf.* [2,5]). Combining (19) and (18) with this formula, we derive the following finite combinatorial sum:

Corollary 2. *Let $n \in \mathbb{N}$. Then we have*

$$n\sum_{j=0}^{n-1}\binom{n-1}{j} D_j \lambda^{j+1} y_{7,n-1-j}^{(v)}(x;\lambda,q,d) = \sum_{j=0}^{n-1}(-1)^j \binom{n}{j+1} j!\lambda^{j+1} y_{7,n-1-j}^{(v)}(x;\lambda,q,d). \tag{20}$$

3.2. Recurrence Relations

Here, we give partial differential equations for generating functions $\mathcal{G}_v(t,x;\lambda,q,d)$. With the help of these equations, two recurrence relations for the polynomials $y_{7,n+1}^{(v)}(x;\lambda,q,d)$ are given.

Differentiating both sides of (9) with respect to t, we obtain the following partial derivative equations:

$$\begin{aligned}\frac{\partial}{\partial t}\{\mathcal{G}_v(t,x;\lambda,q,d)\} &= x\lambda\mathcal{G}_v(t,x-1;\lambda,q,d) \\ &\quad -\frac{vdq^d\lambda^{d+1}}{1+q}\mathcal{G}_{v+1}(t,x+d-1;\lambda,q,d)\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\{\mathcal{G}_v(t,x;\lambda,q,d)\} &= \frac{\lambda x}{1+\lambda t}\mathcal{G}_v(t,x;\lambda,q,d) \\ -\frac{vdq^d\lambda^{d+1}}{1+q}&\mathcal{G}_1(t,d-1;\lambda,q,d)\mathcal{G}_v(t,x;\lambda,q,d).\end{aligned} \tag{22}$$

Combining (9) with (21), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} y_{7,n+1}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} \\ =\;& x\lambda\sum_{n=0}^{\infty} y_{7,n}^{(v)}(x-1;\lambda,q,d)\frac{t^n}{n!} - \frac{vdq^d\lambda^{d+1}}{1+q}\sum_{n=0}^{\infty} y_{7,n}^{(v+1)}(x+d-1;\lambda,q,d)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 11. *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}y_{7,n+1}^{(v)}(x;\lambda,q,d) &= x\lambda y_{7,n}^{(v)}(x-1;\lambda,q,d) \\ &\quad -\frac{vdq^d\lambda^{d+1}}{1+q} y_{7,n}^{(v+1)}(x+d-1;\lambda,q,d).\end{aligned}$$

Assume that $|\lambda t| < 1$. Combining (9) with (22), we get, with $y_{7,n}(d-1;\lambda,q,d) = y_{7,n}^{(1)}(d-1;\lambda,q,d)$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} y_{7,n+1}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} \\
= \quad & \lambda x \sum_{n=0}^{\infty} (-\lambda t)^n \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} \\
& -\frac{v\lambda d(\lambda q)^d}{1+q} \sum_{n=0}^{\infty} y_{7,n}(d-1;\lambda,q,d)\frac{t^n}{n!} \sum_{n=0}^{\infty} y_{7,n}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{n=0}^{\infty} y_{7,n+1}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} \\
= \quad & \lambda x \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j!\lambda^j y_{7,n-j}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!} \\
& -\frac{vdq^d\lambda^{d+1}}{1+q} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} y_{7,j}(d-1;\lambda,q,d) y_{7,n-j}^{(v)}(x;\lambda,q,d)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 12. *Let $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}
y_{7,n+1}^{(v)}(x;\lambda,q,d) \quad = \quad & \lambda x \sum_{j=0}^{n} (-1)^j \binom{n}{j} j!\lambda^j y_{7,n-j}^{(v)}(x;\lambda,q,d) \\
& -\frac{vdq^d\lambda^{d+1}}{1+q} \sum_{j=0}^{n} \binom{n}{j} y_{7,j}(d-1;\lambda,q,d) y_{7,n-j}^{(v)}(x;\lambda,q,d).
\end{aligned}$$

4. Conclusion

In the recent extensive written works about the theory of special functions, especially special numbers and polynomials, there are widespread manuscripts and books including special numbers and polynomials such as combinatorial numbers and polynomials, Apostol type numbers and polynomials, Peters type polynomials and numbers, Boole polynomials and numbers, Stirling numbers, Changhee numbers and Daehee numbers. In this paper, we give some new families of combinatorial numbers, which are generalizations and unifications of the Peters and Boole polynomials and numbers with the help of generating functions. By using these functions and their PDEs and functional equations, we derived various interesting properties and identities of these polynomials and numbers. Appropriate relationships of our polynomials and numbers and the results of this paper are compared with earlier results. Consequently, the results of this paper may potentially be used, not only in analytic number theory and for special numbers and polynomials, but also in other areas.

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