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New Families of Three-Variable Polynomials Coupled with Well-Known Polynomials and Numbers

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Received: 27 December 2018; Accepted: 3 February 2019; Published: 20 February 2019

Abstract: In this paper, firstly the definitions of the families of three-variable polynomials with the new generalized polynomials related to the generating functions of the famous polynomials and numbers in literature are given. Then, the explicit representation and partial differential equations for new polynomials are derived. The special cases of our polynomials are given in tables. In the last section, the interesting applications of these polynomials are found.

Keywords: Fibonacci polynomials; Lucas polynomials; trivariate Fibonacci polynomials; trivariate Lucas polynomials; generating functions

MSC: 11B39; 11B37; 05A19

1. Introduction

In literature, the Fibonacci and Lucas numbers have been studied extensively and some authors tried to enhance and derive some directions to mathematical calculations using these special numbers [1–3]. By favour of the Fibonacci and Lucas numbers, one of these directions verges on the tribonacci and the tribonacci-Lucas numbers. In fact, M. Feinberg in 1963 has introduced the tribonacci numbers and then derived some properties for these numbers in [4–7]. Elia in [4] has given and investigated the tribonacci-Lucas numbers. The tribonacci numbers T_n for any integer $n > 2$ are defined via the following recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \tag{1}$$

with the initial values $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$. Similarly, by way of the initial values $K_0 = 3$, $K_1 = 1$, and $K_2 = 3$, the tribonacci-Lucas numbers K_n are given by the recurrence relation

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}. \tag{2}$$

By dint of the above extensions, the tribonacci and tribonacci-Lucas numbers are introduced with the help of the following generating functions, respectively:

$$\sum_{n=0}^{\infty} T_n t^n = \frac{t}{1-t-t^2-t^3}, \tag{3}$$

and

$$\sum_{n=0}^{\infty} K_n t^n = \frac{3-2t-t^2}{1-t-t^2-t^3}. \tag{4}$$

Moreover, some authors define a large class of polynomials by using the Fibonacci and the tribonacci numbers [6–9]. Firstly, the well-known Fibonacci polynomials are defined via the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x)\,,$$

with $F_0(x) = 0$, $F_1(x) = 1$. The well-known Lucas polynomials are defined with the help of the recurrence relation

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x)\,,$$

with $L_0(x) = 2$, $L_1(x) = x$.

Fibonacci or Fibonacci-like polynomials have been studied by many mathematicians for many years. Recently, in [10], Kim et al. kept in mind the sums of finite products of Fibonacci polynomials and of Chebyshev polynomials of the second kind and obtained Fourier series expansions of functions related to them. In [11], Kim et al. studied the convolved Fibonacci numbers by using the generating functions of them and gave some new identities for the convolved Fibonacci numbers. In [12], Wang and Zhang studied some sums of powers Fibonacci polynomials and Lucas polynomials. In [13], Wu and Zhang obtained the several new identities involving the Fibonacci polynomials and Lucas polynomials.

Afterwards, by giving the Pell and Jacobsthal polynomials, in 1973, Hoggatt and Bicknell [6] introduced the tribonacci polynomials. The tribonacci polynomials are defined by the recurrence relation for $n \geq 0$,

$$t_{n+3}(x) = x^2 t_{n+2}(x) + x t_{n+1}(x) + t_n(x)\,, \tag{5}$$

where $t_0(x) = 0$, $t_1(x) = 1$, and $t_2(x) = x^2$. The tribonacci-Lucas polynomials are defined by the recurrence relation for $n \geq 0$,

$$k_{n+3}(x) = x^2 k_{n+2}(x) + x k_{n+1}(x) + k_n(x)\,, \tag{6}$$

where $k_0(x) = 3$, $k_1(x) = x^2$, and $k_2(x) = x^4 + 2x$, respectively. Here we note that $t_n(1) = T_n$ which is the tribonacci numbers and $k_n(1) = K_n$ which is the tribonacci-Lucas numbers. Also for these polynomials, we have the generating function as follows:

$$\sum_{n=0}^{\infty} t_n(x) t^n = \frac{t}{1 - x^2 t - x t^2 - t^3}\,, \tag{7}$$

and

$$\sum_{n=0}^{\infty} k_n(x) t^n = \frac{3 - 2x^2 t - x t^2}{1 - x^2 t - x t^2 - t^3}. \tag{8}$$

On the other hand, some authors try to define the second and third variables of these polynomials with the help of these numbers. For example [8], for integer $n > 2$, the recurrence relations of the trivariate Fibonacci and Lucas polynomials are as follows:

$$H_n(x,y,z) = xH_{n-1}(x,y,z) + yH_{n-2}(x,y,z) + zH_{n-3}(x,y,z)\,, \tag{9}$$

with $H_0(x,y,z) = 0$, $H_1(x,y,z) = 1$, $H_2(x,y,z) = x$ and

$$K_n(x,y,z) = xK_{n-1}(x,y,z) + yK_{n-2}(x,y,z) + zK_{n-3}(x,y,z)\,, \tag{10}$$

with $K_0(x,y,z) = 3$, $K_1(x,y,z) = x$, $K_2(x,y,z) = x^2 + 2y$, respectively. Also for these, we have the generating functions as follows:

$$\sum_{n=0}^{\infty} H_n(x,y,z) t^n = \frac{t}{1 - xt - yt^2 - zt^3}\,, \tag{11}$$

and

$$\sum_{n=0}^{\infty} K_n(x,y,z)t^n = \frac{3-2xt-yt^2}{1-xt-yt^2-zt^3}. \tag{12}$$

After that, Ozdemir and Simsek [14] give the family of two-variable polynomials, reducing some well-known polynomials and obtaining some properties of these polynomials. In light of these polynomials, we introduce the families of three-variable polynomials with the new generalized polynomials reduced to the generating functions of the famous polynomials and numbers in literature. Then, we obtain the explicit representations and partial differential equations for new polynomials. The special cases of our polynomials are given in tables. Also the last section, we give the interesting applications of these polynomials.

2. The New Generalized Polynomials: Definitions and Properties

Now, we introduce the original and wide generating functions reduce the well-known polynomials and the well-known numbers such as the trivariate Fibonacci and Lucas polynomials, the tribonacci and the tribonacci-Lucas polynomials, the tribonacci and the tribonacci-Lucas numbers, and so on.

Firstly, some properties of these functions are investigated. Then, in the case of the new generating function, we give some properties the particular well-known polynomials as tables.

Via the following generating functions, a new original and wide family of three-variable polynomials denoted by $S_j := S_j(x,y,z;k,m,n,c)$ is defined as follows:

$$T := \; M(t;x,y,z;k,m,n,c) = \sum_{j=0}^{\infty} S_j t^j = \frac{1}{1-x^k t - y^m t^{m+n} - z^c t^{m+n+c}}, \tag{13}$$

where $k,m,n,c \in \mathbb{N}-\{0\}$, and $\left|x^k t + y^m t^{m+n} + z^c t^{m+n+c}\right| < 1$. Now we derive the explicit representation for polynomials S_j. By means of Taylor series of the generating function of the right hand side of (13), we can write

$$T = \sum_{j=0}^{\infty} S_j t^j = \sum_{j=0}^{\infty} \left(x^k t + y^m t^{m+n} + z^c t^{m+n+c}\right)^j.$$

After that, using the binomial expansion and taking $j+s$ instead of j, we get

$$T = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty} \binom{j+s}{s} \left(x^k t\right)^j \left(t^{m+n}\right)^s \left(y^m + z^c t^c\right)^s.$$

Lastly, using the expansion of $\left(y^m + z^c t^c\right)^s$, taking $u+s$ instead of s, taking $j-(m+n+c)u$ instead of j and taking $j-(m+n)s$ instead of j, respectively, we have

$$T = \sum_{j=0}^{\infty} \sum_{s=0}^{\left\lfloor \frac{j}{n+m} \right\rfloor} \sum_{u=0}^{\left\lfloor \frac{j-(m+n)s}{n+m+c} \right\rfloor} \binom{j-(n+m-1)s-(n+m+c-1)u}{s+u} \binom{s+u}{u} \left(x^k\right)^{j-(n+m)(s+u)-cu} z^{cu} y^{ms} t^j.$$

Thus after the equalization of coefficients of t^j, we obtain

$$S_j = \sum_{s=0}^{\left\lfloor \frac{j}{n+m} \right\rfloor} \sum_{u=0}^{\left\lfloor \frac{j-(m+n)s}{n+m+c} \right\rfloor} \binom{j-(n+m-1)s-(n+m+c-1)u}{s+u} \binom{s+u}{u} \left(x^k\right)^{j-(n+m)(s+u)-cu} z^{cu} y^{ms}. \tag{14}$$

Note that for $z=0$, our polynomials reduces to the polynomials Equation (4) [14] .

Remark 1. *As a similar to Theorem 2.3 in [14], we can write the following relation*

$$S_j(2x,-1,0;1,1,1,c)=\sum_{r=0}^{j}P_{j-r}(x)P_r(x),$$

where $P_r(x)$ are the Legendre polynomials.

To obtain other wide family of well-known polynomials, we define the second new generating function for the family of the polynomials $W_j := W_j(x,y,z;k,m,n,c)$ as follows

$$\begin{aligned} R := R(t;x,y,z;k,m,n,c) &= M(t;x,y,z;k,m,n,c)t^n \\ &= \frac{t^n}{1-x^k t - y^m t^{m+n} - z^c t^{m+n+c}} \\ &= \sum_{j=0}^{\infty} W_j t^j, \end{aligned} \tag{15}$$

where $k,m,n,c \in \mathbb{N}-\{0\}$, and $\left|x^k t + y^m t^{m+n} + z^c t^{m+n+c}\right| < 1$. Similarly for $z=0$, our polynomials reduces to the polynomials in (5) in [14]. Now we give some special case. Firstly taking $k=m=n=c=1$ in (15), we give the generating function

$$\frac{t}{1-xt-yt^2-zt^3}=\sum_{j=0}^{\infty}W_j(x,y,z;1,1,1,1)t^j,$$

where $W_j(x,y,z;1,1,1,1)=H_j(x,y,z)$, which are trivariate Fibonacci polynomials in (11). Secondly, writing $k=m=n=c=1$ and $x\to x^2$, $y\to x$, $z\to 1$, we have the generating function

$$\frac{t}{1-x^2t-xt^2-t^3}=\sum_{j=0}^{\infty}W_j(x^2,x,1;1,1,1,1)t^j,$$

where $W_j(x^2,x,1;1,1,1,1)=t_j(x)$ which are the tribonacci polynomials in (7). In the above generating function, for $x=1$, we find the generating function of the tribonacci numbers in (3). Now, we give other special cases as the following table related to (15).

Now, we define a new family of the polynomials denoted by $K_j := K_j(x,y,z;k,m,n,c)$ via the generating function

$$\sum_{j=0}^{\infty}K_j t^j = \frac{\alpha(t;x,y)-\beta(t;x,y)t^n}{1-x^k t-y^m t^{m+n}-z^c t^{m+n+c}}, \tag{16}$$

where $k,m,n,c\in\mathbb{N}-\{0\}$, $\alpha(t;x,y)$ and $\beta(t;x,y)$ are arbitrary polynomials depending on t,x,y and $\left|x^k t + y^m t^{m+n} + z^c t^{m+n+c}\right| < 1$. Thirdly, via (16), we give

$$\begin{aligned} 3M(t;x,y,z;1,1,1,1)-2xR(t;x,y,z;1,1,1,1)-ytR(t;x,y,z;1,1,1,1) &= \tfrac{3-2xt-yt^2}{1-xt-yt^2-zt^3} \\ &= \sum_{j=0}^{\infty}K_j(x,y,z)t^j, \end{aligned}$$

where $K_j(x,y,z)$ are the trivariate Lucas polynomials in (12). Due to the last equation, we have the polynomial representation

$$3S_j(x,y,z;1,1,1,1)-2xW_j(x,y,z;1,1,1,1)-ytW_j(x,y,z;1,1,1,1)=K_j(x,y,z). \tag{17}$$

In (17) substituting $x \to x^2$, $y \to x$, $z \to 1$, we get

$$3S_j(x^2, x, 1; 1, 1, 1, 1) - 2xW_j(x^2, x, 1; 1, 1, 1, 1) - ytW_j(x^2, x, 1; 1, 1, 1, 1) = k_j(x),$$

where $k_j(x)$ are the tribonacci-Lucas polynomials in (6). In the above representation, for $x = 1$, we find the generating function of the tribonacci-Lucas numbers in (4).

Now, we give other special cases as Table 1 and Table 2 related to (15) and (16) respectively.

Table 1. Special cases of W_j.

x	y	z	k	m	n	c	**Special Case**
x	y	z	1	1	1	1	Trivariate Fibonacci Polynomials [8]
x^2	x	1	1	1	1	1	tribonacci Polynomials [8]
x	y	0	1	1	1	c	Bivariate Fibonacci Polynomials [9]
x	1	0	1	p	1	c	Fibonacci $p-$Polynomials [9]
$2x$	1	0	1	p	1	c	Pell $p-$Polynomials [9]
x	1	0	1	1	1	c	Fibonacci Polynomials [9]
$2x$	1	0	1	1	1	c	Pell Polynomials [9]
1	$2y$	0	k	1	1	c	Jacobsthal Polynomials [9]
$3x$	-2	0	1	1	1	c	Fermat Polynomials [15]
x	-2	0	1	1	1	c	First kind of Fermat–Horadam Polynomials [16]
x	$-\alpha$	0	1	1	1	c	Second kind of Dickson Polynomials [17]
$x+2$	-1	0	1	1	1	c	Morgan–Voyce Polynomials [18]
$x+1$	$-x$	0	1	1	1	c	Delannoy Polynomials [19]
$h(x)$	1	0	1	1	1	c	$h(x)-$Fibonacci Polynomials [2]
$p(x)$	$q(x)$	0	1	1	1	c	$(p,q)-$Fibonacci Polynomials [15]
1	1	0	k	1	1	c	Fibonacci Numbers [9]
2	1	0	1	1	1	c	Pell Numbers [9]
1	2	0	k	1	1	c	Jacobsthal Numbers [9]

Table 2. Special cases of K_j

α	β	x	y	z	k	m	n	c	**Special Case**
3	$2x+yt$	x	y	z	1	1	1	1	Trivariate Lucas Polynomials [8]
3	$2x^2+xt$	x^2	x	1	1	1	1	1	tribonacci-Lucas Polynomials [8]
2	xz	x	y	0	1	1	1	c	Bivariate Lucas Polynomials [9]
$p+1$	px	x	1	0	1	p	1	c	Lucas $p-$Polynomials [9]
0	-1	$2x$	1	0	1	p	1	c	Pell Lucas $p-$Polynomials [9]
2	x	x	1	0	1	1	1	c	Lucas Polynomials [9]
2	$2x$	$2x$	1	0	1	1	1	c	Pell Lucas Polynomials [9]
2	1	1	$2y$	0	k	1	1	c	Jacobsthal Lucas Polynomials [9]
2	$3x$	$3x$	-2	0	1	1	1	c	Fermat Lucas Polynomials [15]
2	x	x	-2	0	1	1	1	c	Second kind of Fermat–Horadam P. [16]
2	x	x	$-\alpha$	0	1	1	1	c	First kind of Dickson Polynomials [17]
2	$x+2$	$x+2$	-1	0	1	1	1	c	Morgan–Voyce Polynomials [18]
2	$x+1$	$x+1$	$-x$	0	1	1	1	c	Corona Polynomials [19]
2	$h(x)$	$h(x)$	1	0	1	1	1	c	$h(x)-$Lucas Polynomials [2]
2	$p(x)$	$p(x)$	$q(x)$	0	1	1	1	c	$(p,q)-$Lucas Polynomials [15]
2	1	1	1	0	k	1	1	c	Lucas Numbers [9]
2	2	2	1	0	1	1	1	c	Pell–Lucas Numbers [9]
2	1	1	2	0	k	1	1	c	Jacobsthal–Lucas Numbers [9]
t	t	2	2	-1	1	1	1	1	Squares of Fibonacci Numbers [1]

3. Partial Differential Equations for Polynomials in (13)

With the help of the derivatives of these generating functions with regard to some variable and algebraic arrangements, we derive some partial differential equations for new polynomials. Taking the derivative with regard to x, y, z, t of the generating function in (13), respectively, they hold

$$\frac{\partial}{\partial x}M = kx^{k-1}tM^2, \tag{18}$$

$$\frac{\partial}{\partial y}M = my^{m-1}t^{n+m}M^2, \tag{19}$$

$$\frac{\partial}{\partial z}M = cz^{c-1}t^{n+m+c}M^2, \tag{20}$$

$$\frac{\partial}{\partial t}M = \left(x^k + y^m(n+m)t^{n+m-1} + z^c(n+m+c)t^{n+m+c-1}\right)M^2. \tag{21}$$

From (13) and (18), we get the following theorem.

Theorem 1. *For $j \geq 0$, we have the first relation as follows:*

$$\frac{\partial}{\partial x}S_j = kx^{k-1}\sum_{l=0}^{j-1} S_{j-l-1}S_l.$$

Combining (13) and (19), we have the next theorem.

Theorem 2. *For $j \geq m+n$, we have the second relation as follows:*

$$\frac{\partial}{\partial y}S_j = \sum_{l=0}^{j-m-n} my^{m-1}S_{j-m-n-l}S_l.$$

With the help of considering (13) and (20), we get the next result.

Theorem 3. *For $j \geq m+n+c$, we have the third relation as follows:*

$$\frac{\partial}{\partial z}S_j = cz^{c-1}\sum_{l=0}^{j-m-n-c} S_{j-m-n-c-l}S_l.$$

Lastly, by means of (13) and (21), we get the following result.

Theorem 4.

(i) *For $m+n-1 \leq j \leq m+n+c-1$, then we obtain*

$$(j+1)S_{j+1} = x^k\sum_{l=0}^{j} S_{j-l}S_l + y^m(n+m)\sum_{l=0}^{j-m-n+1} S_lS_{j-m-n-l+1}.$$

(ii) *For $j \leq m+n-1$, then we derive*

$$(j+1)S_{j+1} = x^k\sum_{l=0}^{j} S_{j-l}S_l.$$

(iii) *For* $j \geq m+n+c-1$*, then we get*

$$(j+1)S_{j+1} = x^k \sum_{l=0}^{j} S_{j-l}S_l + y^m(n+m) \sum_{l=0}^{j-m-n+1} S_l S_{j-m-n-l+1}$$
$$+ z^c(n+m+c) \sum_{l=0}^{j-m-n-c+1} S_l S_{j-m-n-c-l+1}.$$

After that, using the partial differential equations in (18)–(21), we get the new partial differential equation for S_j.

Theorem 5. *For* $j \geq 0$*, we have*

$$jS_j = \frac{x}{k}\frac{\partial}{\partial x}S_j + \left(\frac{n+m}{m}\right) y \frac{\partial}{\partial y}S_j + \left(\frac{n+m+c}{c}\right) z \frac{\partial}{\partial z}S_j.$$

Proof. Combining (18)–(21), we get

$$\frac{\partial}{\partial t}M - \frac{x}{kt}\frac{\partial}{\partial x}M = \left(\frac{n+m}{m}\right)\frac{y}{t}\frac{\partial}{\partial y}M + \left(\frac{n+m+c}{c}\right)\frac{z}{t}\frac{\partial}{\partial z}M.$$

In the above, using (13), we get the desired result. □

4. Some Applications of Generating Functions

In this section, by using these functions, some identities connected with these polynomials are derived. Furthermore, in the special case, we show that these identities reduce to the well-known sum identities connected with the well-known numbers in literature.

Case 1. Taking $t = \frac{1}{a}$ in (15) for $|a| > 1$, we get the following equation

$$\sum_{j=0}^{\infty} \frac{W_j}{a^j} = \frac{a^{m+c}}{a^{m+n+c} - x^k a^{m+n+c-1} - y^m a^c - z^c}. \tag{22}$$

(i) Substituting $a = 2$, $x \to x^2$, $y \to x$, $z \to 1$ and $k = m = n = c = 1$ in (22), we obtain the relation for the tribonacci polynomials as

$$\sum_{j=0}^{\infty} \frac{t_j(x)}{2^j} = \frac{4}{7 - 4x^2 - 2x}. \tag{23}$$

Writing $x = 1$ in (23), we have

$$\sum_{j=0}^{\infty} \frac{T_j}{2^j} = 4,$$

where T_j are the tribonacci numbers.

(ii) Taking $a = 10, x \to x^2, y \to x, z \to 1$ and $k = m = n = c = 1$ in (22), we get

$$\sum_{j=0}^{\infty} \frac{T_j(x)}{10^{j+2}} = \frac{1}{999 - 100x^2 - 10x}, \tag{24}$$

and writing $x = 1$ (24), we get for the tribonacci numbers

$$\sum_{j=0}^{\infty} \frac{T_j}{10^{j+2}} = \frac{1}{889}\,.$$

(iii) Substituting $x \to x, y \to 1, z \to 0, a = 2$, and $k = m = n = c = 1$ into (22), we get for the Fibonacci polynomials

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{2^j} = \frac{2}{3-2x}\,, \tag{25}$$

which was given in [14]. Then taking $x = 1$ in (25), we have for Fibonacci numbers

$$\sum_{j=0}^{\infty} \frac{F_j}{2^j} = 2\,,$$

which was given in [14].

(iv) Substituting $x \to x, y \to 1, z \to 0, a = 3$, and $k = m = n = c = 1$ in (22), we get for the Fibonacci polynomials

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{3^{j+1}} = \frac{1}{8-3x}. \tag{26}$$

Taking $x = 1$ in (26), we get for the Fibonacci numbers

$$\sum_{j=0}^{\infty} \frac{F_j}{3^{j+1}} = \frac{1}{5} = \frac{1}{F_5}\,,$$

was given in page 424 in [1].

(v) Substituting $x \to x, y \to 1, z \to 0, a = 8$, and $k = m = n = c = 1$ in (22), we get for the Fibonacci polynomials

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{8^{j+1}} = \frac{1}{63-3x}. \tag{27}$$

Taking $x = 1$ in (27), we get for the Fibonacci numbers

$$\sum_{j=0}^{\infty} \frac{F_j}{8^{j+1}} = \frac{1}{55} = \frac{1}{F_{10}}\,,$$

was given in page 424 in [1].

(vi) Substituting $x \to x, y \to 1, z \to 0, a = -10$, and $k = m = n = c = 1$ in (22), we get for the Fibonacci polynomials

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{(-10)^{j+1}} = \frac{1}{99+10x}\,. \tag{28}$$

Taking $x = 1$ in (28), we get for the Fibonacci numbers

$$\sum_{j=0}^{\infty} \frac{F_j}{(-10)^{j+1}} = \frac{1}{109}\,,$$

was given in page 427 in [1].

(vii) Substituting $x \to 2x, y \to 1, z \to 0, a = 3$, and $k = m = n = c = 1$ in (22), we get

$$\sum_{j=0}^{\infty} \frac{P_j(x)}{3^{j+1}} = \frac{1}{8-6x}\,, \tag{29}$$

where $P_j(x)$ are the Pell polynomials. Then taking $x = 1$ in (29), we have

$$\sum_{j=0}^{\infty} \frac{P_j}{3^{j+1}} = \frac{1}{2},$$

where P_j are the Pell numbers.

(viii) Substituting $x \to 1$, $y \to 2y$, $z \to 0$, $a = 3$, and $k = m = n = c = 1$ in (22), we get

$$\sum_{s=0}^{\infty} \frac{J_s(x)}{3^{s+1}} = \frac{1}{6-2y}, \tag{30}$$

where $J_s(x)$ are the Jacobsthal polynomials. Then taking $y = 1$ in (30), we have

$$\sum_{s=0}^{\infty} \frac{J_s}{3^{s+1}} = \frac{1}{4},$$

where J_s are the Jacobsthal numbers.

Case 2. Taking $t = \frac{1}{a}$ in (16) for $|a| > 1$, we get the following equation

$$\sum_{j=0}^{\infty} \frac{K_j}{a^j} = \frac{a^{m+n+c}\alpha(t;x,y) - a^{m+c}\beta(t;x,y)}{a^{m+n+c} - x^k a^{m+n+c-1} - y^m a^c - z^c}. \tag{31}$$

(i) Substituting $x \to x^2$, $y \to x$, $z \to 1$, $a = 2$, and $k = m = n = c = 1$, $\alpha(t;x,y) = 3$, $\beta(t;x,y) = 2x^2 + xt$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{k_j(x)}{2^j} = \frac{24 - 8x^4 - 2x^2}{7 - 4x^2 - 2x}, \tag{32}$$

where $k_j(x)$ are the tribonacci-Lucas polynomials. Then taking $x = 1$ in (32), we have

$$\sum_{j=0}^{\infty} \frac{k_j}{2^{j+1}} = 7,$$

where $k_j(x)$ are the tribonacci-Lucas numbers.

(ii) Substituting $x \to x$, $y \to 1$, $z \to 0$, $a = 2$, and $k = m = n = c = 1$, $\alpha(t;x,y) = 2$, $\beta(t;x,y) = x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{L_j(x)}{2^{j+1}} = \frac{4-x}{3-2x}, \tag{33}$$

where $L_j(x)$ are the Lucas polynomials. Then taking $x = 1$ in (33), we have

$$\sum_{j=0}^{\infty} \frac{L_j}{2^{j+1}} = 3,$$

where L_j are the Lucas numbers.

(iii) Substituting $x \to x$, $y \to 1$, $z \to 0$, $a = 10$, and $k = m = n = c = 1$, $\alpha(t;x,y) = 2$, $\beta(t;x,y) = x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{L_j(x)}{10^j} = \frac{200 - 10x}{99 - 10x}, \tag{34}$$

where $L_j(x)$ are the Lucas polynomials. Then taking $x = 1$ in (34), for L_j are the Lucas numbers, we have

$$\sum_{j=0}^{\infty} \frac{L_j}{10^{j+1}} = \frac{19}{89} = \frac{L_6 - L_1}{F_{11}}$$

was given in page 427 in [1].

(iv) Substituting $x \to x, y \to 1, z \to 0, a = 3$, and $k = m = n = c = 1, \alpha(t;x,y) = 2, \beta(t;x,y) = x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{L_j(x)}{3^{j+1}} = \frac{6-x}{8-3x}, \tag{35}$$

and taking $x = 1$ in (35), we have

$$\sum_{j=0}^{\infty} \frac{L_j}{3^{j+1}} = 1.$$

(v) Substituting $x \to x, y \to 1, z \to 0, a = 8$, and $k = m = n = c = 1, \alpha(t;x,y) = 2, \beta(t;x,y) = x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{L_j(x)}{8^{j+1}} = \frac{16-x}{63-8x}, \tag{36}$$

and taking $x = 1$ in (36), we have

$$\sum_{j=0}^{\infty} \frac{L_j}{8^{j+1}} = \frac{3}{11} = \frac{L_2}{L_5}.$$

(vi) Substituting $x \to x, y \to 1, z \to 0, a = -10$, and $k = m = n = c = 1, \alpha(t;x,y) = 2, \beta(t;x,y) = x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{L_j(x)}{(-10)^{j+1}} = \frac{-20-x}{99+10x}. \tag{37}$$

Taking $x = 1$ in (37), we have

$$\sum_{j=0}^{\infty} \frac{L_j}{(-10)^{j+1}} = \frac{-21}{109},$$

was given in page 427 in [1].

(vii) Substituting $x \to 2x, y \to 1, z \to 0, a = 5$, and $k = m = n = c = 1, \alpha(t;x,y) = 2, \beta(t;x,y) = 2x$ in (31), we get

$$\sum_{j=0}^{\infty} \frac{Q_j(x)}{5^{j+1}} = \frac{5-x}{12-5x}, \tag{38}$$

where $Q_j(x)$ are the Pell Lucas polynomials. Then taking $x = 1$ in (38), we have

$$\sum_{j=0}^{\infty} \frac{Q_j}{5^{j+1}} = \frac{4}{7},$$

where Q_j are the Pell Lucas numbers.

(viii) Substituting $x \to 1, y \to 2y, z \to 0, a = 3$, and $k = m = n = c = 1, \alpha(t;x,y) = 2, \beta(t;x,y) = 1$ in (31), we get

$$\sum_{s=0}^{\infty} \frac{j_s(y)}{3^{s+1}} = \frac{5}{6-2y}, \tag{39}$$

where $j_s(y)$ are the Jocabsthal Lucas polynomials. Then taking $y = 1$ in (39), we have

$$\sum_{s=0}^{\infty} \frac{j_s}{3^{s+1}} = \frac{5}{4},$$

where j_s is Jocabsthal Lucas number.

(ix) Substituting $x \to 2$, $y \to 2$, $z \to -1$, $a = 4$, and $k = m = n = c = 1$, $\alpha(t;x,y) = t$, $\beta(t;x,y) = t$ in (31), for the square of Fibonacci numbers F_j, we get

$$\sum_{j=0}^{\infty} \frac{F_j^2}{4^j} = \frac{12}{25},$$

was given in page 439 in [1].

Let us give Tables 3 and 4 containing the obtained formulas for simplify reading.

Table 3. Special cases of Equation (22) for $k = m = n = c = 1$.

a	x	y	z	**Formulas**
2	x^2	x	1	$\sum_{j=0}^{\infty} \frac{t_j(x)}{2^j} = \frac{4}{7-4x^2-2x}$
2	1	1	1	$\sum_{j=0}^{\infty} \frac{T_j}{2^j} = 4$
10	x^2	x	1	$\sum_{j=0}^{\infty} \frac{T_j(x)}{10^{j+2}} = \frac{1}{999-100x^2-10x}$
10	1	1	1	$\sum_{j=0}^{\infty} \frac{T_j}{10^{j+2}} = \frac{1}{889}$
2	x	1	0	$\sum_{j=0}^{\infty} \frac{F_j(x)}{2^j} = \frac{2}{3-2x}$
2	1	1	0	$\sum_{j=0}^{\infty} \frac{F_j}{2^j} = 2$
3	x	1	0	$\sum_{j=0}^{\infty} \frac{F_j(x)}{3^{j+1}} = \frac{1}{8-3x}$
3	1	1	0	$\sum_{j=0}^{\infty} \frac{F_j}{3^{j+1}} = \frac{1}{5} = \frac{1}{F_5}$
8	x	1	0	$\sum_{j=0}^{\infty} \frac{F_j(x)}{8^{j+1}} = \frac{1}{63-3x}$
8	1	1	0	$\sum_{j=0}^{\infty} \frac{F_j}{8^{j+1}} = \frac{1}{55} = \frac{1}{F_{10}}$
-10	x	1	0	$\sum_{j=0}^{\infty} \frac{F_j(x)}{(-10)^{j+1}} = \frac{1}{99+10x}$
-10	1	1	0	$\sum_{j=0}^{\infty} \frac{F_j}{(-10)^{j+1}} = \frac{1}{109}$
3	$2x$	1	0	$\sum_{j=0}^{\infty} \frac{P_j(x)}{3^{j+1}} = \frac{1}{8-6x}$
3	2	1	0	$\sum_{j=0}^{\infty} \frac{P_j}{3^{j+1}} = \frac{1}{2}$
3	1	$2y$	0	$\sum_{s=0}^{\infty} \frac{J_s(x)}{3^{s+1}} = \frac{1}{6-2y}$
3	1	2	0	$\sum_{s=0}^{\infty} \frac{J_s}{3^{s+1}} = \frac{1}{4}$

Table 4. Special cases of Equation (31) for $k = m = n = c = 1$.

a	x	y	z	α	β	**Formulas**
2	x^2	x	1	3	$2x^2 + xt$	$\sum_{j=0}^{\infty} \frac{k_j(x)}{2^j} = \frac{24-8x^4-2x^2}{7-4x^2-2x}$
2	1	1	1	3	$2+t$	$\sum_{j=0}^{\infty} \frac{k_j}{2^{j+1}} = 7$
2	x	1	0	2	x	$\sum_{j=0}^{\infty} \frac{L_j(x)}{2^{j+1}} = \frac{4-x}{3-2x}$
2	1	1	0	2	1	$\sum_{j=0}^{\infty} \frac{L_j}{2^{j+1}} = 3$
10	x	1	0	2	x	$\sum_{j=0}^{\infty} \frac{L_j(x)}{10^j} = \frac{200-10x}{99-10x}$
10	1	1	0	2	1	$\sum_{j=0}^{\infty} \frac{L_j}{10^{j+1}} = \frac{19}{89} = \frac{L_6-L_1}{F_{11}}$
3	x	1	0	2	x	$\sum_{j=0}^{\infty} \frac{L_j(x)}{3^{j+1}} = \frac{6-x}{8-3x}$
3	1	1	0	2	1	$\sum_{j=0}^{\infty} \frac{L_j}{3^{j+1}} = 1$
8	x	1	0	2	x	$\sum_{j=0}^{\infty} \frac{L_j(x)}{8^{j+1}} = \frac{16-x}{63-8x}$
8	1	1	0	2	1	$\sum_{j=0}^{\infty} \frac{L_j}{8^{j+1}} = \frac{3}{11} = \frac{L_2}{L_5}$
-10	x	1	0	2	x	$\sum_{j=0}^{\infty} \frac{L_j(x)}{(-10)^{j+1}} = \frac{-20-x}{99+10x}$
-10	1	1	0	2	1	$\sum_{j=0}^{\infty} \frac{L_j}{(-10)^{j+1}} = \frac{-21}{109}$
5	$2x$	1	0	2	$2x$	$\sum_{j=0}^{\infty} \frac{Q_j(x)}{5^{j+1}} = \frac{5-x}{12-5x}$
5	2	1	0	2	2	$\sum_{j=0}^{\infty} \frac{Q_j}{5^{j+1}} = \frac{4}{7}$
3	1	$2y$	0	2	1	$\sum_{s=0}^{\infty} \frac{j_s(y)}{3^{s+1}} = \frac{5}{6-2y}$
3	1	2	0	2	1	$\sum_{s=0}^{\infty} \frac{j_s}{3^{s+1}} = \frac{5}{4}$
4	2	2	-1	$1/4$	$1/4$	$\sum_{j=0}^{\infty} \frac{F_j^2}{4^j} = \frac{12}{25}$

5. Conclusions

In the present paper, we considered the families of three-variable polynomials with the generalized polynomials reduce to generating function of the polynomials and numbers in the literature. In Section 2, we gave special polynomials and numbers as the tables related to (15) and (16). Then we obtained the explicit representations and partial differential equations for new polynomials. In the last section, we gave the interesting sum identities related to the well-known numbers and polynomials in the literature.

For all of the resuts, if the appropriate values given in the tables are taken, many infinite sums including various polynomials are obtained.

In recent years, some authors use the well-known polynomials and numbers in the applications of ordinary and fractional differential equations and difference equations (for example [20–23]). Therefore, our new families of three variables polynomials could been used for future works of some application areas such as mathematical modelling, physics, engineering, and applied sciences.

Author Contributions: All authors contributed equally to this work. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to express their sincere gratitude to the referees for their valuable comments which have significantly improved the presentation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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