symmetry

MDPI

Article

On Suzuki Mappings in Modular Spaces

Andreea Bejenaru [1] and Mihai Postolache [1,2,3,*]

[1] Department of Mathematics and Informatics, University Politehnica of Bucharest, 060042 Bucharest, Romania; bejenaru.andreea@yahoo.com
[2] Center for General Education, China Medical University, Taichung 40402, Taiwan
[3] Romanian Academy, Gh. Mihoc—C. Iacob Institute of Mathematical Statistics and Applied Mathematics, 050711 Bucharest, Romania
* Correspondence: mihai@mathem.pub.ro

Received: 12 January 2019; Accepted: 1 March 2019; Published: 3 March 2019

check for updates

Abstract: Inspired by Suzuki's generalization for nonexpansive mappings, we define the (C)-property on modular spaces, and provide conditions concerning the fixed points of newly introduced class of mappings in this new framework. In addition, Kirk's Lemma is extended to modular spaces. The main outcomes extend the classical results on Banach spaces. The major contribution consists of providing inspired arguments to compensate the absence of subadditivity in the case of modulars. The results herein are supported by illustrative examples.

Keywords: Suzuki mapping; modular space; Kirk's lemma

MSC: 47H10; 54H25; 37C25

1. Introduction

The first idea regarding the concept of modular space was initiated by Orlicz in [1] through a remarkable example. Later, Nakano [2] settled a more formal framework by defining the modular as a generalization for a norm-type function. His work was extended further by Musielak and Orlicz in [3]. A consistent approach on modular function spaces was realized by Kozlowski in [4] (see also [5]). In time, the notion was reactivated in the expanded framework of vector spaces and one important direction was settled by Khamsi [6] in connection with the fixed point theory. Nowadays, this approach is fructified in several works: Okeke et al. [7], Khan [8], Abbas et al. [9], Abdou and Khamsi [10], Alfuraidan et al. [11] and the papers referenced there.

By properly defining the modular convergency and the modular Cauchy sequences, the modular vectorial structures become natural extensions of Banach spaces. Moreover, many of the fixed point theory outcomes on Banach spaces can be extended, as the above-mentioned references prove, to modular structures. This paper continues this approach by extending Suzuki's concept of generalized nonexpasive mapping to modular vector spaces and by analyzing the existence of fixed points. The resulting main outcomes approach to the conclusions from Suzuki [12] and Thakur et al. [13] but under weaker assumptions. The notable breakthrough lies in the fact that the arguments for the central theorem are provided in the absence of the subadditivity (a modular function lacks subadditivity; it has at most convexity) and other properties deriving from it, which were used multiple times in the proof of Suzuki's original outcome. In addition, some partial extensions of the original Kirk's Lemma [14] and the Goebel and Kirk Lemma [15] are provided, bypassing, again, the subadditivity.

2. Preliminaries on Modular Vector Spaces

Along this paper, the general framework will be provided by a real linear (vector) space X.

Definition 1. *Assume that $\rho\colon X \to [0,\infty]$ is a function satisfying the following properties [2]:*

(1) $\rho(x) = 0 \Leftrightarrow x = 0$;
(2) $\rho(-x) = \rho(x), \quad \forall x \in X$;
(3) $\rho(\alpha x + (1-\alpha y)) \leq \rho(x) + \rho(y), \quad \forall \alpha \in [0,1]$.

Then, ρ is called modular on X.
If, instead of condition (3), the following stronger requirement is fulfilled:

$$\rho(\alpha x + (1-\alpha y)) \leq \alpha\rho(x) + (1-\alpha)\rho(y),\ \forall \alpha \in [0,1], \quad \forall x, y \in X,$$

then the modular ρ is called convex.

Definition 2. *Let X be endowed with a modular ρ. The set*

$$X_\rho = \left\{x \in X \,:\, \lim_{\alpha\to 0} \rho(\alpha x) = 0\right\}$$

will be referred to as modular space.

Definition 3. *Let X be endowed with a modular ρ and consider a sequence $\{x_\ell\} \subset X_\rho$.*

(a) *$\{x_\ell\}$ is said to be modular-convergent (or ρ-convergent) to $x \in X_\rho$ if $\lim_{\ell\to\infty} \rho(x_\ell - x) = 0$. It is worth mentioning the uniqueness of the ρ-limit, whenever it exists.*
(b) *$\{x_\ell\} \in X_\rho$ is called modular or ρ-Cauchy if $\lim_{l,k\to\infty} \rho(x_\ell - x_k) = 0$.*
(c) *If all the modular-Cauchy sequences are also modular-convergent, then the modular space X_ρ is called ρ-complete.*
(d) *A subset $S \subset X_\rho$ which contains the ρ-limits of all its modular-convergent sequences $\{x_\ell\} \subset S$ is called ρ-closed.*
(e) *A subset $S \subset X_\rho$ satisfying $\operatorname{diam}_\rho(S) = \sup\{\rho(x-y) \,:\, x, y \in S\} < \infty$ is referred to as ρ-bounded.*
(f) *ρ satisfies the Fatou property if, for any sequence $\{x_\ell\}$ in X_ρ which is ρ-convergent to $x \in X_\rho$, the inequality $\rho(x) \leq \liminf_{\ell\to\infty} \rho(x_\ell)$ holds.*
(g) *ρ is said to satisfy condition Δ_2 if $\rho(2x) \leq K\rho(x), \ \forall x \in X_\rho$, for a constant element $K \geq 0$. The minimal possible value of K is usually denoted by $\omega(2)$.*

Remark 1. *More extensively, if ρ is assumed to be convex and to satisfy condition Δ_2, we can define the growth function (see [11]):*

$$\omega : [0,\infty) \to [0,\infty), \quad \omega(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)} \,:\, 0 < \rho(x) < \infty\right\}.$$

Then, $1 < \omega(2)$. The arguments for this statement can be extracted from Lemma 2.6 in [11]. In addition, let us notice that $\rho(\alpha x) \leq \omega(\alpha)\rho(x), \ \forall \alpha \geq 0, \ \forall x \in X_\rho$ and also that, for each positive integer l and arbitrary elements $x_1, x_2, \ldots, x_l \in X_\rho$,

$$\rho(x_1 + x_2 + \cdots + x_l) \leq \frac{\omega(l)}{l}\left[\rho(x_1) + \rho(x_2) + \cdots + \rho(x_l)\right].$$

Definition 4. *Let $r, \epsilon > 0$. Define (see [10], cf. [6])*

$$D_1(r,\epsilon) = \{(x,y) : x, y \in X_\rho, \ \rho(x) \leq r, \ \rho(y) \leq r, \ \rho(x-y) \geq \epsilon r\}$$

and

$$\delta_1(r,\epsilon) = \begin{cases} \inf\left\{1 - \frac{1}{r}\rho\left(\frac{x+y}{2}\right) : x,y \in D_1(r,\epsilon)\right\}, & \text{if } D_1(r,\epsilon) \neq \emptyset; \\ 1, & \text{if } D_1(r,\epsilon) = \emptyset. \end{cases}$$

The modular ρ has the $(UC1)$-property if $\delta_1(r,\epsilon) > 0$, for every $r > 0$ and $\epsilon > 0$. Moreover, ρ has the $(UUC1)$-property if, for each $s \geq 0$ and each $\epsilon > 0$, one may find $\eta_1(s,\epsilon) > 0$ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$, for $r > s$.

Lemma 1. *Suppose that ρ satisfies property $(UUC1)$ and let $\{\alpha_\ell\} \subset [a,b]$, where $0 < a \leq b < 1$. If there exists a positive real number r such that $\limsup_{\ell\to\infty} \rho(\alpha_\ell x_\ell + (1-\alpha_\ell)y_\ell) = r$, $\limsup_{\ell\to\infty} \rho(x_\ell) \leq r$ and $\limsup_{\ell\to\infty} \rho(y_\ell) \leq r$, then $\lim_{\ell\to\infty} \rho(x_\ell - y_\ell) = 0$ ([8], cf. [5]).*

Definition 5. *Given a sequence $\{x_\ell\}$ in X_ρ and a nonempty subset $S \subset X_\rho$, the following elements may be defined in connection with them (see [10]):*

(1) the ρ-type function defined as

$$\tau : S \to [0,\infty], \;\; \tau(x) = \limsup_{\ell\to\infty} \rho(x - x_\ell);$$

(2) the asymptotic radius of $\{x_\ell\}$ with respect to S, meaning the value $r(S) = \inf\{\tau(x) : x \in S\}$;
(3) the asymptotic center of $\{x_\ell\}$ with respect to S defined as the set $A(S) = \{x \in S : \tau(x) = r(S)\}$;
(4) the minimizing sequences of the ρ-type function, namely sequences $\{c_\ell\}$ in S satisfying $\lim_{n\to\infty} \tau(c_\ell) = r(S)$.

Lemma 2. *Assume that X_ρ is a ρ-complete modular space. Let ρ satisfy the Fatou property. Let S be a nonempty ρ-closed convex subset of X_ρ and $\{x_\ell\}$ be a sequence in X_ρ with a finite asymptotic radius relative to S (i.e., $r(S) < \infty$). If ρ satisfies the (UUC1)-condition, then all the minimizing sequences of τ are modular-convergent, having the same ρ-limit [10].*

3. Condition (C) of Suzuki in Modular Spaces

Let us start by recalling the concept of generalized nonexpansive mapping as it was phrased by Suzuki in [12] via the so called condition (C).

Definition 6. *Let $S \subset X$ be a nonempty subset of a Banach space $(X, \|\cdot\|)$. A mapping $\mathcal{M}\colon S \to S$ is said to satisfy condition (C) (or to be a Suzuki nonexpansive mappings) if the inequality $\frac{1}{2}\|x - \mathcal{M}x\| \leq \|x-y\|$ for some $x,y \in S$ leads to $\|\mathcal{M}x - \mathcal{M}y\| \leq \|x-y\|$ [12].*

The following Lemma refers to an essential property of nonexpansive mappings under condition (C); it provided the key element in [12,13] for proving the results regarding the existence of fixed points.

Lemma 3. *If $S \subset X$ is a nonempty subset of a Banach space $(X, \|\cdot\|)$ and $\mathcal{M}\colon S \to S$ is a Suzuki nonexpansive mapping, then $\|x - \mathcal{M}y\| \leq 3\|\mathcal{M}x - x\| + \|x-y\|$, $\forall x,y \in X$ [12].*

Nevertheless, the Lemma above is directly related to the triangle inequality, which is no longer among the properties of a modular space. Therefore, we are forced to find an alternative way of proving the corresponding results on modular spaces.

Inspired by Suzuki's definition, we adapt it to modular structures resulting in the modular-(C) property, as it follows.

Definition 7. *Let ρ denote a modular satisfying condition Δ_2 on a linear (vector) space X and let $S \subset X_\rho$ be a nonempty subset. A mapping $\mathcal{M}\colon S \to S$ is said to satisfy condition (ρC) if $\forall\ x, y \in S$,*

$$\frac{1}{\omega(2)}\rho(x-\mathcal{M}x) \leq \rho(x-y) \text{ leads to } \rho(\mathcal{M}x-\mathcal{M}y) \leq \rho(x-y).$$

Lemma 4. *Let ρ denote a convex modular with property Δ_2 and $S \subset X_\rho$ be a nonempty subset. Then,*

(i) for each $x \in X_\rho$, one has $\rho(\mathcal{M}x-\mathcal{M}^2x) \leq \rho(x-\mathcal{M}x)$;
(ii) for any $x, y \in X_\rho$ either $\frac{1}{\omega(2)}\rho(x-\mathcal{M}x) \leq \rho(x-y)$ or $\frac{1}{\omega(2)}\rho(\mathcal{M}x-\mathcal{M}^2x) \leq \rho(\mathcal{M}x-y)$.

Proof. According to Remark 1, one has $\frac{1}{\omega(2)}\rho(x-\mathcal{M}x) < \rho(x-\mathcal{M}x)$, which, using the definition of the condition (ρC), leads to $\rho(\mathcal{M}x-\mathcal{M}^2x) \leq \rho(x-\mathcal{M}x)$.

We prove the second statement by assuming the contrary. Suppose that $\frac{1}{\omega(2)}\rho(x-\mathcal{M}x) > \rho(x-y)$ and $\frac{1}{\omega(2)}\rho(\mathcal{M}x-\mathcal{M}^2x) > \rho(\mathcal{M}x-y)$. Then, using the Δ_2-condition and the convexity of ρ, one finds

$$\begin{aligned}
\rho(x-\mathcal{M}x) &= \rho\left(2\left(\frac{1}{2}(x-y)+\frac{1}{2}(y-\mathcal{M}x)\right)\right) \\
&\leq \omega(2)\rho\left(\frac{1}{2}(x-y)+\frac{1}{2}(y-\mathcal{M}x)\right) \\
&\leq \frac{\omega(2)}{2}\left(\rho(x-y)+\rho(y-\mathcal{M}x)\right) \\
&< \frac{\omega(2)}{2}\left(\frac{1}{\omega(2)}\rho(x-\mathcal{M}x)+\frac{1}{\omega(2)}\rho(\mathcal{M}x-\mathcal{M}^2x)\right) \\
&\leq \rho(x-\mathcal{M}x).
\end{aligned}$$

We have found $\rho(x-\mathcal{M}x) < \rho(x-\mathcal{M}x)$, which is not possible. Therefore, the initial assumption does not hold, hence the proof. □

Before stating and proving our main outcome, we recall some elementary properties of lim sup.

Lemma 5. *Let $\{\mu_\ell\}$ and $\{\nu_\ell\}$ be two bounded real sequences. Then:*

(i) $\limsup\limits_{\ell\to\infty} \max\{\mu_\ell, \nu_\ell\} = \max\{\limsup\limits_{\ell\to\infty} \mu_\ell, \limsup\limits_{\ell\to\infty} \nu_\ell\}$;
(ii) let $\eta_\ell = \alpha_\ell\mu_\ell + (1-\alpha_\ell)\nu_\ell$, with $\alpha_\ell \in [0,1]$ convergent to a real number $\alpha \in [0,1]$. Then, $\limsup\limits_{\ell\to\infty} \eta_\ell \leq \alpha \limsup\limits_{\ell\to\infty} \mu_\ell + (1-\alpha)\limsup\limits_{\ell\to\infty} \nu_\ell$.

Theorem 1. *Assume that X_ρ is a ρ-complete modular space. Assume also that ρ satisfies the Δ_2 condition, is convex, has the Fatou and the $(UUC1)$ properties. Let S denote a nonempty, ρ-bounded, ρ-closed and convex subset of X_ρ and let $\mathcal{M}\colon S \to S$ satisfy condition (ρC). If $\{\alpha_\ell\}$, with $0 < a \leq \alpha_\ell \leq b < 1$ is a real sequence convergent to α, consider the iterative process $x_{\ell+1} = \alpha_\ell x_\ell + (1-\alpha_\ell)\mathcal{M}x_\ell$, for given $x_0 \in S$. Then,* $\mathrm{Fix}\,(\mathcal{M}) \neq \emptyset$ *if and only if $\lim\limits_{\ell\to\infty} \rho(\mathcal{M}x_\ell - x_\ell) = 0$.*

Proof. We start with the direct implication. Let $p \in \mathrm{Fix}\,(\mathcal{M})$. Then,

$$\frac{1}{\omega(2)}\rho(p-\mathcal{M}p) = 0 \leq \rho(p-x),\ \forall x \in S.$$

Therefore, by applying the (ρC) condition, one obtains

$$\rho(\mathcal{M}x - \mathcal{M}p) \leq \rho(x - p), \quad \forall x \in S. \tag{1}$$

Using the convexity of the modular and inequality (1), one finds

$$\begin{aligned} \rho(x_{\ell+1} - p) &= \rho(\alpha_\ell(x_\ell - p) + (1-\alpha_\ell)(\mathcal{M}x_\ell - \mathcal{M}p)) \\ &\leq \alpha_\ell \rho(x_\ell - p) + (1-\alpha_\ell)\rho(\mathcal{M}x_\ell - \mathcal{M}p) \\ &\leq \alpha_\ell \rho(x_\ell - p) + (1-\alpha_\ell)\rho(x_\ell - p) \\ &= \rho(x_\ell - p). \end{aligned}$$

It follows that $\{\rho(x_\ell - p)\}$ is a decreasing nonnegative sequence. Moreover, since S is convex, the sequence $\{x_\ell\}$ lies entirely in S and also, since S is ρ-bounded, $\rho(x_\ell - p) < \infty$, $\forall n \in \mathbb{N}$. These lead, on one hand, to the conclusion that $\{\rho(x_\ell - p)\}$ is convergent. Let

$$r = \lim_{\ell\to\infty} \rho(x_\ell - p). \tag{2}$$

By denoting $y_\ell = \mathcal{M}x_\ell$, one finds $\rho(y_\ell - p) = \rho(\mathcal{M}x_\ell - \mathcal{M}p) \leq \rho(x_\ell - p)$. Therefore,

$$\limsup_{\ell\to\infty} \rho(y_\ell - p) \leq r. \tag{3}$$

In addition,

$$\lim_{n\to\infty} \rho(\alpha_\ell(x_\ell - p) + (1-\alpha_\ell)(y_\ell - \mathcal{M}p)) = \lim_{\ell\to\infty} \rho(x_{\ell+1} - p) = r. \tag{4}$$

Using inequalities (2), (3) and (4) and the fact that ρ is (UUC1), it follows, according to Lemma 1, that $\lim_{\ell\to\infty} \rho(x_\ell - y_\ell) = \lim_{\ell\to\infty} \rho(x_\ell - \mathcal{M}x_\ell) = 0$, which ends the proof.

In the following, let us prove the converse statement. Let $\tau, \bar{\tau}\colon S \to [0,\infty]$ denote the ρ-type functions corresponding to sequences $\{x_\ell\}$ and $\{y_\ell\}$, respectively. We shall prove first that, for each $p \in S$, $\bar{\tau}(\mathcal{M}p) \leq \tau(p)$. Indeed, for each $n \in \mathbb{N}$, according to Lemma 4, one has either $\frac{1}{\omega(2)}\rho(x_\ell - \mathcal{M}x_\ell) \leq \rho(x_\ell - p)$ or $\frac{1}{\omega(2)}\rho(y_\ell - \mathcal{M}y_\ell) \leq \rho(y_\ell - p)$.

Case 1. Suppose that $\frac{1}{\omega(2)}\rho(x_\ell - \mathcal{M}x_\ell) \leq \rho(x_\ell - p)$. Then, using the (ρC) condition, it follows $\rho(\mathcal{M}x_\ell - \mathcal{M}p) \leq \rho(x_\ell - p)$, i.e.,

$$\rho(y_\ell - \mathcal{M}p) \leq \rho(x_\ell - p). \tag{5}$$

Case 2. Suppose that $\frac{1}{\omega(2)}\rho(x_\ell - \mathcal{M}x_\ell) > \rho(x_\ell - p)$. Then, $\frac{1}{\omega(2)}\rho(y_\ell - \mathcal{M}y_\ell) \leq \rho(y_\ell - p)$ and, due to (ρC) condition, $\rho(\mathcal{M}y_\ell - \mathcal{M}p) \leq \rho(y_\ell - p)$. We use these together with the Δ_2 condition and the convexity of ρ to derive the following chain of inequalities:

$$\begin{aligned} \rho(y_\ell - \mathcal{M}p) &\leq \frac{\omega(2)}{2}\left(\rho(y_\ell - \mathcal{M}y_\ell) + \rho(\mathcal{M}y_\ell - \mathcal{M}p)\right) \\ &\leq \frac{\omega(2)}{2}\left(\rho(y_\ell - \mathcal{M}y_\ell) + \rho(y_\ell - p)\right) \\ &\leq \frac{\omega(2)}{2}\left[\rho(y_\ell - \mathcal{M}y_\ell) + \frac{\omega(2)}{2}\left(\rho(y_\ell - x_\ell) + \rho(x_\ell - p)\right)\right] \\ &< \frac{\omega(2)}{2}\left[\rho(y_\ell - \mathcal{M}y_\ell) + \frac{\omega(2)}{2}\rho(y_\ell - x_\ell) + \frac{\omega(2)}{2}\frac{1}{\omega(2)}\rho(x_\ell - y_\ell)\right]. \end{aligned}$$

Let us evaluate the first term inside the square bracket. Knowing that $y_\ell = \mathcal{M}x_\ell$ and using Lemma 4 (i), we find $\rho(y_\ell - \mathcal{M}y_\ell) = \rho(\mathcal{M}x_\ell - \mathcal{M}^2 x_\ell) \le \rho(x_\ell - \mathcal{M}x_\ell) = \rho(x_\ell - y_\ell)$ and, by turning back in the above inequality, we obtain

$$\rho(y_\ell - \mathcal{M}p) < \frac{(3+\omega(2))\omega(2)}{4}\rho(x_\ell - y_\ell). \tag{6}$$

Overall, from inequalities (5) and (6), it follows that $\rho(y_\ell - \mathcal{M}p) \le \max\left\{\rho(x_\ell - p), \frac{(3+\omega(2))\omega(2)}{4}\rho(x_\ell - y_\ell)\right\}$. Applying $\limsup\limits_{\ell\to\infty}$ and using Lemma 5 (i), one finds

$$\bar{\tau}(\mathcal{M}p) \le \max\{\tau(p), 0\} = \tau(p). \tag{7}$$

In addition, $\rho(x_{\ell+1} - p) = \rho(\alpha_\ell(x_\ell - p) + (1-\alpha_\ell)(p - y_\ell)) \le \alpha_\ell\rho(x_\ell - p) + (1-\alpha_\ell)\rho(p - y_\ell)$. Again, from Lemma 5, it follows that $\tau(p) \le \alpha\tau(p) + (1-\alpha)\bar{\tau}(p)$, where $\alpha \in (0,1)$ is the limit of the sequence $\{\alpha_\ell\}$, thus

$$\tau(p) \le \bar{\tau}(p). \tag{8}$$

Combining relations (7) and (8), one finds

$$\tau(\mathcal{M}p) \le \bar{\tau}(\mathcal{M}p) \le \tau(p) \le \bar{\tau}(p),\ \forall p \in S. \tag{9}$$

Finally, let $\{c_\ell\}$ be a minimizing sequence of τ. Then, $\lim\limits_{n\to\infty}\tau(c_\ell) = r(S)$. Since, according to inequality (9) $\tau(\mathcal{M}c_\ell) \le \tau(c_\ell)$, it follows that $\{\mathcal{M}c_\ell\}$ is also a minimizing sequence of τ. By similar arguments, $\{\mathcal{M}^2 c_\ell\}$ is a minimizing sequence too. According to Lemma 2, all of the minimizing sequences are ρ-convergent to the same limit c, i.e.,

$$\lim_{\ell\to\infty}\rho(c_\ell - c) = \lim_{\ell\to\infty}\rho(\mathcal{M}c_\ell - c) = \lim_{\ell\to\infty}\rho(\mathcal{M}^2 c_\ell - c) = 0.$$

On the other hand, from Lemma 4, $\rho(\mathcal{M}c_\ell - \mathcal{M}c) \le \rho(c_\ell - c)$ or $\rho(\mathcal{M}^2 c_\ell - \mathcal{M}c) \le \rho(\mathcal{M}c_\ell - c)$, meaning that $\{\mathcal{M}c_\ell\}$ or $\{\mathcal{M}^2 c_\ell\}$ is also ρ-convergent to $\mathcal{M}c$. Since the ρ-limit is unique, it follows that $\mathcal{M}c = c$. $\square$

The following outcome states that each Suzuki nonexpansive mapping on a nonempty, convex, ρ-bounded and ρ-closed subset of $\mathbb{R}$ has fixed points. In order to prove that, we use the Lemma of Goebel and Kirk.

Lemma 6. *Consider two bounded sequences $\{x_\ell\}$ and $\{y_\ell\}$ in a Banach space $(X, ||\cdot||)$ and $\alpha \in (0,1)$. Assume that $x_{\ell+1} = (1-\alpha)x_\ell + \alpha y_\ell$ and $||y_{\ell+1} - y_\ell|| \le ||x_{\ell+1} - x_\ell||$ for all ℓ. Then, $\lim\limits_{\ell\to\infty}||y_\ell - x_\ell|| = 0$ (Goebel and Kirk [15]).*

Corollary 1. *Let X_ρ be a ρ-complete modular space with $X_\rho \subset \mathbb{R}$. Assume that ρ, S and $\mathcal{M}\colon S \to S$ satisfy all the conditions from Theorem 1. Then,* Fix $(\mathcal{M}) \neq \emptyset$.

Proof. When ρ denotes a modular on $\mathbb{R}$, the following features can be immediately extracted from the properties of the modular function:

1. $\rho(x) = \rho(|x|),\ \forall x \in \mathbb{R}$;
2. ρ is a nondecreasing function on $\mathbb{R}_+$;
3. ρ is a continuous function.

Suppose that $\alpha \in \left[\frac{1}{2}, 1\right)$ and consider the iterative process $x_{\ell+1} = (1-\alpha)x_\ell + \alpha\mathcal{M}x_\ell$, for given $x_0 \in S$. Then, $\mathcal{M}x_\ell - x_\ell = \frac{1}{\alpha}(x_{\ell+1} - x_\ell)$. Using properties 1 and 2 of ρ, we obtain

$$\frac{1}{\omega(2)}\rho(x_\ell - \mathcal{M}x_\ell) = \frac{1}{\omega(2)}\rho\left(\frac{1}{\alpha}(x_{\ell+1} - x_\ell)\right) \leq \frac{1}{\omega(2)}\rho\left(2(x_{\ell+1} - x_\ell)\right) \leq \rho(x_{\ell+1} - x_\ell)$$

and, since ρ satisfies condition (ρC), it follows that $\rho(\mathcal{M}x_{\ell+1} - \mathcal{M}x_\ell) \leq \rho(x_{\ell+1} - x_\ell)$. Let $y_\ell = \mathcal{M}x_\ell$. Then, $\rho(y_{\ell+1} - y_\ell) \leq \rho(x_{\ell+1} - x_\ell)$, which, by considering again properties 1 and 2, leads to the conclusion that $|y_{\ell+1} - y_\ell| \leq |x_{\ell+1} - x_\ell|$. Applying Lemma 6, one finds that $\lim_{\ell\to\infty} |y_\ell - x_\ell| = 0$. Using the continuity of ρ stated in property 3, it follows that $\lim_{\ell\to\infty} \rho(y_\ell - x_\ell) = \lim_{\ell\to\infty} \rho(\mathcal{M}x_\ell - x_\ell) = 0$, which, according to Theorem 1, leads to the desired conclusion. □

Remark 2. *The above outcome may be extended to arbitrary Banach spaces, whenever the modular ρ is defined in connection with the norm, via a continuous, nondecreasing function φ, i.e., $\rho(x) = \varphi(||x||)$. A natural question rises then: is it possible to extend the results above to modulars which are not necessarily connected to preexisting norms? Obviously, this would be possible if Lemma 6 could be extended to arbitrary modular spaces.*

4. Kirk's Lemma in Modular Spaces

In order to prove the unrestricted existence of fixed points of the generalized nonexpansive mappings, in [12], Suzuki used the famous Lemma of Goebel and Kirk, initiated by Kirk in [14] in connection with spaces of hyperbolic type and extended afterwards by Goebel and Kirk in [15], by Ishikawa in [16] or by Suzuki in [17]. Obviously, we aim to find a similar outcome on modular spaces. The main challenge is to obtain generalization without using the triangle inequality. This inequality is used several times in the proof of the original outcome on spaces of hyperbolic type, but it is missing from the properties of a modular space.

Let us start with recalling the initial result of Kirk as it was phrased in [14]. Assume that X is endowed with a metric d and contains a family of metric lines, and any two distinct points belong to one and only member of that family. Denote by $\sigma[x, y]$ the metric segment connecting x and y. In addition, assume that, for all $x, y, z \in X$, one has $d(m_1, m_2) \leq \frac{1}{2}d(x, y)$, where m_1 and m_2 are the midpoints of the metric segments $\sigma[z, x]$ and $\sigma[z, y]$. Then, (X, d) is called a hyperbolic type. A more detailed approach on the geometric properties deriving from metric structures is provided in [18].

Lemma 7. *Let $\{x_\ell\}$ and $\{y_\ell\}$ be two sequences in the hyperbolic type space (X, d) and $\alpha \in (0, 1)$ and suppose that, for all $\ell \in \mathbb{N}$, the following conditions are satisfied [14]:*

(i) $x_{\ell+1}$ *is the point of* $\sigma[x_\ell, y_\ell]$ *for which* $d(x_\ell, x_{\ell+1}) = \alpha d(x_\ell, y_\ell)$;
(ii) $d(y_\ell, y_{\ell+1}) \leq d(x_\ell, x_{\ell+1})$.

Then, for all $i, \ell \in \mathbb{N}$,

$$d(y_{i+\ell}, x_i) \geq (1-\alpha)^{-\ell}\left[d(y_{i+\ell}, x_{i+\ell}) - d(y_i, x_i)\right] + (1+\ell\alpha)d(y_i, x_i).$$

In the following, we extend the above result to modular spaces.

Lemma 8. *Let ρ be a convex modular with Δ_2 property. Let $\{x_\ell\}$ and $\{y_\ell\}$ be two sequences in X_ρ such that, for each $\ell \in \mathbb{N}$,*

(i) $x_{\ell+1} = (1-\alpha)x_\ell + \omega(\alpha)y_\ell$;
(ii) $\rho(y_{\ell+1} - y_\ell) \leq \rho(x_{\ell+1} - x_\ell)$.

Then, for all $i, \ell \in \mathbb{N}$, $\ell \neq 0$,

$$\begin{aligned}\rho(\mu y_{i+\ell} - x_i) \quad &\geq \quad (1-\alpha)^{-\ell}\rho(\mu y_{i+\ell} - x_{i+\ell}) \\ &- \quad \frac{\alpha\omega(\mu)\omega(\alpha)}{(1-\alpha)}\sum_{k=0}^{\ell-1}\left(\sum_{p=0}^{k}\frac{\omega(\ell-p)}{\ell-p}(1-\alpha)^{-p}\right)\rho(\mu y_{i+k} - x_{i+k}),\end{aligned} \tag{10}$$

where $\mu = \dfrac{\omega(\alpha)}{\alpha}$.

Proof. For simplicity, let us denote $A_\ell^k = \displaystyle\sum_{p=0}^{k}\frac{\omega(\ell-p)}{\ell-p}(1-\alpha)^{-p}$ for $\ell \geq 1$, $k \leq \ell - 1$. Then, inequality (10) can be rewritten as

$$P_i(\ell): \ \rho(\mu y_{i+\ell} - x_i) \geq (1-\alpha)^{-\ell}\rho(\mu y_{i+\ell} - x_{i+\ell}) - \frac{\alpha\omega(\mu)\omega(\alpha)}{(1-\alpha)}\sum_{k=0}^{\ell-1}A_\ell^k\rho(\mu y_{i+k} - x_{i+k}). \tag{11}$$

Let us prove it by induction on ℓ. If $\ell = 1$, one has $A_1^0 = \omega(1) = 1$ and relation (11) becomes in this particular case

$$P_i(1): \ \rho(\mu y_{i+1} - x_i) \geq \frac{1}{1-\alpha}\rho(\mu y_{i+1} - x_{i+1}) - \frac{\alpha\omega(\mu)\omega(\alpha)}{(1-\alpha)}\rho(\mu y_i - x_i),$$

which can be easily checked to be true. Indeed, by evaluating the element $\mu y_{i+1} - x_{i+1}$, we notice the possibility of rewriting it in several equivalent forms, based on condition (i) and the definition of μ. More precisely,

$$\begin{aligned}\mu y_{i+1} - x_{i+1} \quad &= \quad \mu y_{i+1} - ((1-\alpha)x_i + \omega(\alpha)y_i) \\ &= \quad (1-\alpha)\mu y_{i+1} + \alpha\mu y_{i+1} - (1-\alpha)x_i - \alpha\mu y_i \\ &= \quad (1-\alpha)(\mu y_{i+1} - x_i) + \alpha\mu(y_{i+1} - y_i).\end{aligned}$$

Moreover, by using the fact that ρ is convex and from condition (ii), and also by invoking the property $\rho(\alpha x) \leq \omega(\alpha)\rho(x)$, $\forall\alpha \geq 0$, $\forall x \in X_\rho$ of the growth function (see Remark 1), one finds

$$\begin{aligned}\rho(\mu y_{i+1} - x_{i+1}) \quad &= \quad \rho((1-\alpha)(\mu y_{i+1} - x_i) + \alpha\mu(y_{i+1} - y_i)) \\ &\leq \quad (1-\alpha)\rho(\mu y_{i+1} - x_i) + \alpha\omega(\mu)\rho(y_{i+1} - y_i) \\ &\leq \quad (1-\alpha)\rho(\mu y_{i+1} - x_i) + \alpha\omega(\mu)\rho(x_{i+1} - x_i) \\ &= \quad (1-\alpha)\rho(\mu y_{i+1} - x_i) + \alpha\omega(\mu)\rho(\alpha(\mu y_i - x_i)) \\ &\leq \quad (1-\alpha)\rho(y_{i+1} - x_i) + \alpha\omega(\mu)\omega(\alpha)\rho(\mu y_i - x_i),\end{aligned}$$

leading to the announced relation.

Let assume next that $P_i(\ell)$ is true for a given integer ℓ and for all i. Replacing i with $i+1$ in relation (11) leads to

$$P_{i+1}(\ell): \ \rho(\mu y_{i+\ell+1} - x_{i+1}) \geq (1-\alpha)^{-\ell}\rho(\mu y_{i+\ell+1} - x_{i+\ell+1}) - \frac{\alpha\omega(\mu)\omega(\alpha)}{(1-\alpha)}\sum_{k=0}^{\ell-1}A_\ell^k\rho(\mu y_{i+k+1} - x_{i+k+1}). \tag{12}$$

On the other side, using again the properties of the modular being convex and having the Δ_2 property, together with condition (ii) and Remark 1, one finds

$$\begin{aligned}
\rho(\mu y_{i+\ell+1} - x_{i+1}) &\leq (1-\alpha)\rho(\mu y_{i+\ell+1} - x_i) + \alpha\omega(\mu)\rho(y_{i+\ell+1} - y_i) \\
&\leq (1-\alpha)\rho(\mu y_{i+\ell+1} - x_i) + \alpha\omega(\mu)\frac{\omega(\ell+1)}{\ell+1}\sum_{k=0}^{\ell}\rho(\mu y_{i+k+1} - y_{i+k}) \\
&\leq (1-\alpha)\rho(\mu y_{i+\ell+1} - x_i) + \alpha\omega(\mu)\frac{\omega(\ell+1)}{\ell+1}\sum_{k=0}^{\ell}\rho(x_{i+k+1} - x_{i+k}) \\
&\leq (1-\alpha)\rho(\mu y_{i+\ell+1} - x_i) + \alpha\omega(\mu)\omega(\alpha)\frac{\omega(\ell+1)}{\ell+1}\sum_{k=0}^{\ell}\rho(\mu y_{i+k} - x_{i+k}). \quad (13)
\end{aligned}$$

Combining the inequalities (12) and (13) leads to

$$\begin{aligned}
\rho(\mu y_{i+\ell+1} - x_i) &\geq (1-\alpha)^{-(\ell+1)}\rho(\mu y_{i+\ell+1} - x_{i+\ell+1}) - \frac{\alpha\omega(\mu)\omega(\alpha)}{(1-\alpha)^2}\sum_{k=1}^{\ell}A_\ell^{k-1}\rho(\mu y_{i+k} - x_{i+k}) \\
&- \frac{\alpha\omega(\mu)\omega(\alpha)}{1-\alpha}\frac{\omega(\ell+1)}{\ell+1}\sum_{k=0}^{\ell}\rho(\omega(\mu)y_{i+k} - x_{i+k}) \\
&= (1-\alpha)^{-(\ell+1)}\rho(\mu y_{i+\ell+1} - x_{i+\ell+1}) - \frac{\alpha\omega(\mu)\omega(\alpha)}{1-\alpha}\left[\frac{\omega(\ell+1)}{\ell+1}\rho(\mu y_i - x_i)\right. \\
&- \left.\sum_{k=1}^{\ell}\left(\frac{A_\ell^{k-1}}{1-\alpha} + \frac{\omega(\ell+1)}{\ell+1}\right)\rho(\mu y_{i+k} - x_{i+k})\right]. \quad (14)
\end{aligned}$$

On the other side, $\dfrac{\omega(\ell+1)}{\ell+1} = A_{\ell+1}^0$ and

$$\begin{aligned}
\frac{A_\ell^{k-1}}{1-\alpha} + \frac{\omega(\ell+1)}{\ell+1} &= \frac{1}{1-\alpha}\left(\sum_{p=0}^{k-1}\frac{\omega(\ell-p)}{\ell-p}(1-\alpha)^{-p}\right) + \frac{\omega(\ell+1)}{\ell+1} \\
&= \sum_{p=1}^{k}\frac{\omega(\ell+1-p)}{\ell+1-p}(1-\alpha)^{-p} + \frac{\omega(\ell+1)}{\ell+1} \\
&= \sum_{p=0}^{k}\frac{\omega(\ell+1-p)}{\ell+1-p}(1-\alpha)^{-p} \\
&= A_{\ell+1}^k.
\end{aligned}$$

Substituting these in relation (14), one finds precisely $P_i(\ell+1)$, which completes the proof. □

The following statement refers to an important inequality specific to ρ-bounded subsets.

Corollary 2. *Let X_ρ be a modular space and let $S \subset X_\rho$ be a ρ-bounded subset with* $\mathrm{diam}_\rho(S) = M$. *Suppose that ρ satisfies condition Δ_2 and is convex. Let $\{x_\ell\}$ and $\{y_\ell\}$ be two sequences in S satisfying the conditions from Lemma 8 and denote $r = \limsup_{\ell\to\infty}\rho(\mu y_\ell - x_\ell)$. Then,*

$$M \geq \frac{r}{(1-\alpha)^\ell}\left[1 - \alpha\omega(\mu)\omega(\alpha)\sum_{k=0}^{\ell-1}\omega(k+1)(1-\alpha)^k\right]. \quad (15)$$

Proof. The inequality is a direct consequence of taking $\limsup_{i\to\infty}$ in relation (10). □

The following two examples are meant to illustrate the applicability of inequality (15). In particular, when dealing with Banach spaces, Corollary 2 leads directly to Lemma 6 of Goebel and Kirk.

Example 1. *If $(X, ||\cdot||)$ is a normed space, by taking the modular ρ to be exactly the norm, we obtain the growth function $\omega(t) = t$, $\forall t \geq 0$. Consider $\{x_\ell\}$ and $\{y_\ell\}$ in X such that, for each $\ell \in \mathbb{N}$,*

(i) $x_{\ell+1} = (1-\alpha)x_\ell + \alpha y_\ell$;
(ii) $\rho(y_{\ell+1} - y_\ell) \leq \rho(x_{\ell+1} - x_\ell)$.

When substituting these in inequality (15), one finds

$$\begin{aligned} M &\geq \frac{r}{(1-\alpha)^\ell}\left[1 - \alpha^2 \sum_{k=0}^{\ell-1}(k+1)(1-\alpha)^k\right] \\ &= \frac{r}{(1-\alpha)^\ell}\left[1 - \alpha^2 \frac{\ell(1-\alpha)^{\ell+1} - (\ell+1)(1-\alpha)^\ell + 1}{\alpha^2}\right] \\ &= r(\ell\alpha - 1),\ \forall \ell, \end{aligned}$$

and this is true if and only if $r = \limsup\limits_{\ell\to\infty} \rho(y_\ell - x_\ell) = 0$, *leading to* $\lim\limits_{\ell\to\infty} \rho(y_\ell - x_\ell) = 0$, *which is the conclusion of Lemma 6.*

Let us recall that Remark 2 emphasized the fact that, whenever we deal with modulars defined in connection with a particular norm via a continuous, nondecreasing function, then the (original) Lemma 6 of Goebel and Kirk, together with Theorem 1, provide enough arguments to state the (unrestricted) existence of fixed points (in particular, when the modular is exactly the norm of a Banach space, we find the Suzuki's original outcome). In the following, we provide an example of a modular function which can not be connected directly (as described above) with a specific norm. Moreover, we analyze the effect of this modular from the perspective of Corollary 2.

Example 2. *Consider the modular* $\rho\colon \mathbb{R}^2 \to [0,\infty]$, $\rho(x_1, x_2) = |x_1| + x_2^2$. *The growth function corresponding to this modular is* $\omega(t) = t^2,\ t \geq 0$. *Consider also two sequences* $\{x_\ell\}$ *and* $\{y_\ell\}$ *in* $\mathbb{R}^2$ *such that, for a given parameter* $\alpha \in (0,1)$ *and for each* $\ell \in \mathbb{N}$,

(i) $x_{\ell+1} = (1-\alpha)x_\ell + \alpha^2 y_\ell$;
(ii) $\rho(y_{\ell+1} - y_\ell) \leq \rho(x_{\ell+1} - x_\ell)$.

Substituting in relation (15) leads to

$$\begin{aligned} M &\geq \frac{r}{(1-\alpha)^\ell}\left[1 - \alpha^5 \sum_{k=0}^{\ell-1}(k+1)^2(1-\alpha)^k\right] \\ &= \frac{r}{(1-\alpha)^\ell}\left[1 - \alpha^5 \frac{2 - \alpha - (1-\alpha)^\ell[(\ell\alpha+1)^2 + 1 - \alpha]}{\alpha^3}\right] \\ &= [1 - \alpha^2(2-\alpha)]\frac{r}{(1-\alpha)^\ell} + r\alpha^2[(\ell\alpha+1)^2 + 1 - \alpha] \\ &= (1-\alpha)(1+\alpha-\alpha^2)\frac{r}{(1-\alpha)^\ell} + r\alpha^2[(\ell\alpha+1)^2 + 1 - \alpha],\ \forall \ell. \end{aligned}$$

Since $(1-\alpha)(1+\alpha-\alpha^2) \geq 0,\ \forall \alpha \in (0,1)$ *we find* $r = \limsup\limits_{\ell\to\infty} \rho(\mu y_\ell - x_\ell) = 0$, *which is* $\lim\limits_{\ell\to\infty} \rho(\alpha y_\ell - x_\ell) = 0$.

5. Conclusions

This paper provides two important outcomes: a necessary and sufficient condition for a Suzuki nonexpansive mapping on a modular space to have fixed points and an extension of Goebel and Kirk's Lemma. In particular, when dealing with Banach spaces where the modular is precisely the norm (and the growth function is precisely the identity), combining the two outcomes leads to the original results of Suzuki. Nevertheless, on arbitrary modular spaces, the combination of those two does not provide enough arguments to state the unrestricted existence of fixed points. In order to be able to do so, stronger versions for Lemma 8 and its corollary would be necessary. More precisely, it would be most advantageous if the modular distances involved in inequality (10) would not include the parameter μ. The proof of such an outcome would provide a valuable breakthrough.

Author Contributions: The authors contributed equally to the conceptualization, formal analysis and validation of the content. Both authors read and approved the final manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Orlicz, W. Uber konjugierte exponentenfolgen. *Studia Math.* **1931**, *3*, 200–211. [CrossRef]
2. Nakano, H. *Modulared Semi-Ordered Linear Spaces*; Maruzen Co. Ltd.: Tokyo, Japan, 1950.
3. Musielak, J.; Orlicz, W. On modular spaces. *Studia Math.* **1959**, *18*, 591–597. [CrossRef]
4. Kozlowski, W.M. Modular Function Spaces. In *Series of Monographs and Textbooks in Pure and Applied Mathematics*; Dekker: New York, NY, USA, 1988; Volume 122.
5. Dehaish, B.A.B.; Kozlowski, W.M. Fixed point iteration processes for asymptotic pointwise nonexpansive mapping in modular function spaces. *Fixed Point Theory Appl.* **2012**, *118*, 1–23. [CrossRef]
6. Khamsi, M.A.; Kozlowski, W.M. *Fixed Point Theory in Modular Function Spaces*; Birkhauser: New York, NY, USA, 2015.
7. Okeke, G.A.; Bishop, S.A.; Khan, S.H. Iterative approximation of fixed point of multivalued ρ-quasi-nonexpansive mappings in modular function spaces with applications. *J. Funct. Spaces* **2018**, 1–9. [CrossRef]
8. Khan, S.H. Approximating fixed points of (λ, ρ)-firmly nonexpansive mappings in modular function spaces. *Arab. J. Math.* **2018**, *7*, 281–287. [CrossRef]
9. Abbas, M.; Khan, S.H.; Razani, A. Fixed point theorems of expansive type mappings in modular function spaces. *Fixed Point Theory* **2011**, *12*, 235–240.
10. Abdou, A.A.N.; Khamsi, M.A. Fixed point theorems in modular vector spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4046–4057. [CrossRef]
11. Alfuraidan, M.R.; Khamsi, M.A.; Manav, N. A fixed point theorem for uniformly Lipschitzian mappings in modular vector spaces. *Filomat* **2017**, *31*, 5435–5444. [CrossRef]
12. Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095. [CrossRef]
13. Thakur, B.S.; Thakur, D.; Postolache, M. A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings. *Appl. Math. Comput.* **2016**, *275*, 147–155. [CrossRef]
14. Kirk, W.A. Krasnoselskii's iteration process in hyperbolic space. *Numer. Funct. Anal. Optim.* **1982**, *4*, 371–381. [CrossRef]
15. Goebel, K.; Kirk, W.A. Iteration processes for nonexpansive mappings. In *Topological Methods in Nonlinear Functional Analysis*; Singh, S.P., Thomeier, S., Watson, B., Eds.; American Mathematical Soc.: Providence, RI, USA, 1983; Volume 21, pp. 115–123.
16. Ishikawa, S. Fixed points and iteration of a nonexpansive mapping in a Banach space. *Proc. Am. Math. Soc.* **1976**, *59*, 65–71. [CrossRef]
17. Suzuki, T. Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. *Fixed Point Theory Appl.* **2005**, *2005*, 685918. [CrossRef]
18. Andalafte, E.Z.; Blumenthal, L.M. Metric characterizations of Banach and Euclidean spaces. *Fund. Math.* **1964**, *55*, 23–55. [CrossRef]