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β–Hyers–Ulam–Rassias Stability of Semilinear Nonautonomous Impulsive System

Xiaoming Wang [1,†], Muhammad Arif [2,†] and Akbar Zada [2,*]

[1] School of Mathematics & Computer Science, Shangrao Normal University, Shangrao 334001, China; wxmsuda03@163.com

[2] Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan; arifjanmath@gmail.com

* Correspondence: zadababo@yahoo.com or akbarzada@uop.edu.pk; Tel.: +92-345-9515060

† These authors contributed equally to this work.

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Abstract: In this paper, we study a system governed by impulsive semilinear nonautonomous differential equations. We present the β–Ulam stability, β–Hyers–Ulam stability and β–Hyers–Ulam–Rassias stability for the said system on a compact interval and then extended it to an unbounded interval. We use Grönwall type inequality and evolution family as a basic tool for our results. We present an example to demonstrate the application of the main result.

Keywords: semilinear nonautonomous system; instantaneous impulses; mild solution; β–Hyers–Ulam–Rassias stability

1. Introduction

Differential equations are the key tools for modeling the physical problems in nature. To understand the sudden changes in physical problems, differential equations are the best option for use. Examples of these sudden changes are Plague deforestation, volcano eruption and rivers overflow [1]. Physical problems which have rapid changes are blood flows, biological systems such as heart beats, theoretical physics, engineering, control theory, population dynamics, mechanical systems with impact, pharmacokinetics, biotechnology processes, mathematical economy, chemistry, medicine and many more. These problems can be modeled by systems of differential equations with impulses. One can obtain the impulsive conditions by taking the short-term perturbation parameters and the initial value problem. For the details of the impulsive differential equations see the results by Ahmad et al. [2], Bainov et al. [3], Benchohra et al. [4], Berger et al. [5], Bianca et al. [6], Gala et al. [7], Hernandez et al. [8], Pierri et al. [9], Samoilenko et al. [10,11], Tang et al. [12] and Wang et al. [13,14].

Ulam stability problem was put forward for the first time at Wisconsin University in 1940. The problem was to discuss the relationship between approximate solution of homomorphism from a group H_1 to a metric group H_2 [15]. Considering H_1 and H_2 as Banach spaces, Hyers solved the above problem with the help of direct method [16]. The extension of the famous work of Hyers and Ulam can be seen in Aoki [17] and Rassias [18] work. In this work they found the bound for the norm of difference, Cauchy difference, $f(t+s) - f(t) - f(s)$. Answers to this problem, its inductions and attractions for different categories of equations, is a vast region of research and has well elaborated of what is now called Ulam's type stability.

In 2012, Ulam type stability of impulsive differential equations were discussed by Wang et al. [19]. They used the concept of bounded interval with finite impulses and proved the Ulam type stability for first order nonlinear impulsive differential equations. In 2014, Wang et al. proved the Hyers–Ulam–Rassias stability and generalized Hyers–Ulam–Rassias stability for impulsive evolution equations on a closed and bounded interval [20]. In 2015, Zada et al. proved the Hyers–Ulam stability of differential system in terms of dichotomy [21]. For more details about Hyers–Ulam stability, see [16,18,22–35].

Recently, Yu et al. [36] studied β–Hyers–Ulam stability of the system

$$\begin{cases} \Theta'(t) - \mathcal{H}(t)\Theta(t) = f(t,\Theta(t)), \ \ t \in \mathcal{I}, \ t \neq t_k \\ \Theta(t_k^+) - \Theta(t_k^-) = \mathfrak{I}_k(t_k,\Theta(t_k)), \ \ k = 1,2,\ldots,m. \end{cases} \tag{1}$$

Motivated from the above work, we investigate the β–Hyers–Ulam–Rassias stability of the system:

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)\zeta(t) + f(t,\Theta(t),\zeta(t)), \ \ t \in [0,\tau], t \neq t_k \\ \Theta(0) = \Theta_0, \\ \Theta(t_k^+) = \Theta(t_k^-) + \mathfrak{I}_k(t,\Theta(t_k^-),\zeta(t_k^-)), \ \ , k = 1,2,\ldots,m, \end{cases} \tag{2}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = \tau$, $\Theta(t) \in \mathfrak{R}^n$, $\mathcal{H}(t)$, $\mathcal{B}(t)$ are continuous matrices of dimension $n \times n$ and $n \times m$ respectively, $\zeta \in \mathcal{C}([0,\tau];\mathfrak{R}^m)$ is the control function and f, $\mathfrak{I}_k \in \mathcal{C}([0,\tau] \times \mathfrak{R}^n \times \mathfrak{R}^m;\mathfrak{R}^n)$, $k = 1,2,3,\ldots,m$ are suitable functions.

In this article, we present four different types of β–Ulam type stability for the system of semilinear nonautonomous impulsive differential equations. Our main objective of this work is to discuss the uniqueness of solution for the given system and analyze the β–Hyers–Ulam–Rassias stability of semilinear nonautonomous system (2) with the help of evolution family. Evolution family has its great importance in every field of research. Different researchers are working to discuss stability analysis of different systems using evolution family. For more details of evolution family we prefer [20,28,37–44].

2. Results

2.1. Basic

Here we present basic concepts and definitions. For any interval $\mathcal{I} = [0,\tau] \subseteq \mathfrak{R}$ and $\mathcal{S} \subseteq \mathfrak{R}^k$, $1 \leq k \leq n$, we define the Banach space $\mathcal{C}(\mathcal{I},\mathcal{S})$ the space of all continuous functions from $\mathcal{I}$ to $\mathcal{S}$ with the norm $||\Theta||_{\mathcal{C}} = \{\sup_{t\in\mathcal{I}} ||\Theta(t)||$, for all $\Theta \in \mathcal{C}(\mathcal{I},\mathcal{S})\}$. Denote $\mathcal{C}'(\mathcal{I},\mathcal{S}) = \{\Theta \in \mathcal{C}(\mathcal{I},\mathcal{S}) : \Theta' \in \mathcal{C}(\mathcal{I},\mathcal{S})\}$. We also introduce the Banach space $\mathcal{PC}(\mathcal{I},\mathcal{S}) := \Big\{\Theta : \mathcal{I} \to \mathcal{S},\ \Theta \in \mathcal{C}((t_k,t_{k+1}),\mathcal{S}),\ k = 0,1,\ldots,m\Big\}$ and there exist $\Theta(t_k^+)$, $\Theta(t_k^-)$ such that $\Theta(t_k^+) = \Theta(t_k^-)$, $k = 1,2,\ldots,m$, with the norm $||\Theta||_{\mathcal{PC}} = \Big\{\sup ||\Theta(t)||$, for all $t \in \mathcal{I}\Big\}$.

Definition 1. *Consider $\mathcal{V}$ to be a vector space over some field K. A function $||.||_\beta : \mathcal{V} \to [0,\infty)$ is called β-norm if: (i) $||\Theta||_\beta = 0$ if and only if $\Theta = 0$, (ii) $||\eta\Theta||_\beta = |\eta|^\beta ||\Theta||_\beta$ for each $\eta \in K$ and $\Theta \in \mathcal{V}$, (iii) $||\Theta + z||_\beta \leq ||\Theta||_\beta + ||z||_\beta$. Then ($\mathcal{V}$, $||.||_\beta$) is known as β–normed space.*

Our space will be $P\beta$-Banach space with norm $||\Theta||_{P\beta} = \sup\{||\Theta(t)||^\beta\}$, where $t \in \mathcal{I} = [0,\tau]$ and $0 < \beta < 1$. To define $P\beta$-Banach space we consider the space $\mathcal{PC}(I,\mathcal{S})$. Choose another interval $t \in \mathcal{I}' = [0,\tau]$, $t \neq t_k$, $k = 1,2,\ldots,m$.

Definition 2. *$\mathcal{PC}(\mathcal{I},\mathcal{S}) := \Big\{\Theta : \Theta \in \mathcal{C}((t_k, t_{k+1}), \mathcal{S})\Big\}$, there exist $\Theta(t_k^-)$ and $\Theta(t_k^+)$ such that $\Theta(t_k^+) = \Theta(t_k^-)$ for (any) $k \in M_0 = \{0\} \cup M$, where $M = \{1, 2, \ldots, m\}$ with norm*

$$||\Theta||_{P\beta} = \sup\{||\Theta(t)||^{\beta}\},$$

where $t \in \mathcal{I}$ and $1 > \beta > 0$. So $(\mathcal{PC}(\mathcal{I},\mathcal{S}), ||.||_{P\beta})$ is $P\beta$-Banach space.

Definition 3. *The family $\mathcal{W} := \{Q(t,s) : t \geq s \geq 0\}$ of bounded linear operators is called bounded evolution family from the Banach space $\mathcal{X}$ to itself, if:*

- $Q(t,t) = I$, *for all* $t \geq 0$.
- $Q(t,s)Q(s,r) = Q(t,r)$, *for all* $t \geq s \geq r$, $t, s, r \geq 0$.
- $Q(t+q, s+q) = Q(t,s)$, *for all* $t \geq s$, $t, s \geq 0$ *for some* $q \in \{2, 3, \ldots\}$.
- $||Q(t,s)|| \leq \mathcal{M}e^{\kappa(t-s)}$ $\exists$ $\mathcal{M} \geq 1$, $\kappa \in R$ *not depends on* $s, t \geq 0$.

Definition 4 ([45])**.** *The semilinear nonautonomous system of differential equations with impulses*

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t) + f(t, \Theta(t), u(t)), \;\; t \in [0,\tau], t \neq t_k \\ \Theta(0) = \varrho_0, \\ \Theta(t_k^+) = \Theta(t_k^-) + \Im_k(t, \Theta(t_k^-), u(t_k^-)), \;\; , \; k = 1, 2, \ldots, m, \end{cases}$$

gives the solution in the form

$$\begin{aligned} \Theta(t) &= Q(t,0)\rho_0 + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\ &\quad + \int_0^t Q(t,s)f(s, \Theta(s), u(s))ds + \sum_{0<t_k<t} Q(t,t_k)\Im_k(t_k, \Theta(t_k), u(t_k)), \end{aligned}$$

where $Q(t,s) = \Upsilon(t)\Upsilon^{-1}(s)$ and is known as evolution family and $\Upsilon(t)$ is the fundamental matrix of $\Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t)$.

Definition 5. *If $\Upsilon(t)$ is the fundamental matrix of*

$$\begin{aligned} \Theta'(t) &= \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t), \;\; \Theta \in \mathcal{S} \\ w(0) &= \varrho_0. \end{aligned}$$

The above system is exponentially bounded if we can find some constants $\mathcal{M} > 0$ and $\kappa < 0$ such that

$$||Q(t,s)|| \leq \mathcal{M}e^{\kappa(t-s)}, \;\; 0 \leq s \leq t \leq \tau. \tag{3}$$

Choose $\epsilon > 0$, $\psi \geq 0$ and φ from $\mathcal{PC}(\mathcal{I},\mathcal{S})$. Take the inequality

$$\begin{cases} ||\Theta'(t) - \mathcal{H}(t)\Theta(t) - \mathcal{B}(t)u(t) - f(t, \Theta(t), u(t))|| \leq \epsilon\varphi(t), \;\; t \in [0,\tau], t \neq t_k \\ ||\Theta(0) - \varrho_0|| \leq \epsilon\psi, \\ ||\Theta(t_k^+) - \Theta(t_k^-) - \Im_k(t_k, \Theta(t_k), u(t_k))|| \leq \epsilon\psi, \;\; k = 1, 2, \ldots, m. \end{cases} \tag{4}$$

With the help of inequality (4) we will define β–Hyers–Ulam–Rassias stability for the system (2).

Definition 6. *(2) is said to be β–Hyers–Ulam–Rassias stable with respect to $(\psi^{\beta}, \varphi^{\beta})$ if $\exists$ positive $\mathcal{K}_{f,\mathcal{M},\varphi,\beta}$ such that for any $\epsilon > 0$ and for any solution $\Theta \in \mathcal{PC}(\mathcal{I}', \mathcal{S}) \bigcap \mathcal{C}(\mathcal{I}', \mathcal{S})$ of (4) $\exists$ a solution y of (2) in $\mathcal{PC}(\mathcal{I}', \mathcal{S})$ satisfying*

$$||y(t) - \Theta(t)||^{\beta} \leq \mathcal{K}_{f,\mathcal{M},\varphi,\beta}\epsilon^{\beta}\left(\varphi^{\beta}(t) + \psi^{\beta}\right), \ \ t \in \mathcal{I}.$$

Remark 1. *It is direct consequence of inequality (4) that a function $y \in \mathcal{PC}(\mathcal{I}', \mathcal{S}) \bigcap \mathcal{C}(\mathcal{I}', \mathcal{S})$ is the solution for the inequality (4) if and only if we can find $h \in \mathcal{C}(\mathcal{I}')$, $\psi \geq 0$ and a sequence h_k, $k \in M$ satisfying*

$$\begin{cases} ||h(t)|| \leq \epsilon\varphi(t) \text{ and } ||h_k|| \leq \epsilon\psi, \ t \in \mathcal{I}' \text{ and } k \in M, \\ y'(t) = \mathcal{H}(t)y(t) + \mathcal{B}(t)u(t) + f(t, y(t), u(t)) + h(t), \ \ t \in \mathcal{I}', \\ y(0) = \Theta_0 + h(t), \\ y(t_k^{+}) = y(t_k^{-}) + \Im_k(t_k, y(t_k), u(t_k)) + h(t_k), \ \ k = 1, 2, 3, \ldots m. \end{cases}$$

Assume that

$$\mathcal{M} = \sup_{0 \leq s \leq t \leq \tau} ||Q(t,s)||. \tag{5}$$

On the basis of Remak 1 we can say that the solution of the system

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t) + f(t, \Theta(t), u(t)) + h(t), \ \ t \in [0, \tau], t \neq t_k \\ \Theta(0) = \varrho_0 + h(t), \\ \Theta(t_k^{+}) = \Theta(t_k^{-}) + \Im_k(t, \Theta(t_k^{-}), u(t_k^{-})) + h(t_k), \ \ \ k = 1, 2, \ldots, m, \end{cases}$$

is

$$\begin{aligned} \Theta(t) \ &= \ Q(t,0)(\Theta_0 + h(t)) + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\ &+ \int_0^t Q(t,s)(f(s, \Theta(s), u(s)) + h(s))ds \\ &+ \sum_{0<k<m} Q(t, t_k)(\Im_k(t_k, \Theta(t_k), u(t_k)) + h(t_k)), \ \ t \in [0, \tau]. \end{aligned}$$

For the inequality (4) we obtain

$$
\begin{aligned}
& ||\Theta(t) - Q(t,0)\Theta_0 - \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\
& - \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds - \sum_{0<k<m} Q(t,t_k)\mathfrak{I}_k(t_k,\Theta(t_k),u(t_k))|| \\
= \quad & ||Q(t,0)h(t) + \int_0^t Q(t,s)h(s)ds + \sum_{0<k<m} Q(t,t_k)h(t_k)|| \\
\leq \quad & \mathcal{M}||h(t)|| + \int_0^t \mathcal{M}||h(s)||ds + \sum_{0<k<m} \mathcal{M}||h(t_k)|| \\
\leq \quad & \epsilon\mathcal{M}\psi + \int_0^t \epsilon\mathcal{M}\varphi(s)ds + \sum_{0<k<m} \epsilon\mathcal{M}\psi \\
\leq \quad & \sum_{0\leq k<m} \epsilon\mathcal{M}\psi + \int_0^t \epsilon\mathcal{M}\varphi(s)ds \\
\leq \quad & \epsilon\mathcal{M}\Big(m\psi + \int_0^t \varphi(s)ds\Big), \text{ where } t \in (t_k, t_{k+1}].
\end{aligned}
$$

Now we state an important lemma known as Grönwall lemma, which is used in our main result.

Lemma 1 (Grönwall lemma [10])**.** *For any $t \geq 0$ with*

$$
\mathrm{u}(t) \quad \leq \quad q(t) + \int_0^t p(s)\mathrm{u}(s)ds + \sum_{0<t_k<t} \gamma_k \mathrm{u}(t_k^-), \tag{6}
$$

where u, q, $p \in \mathcal{PC}(\mathfrak{R}^+, \mathfrak{R}^+)$, q is nondecreasing and $\gamma > 0$. Then for $t \in \mathfrak{R}^+$ we have:

$$
\mathrm{u}(t) \quad \leq \quad q(t)\Big(1+\gamma_k\Big)^k \exp\Big(\int_0^t p(s)ds\Big), \text{ where } k \in M. \tag{7}
$$

Remark 2. *If we replace γ_k by $\gamma_k(t)$ then*

$$
\mathrm{u}(t) \quad \leq \quad q(t) \prod_{0<t_k<t} \Big(1+\gamma_k(t)\Big) \exp\Big(\int_0^t p(s)ds\Big), \text{ where } k \in M. \tag{8}
$$

Definition 7. *The function f from $\mathcal{X}$ to $\mathcal{X}$ is called contraction if for every $\Theta, z \in \mathcal{X}$, $\exists\, 0 \leq k < 1$ such that*

$$
d(f(\Theta), f(z)) \leq kd(\Theta, z).
$$

where $(\mathcal{X}, d)$ is a metric space.

Definition 8. *The function f from $\mathcal{X}$ to $\mathcal{X}$, has a unique fixed point if it is a contraction, where $(\mathcal{X}, d)$ is complete metric space.*

To discuss β–Hyers–Ulam–Rassias stability of the given system, we need some assumptions which can be used later on. The assumptions are:
$[A_1]$: The linear system $\Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t)$ is well posed.
$[A_2]$: $f : \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ which satisfies Caratheodory conditions and $\exists$ constant $\mathcal{L}_f > 0$ such that

$$||f(t, \Theta, u) - f(t, \Theta', u)|| \leq \mathcal{L}_f ||\Theta - \Theta'||,$$

for every $\Theta, \Theta' \in \mathcal{S}$.
$[A_3]$: $\mathfrak{I}_k \in \mathcal{C}(\mathcal{I}, \mathcal{S}) : \mathcal{S} \to \mathcal{S}$, for $k = 1, 2, ..., m$ and $\exists$ constants $\mathcal{L}_{\mathfrak{I}_k} > 0$ such that

$$||\mathfrak{I}_k(t, \Theta_k, u_k) - \mathfrak{I}_k(t, \Theta'_k, u_k)|| \leq \mathcal{L}_{\mathfrak{I}_k} ||\Theta_k - \Theta'_k||,$$

for each $\Theta_k, \Theta'_k \in \mathcal{S}$.
$[A_4]$: The inequality $\mathcal{M}\left\{\sum\limits_{k=1}^{m} \mathcal{L}_{\mathfrak{I}_k} + \mathcal{L}_f \tau\right\} < 1$ holds.

Now we are able to prove that the nonautonomous differential system (2) has only one solution.

Theorem 1. *If the assumptions $[A_1] - [A_4]$ along with (5) holds then the system (2) has only one solution $\Theta \in \mathcal{PC}(I)$ with $\Theta(0) = \varrho_0$.*

Proof of Theorem 1. Define an operator $\mathcal{F} : \mathcal{PC}(\mathcal{I}, \mathcal{S}) \to \mathcal{PC}(\mathcal{I}, \mathcal{S})$ by:

$$\begin{aligned} (\mathcal{F}\Theta)(t) &= Q(t,0)\varrho_0 + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\ &\quad + \int_0^t Q(t,s) f(s, \Theta(s), u(s))ds + \sum_{0<t_k<t} Q(t, t_k)\mathfrak{I}_k(t_k, \Theta(t_k), u(t_k)). \end{aligned}$$

Now for any $\Theta, \Theta' \in \mathcal{PC}(\mathcal{I}, \mathcal{S})$ we have

$$\begin{aligned} ||(\mathcal{F}\Theta)(t) - (\mathcal{F}\Theta')(t)|| &\leq \mathcal{M} \int_0^t ||f(s, \Theta(s), u(s)) - f(s, \Theta'(s), u(s))||ds \\ &\quad + \sum_{0<t_k<t} \mathcal{M}||\mathfrak{I}_k(t_k, \Theta(t_k), u(t_k)) - \mathfrak{I}_k(t_k, \Theta'(t_k), u(t_k))|| \\ &\leq \mathcal{M} \int_0^t \mathcal{L}_f ||\Theta(s) - \Theta'(s)||ds \\ &\quad + \sum_{0<t_k<t} \mathcal{M}\mathcal{L}_{\mathfrak{I}_k} ||\Theta(t_k) - \Theta'(t_k)|| \\ &\leq \mathcal{M}\left\{\sum_{k=1}^{m} \mathcal{L}_{\mathfrak{I}_k} + \mathcal{L}_f \tau\right\} ||\Theta - \Theta'||_{\mathcal{PC}} \\ &< ||\Theta - \Theta'||_{\mathcal{PC}}. \end{aligned}$$

Then, $\mathcal{F}$ is contractive with respect to $||.||_{\mathcal{PC}}$. By using contraction mapping theorem, which shows that the mapping $\mathcal{F}$ has a unique fixed point which is the solution of the system (2). □

2.2. β–Hyers–Ulam–Rassias Stability on a Compact Interval

To discuss β–Hyers–Ulam–Rassias stability of system (2) on a compact interval, we need to introduce other conditions along with $[A_1]$, $[A_3]$ and $[A_4]$, which can be used to prove our required results. The assumptions are given as follows:
$[A_2^*]$: $f : \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ which satisfies Caratheodory conditions and $\exists$ function $\mathcal{L}_f \in \mathcal{C}(\mathcal{I}, \mathcal{S})$ so that

$$||f(t,\Theta,u) - f(t,\Theta',u)|| \leq \mathcal{L}_f(t)||\Theta - \Theta'||,$$

for every $t \in \mathcal{I}$ and $\Theta, \Theta' \in \mathcal{S}$.
$[A_5]$: there exists a non decreasing function $\varphi \in \mathcal{PC}(\mathcal{I}, \mathcal{S})$ with $\varphi(t) \geq 0$ and a constant η_φ so that

$$\int_0^t \varphi(s)ds \leq \eta_\varphi \varphi(t), \; for\ each\ t \in \mathcal{I}.$$

By considering the inequality (4) and above assumptions, we present our first result as follows.

Theorem 2. *If $[A_1]$, $[A_2^*]$ and $[A_3] - [A_5]$ along with (5) hold. Then the system (2) is β–Hyers–Ulam–Rassias stable with respect to $(\psi^\beta, \varphi^\beta)$.*

Proof of Theorem 2. Unique solution of the impulsive Cauchy problem

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t) + f(t,\Theta(t),u(t)), \;\; t \in [0,\tau], t \neq t_k \\ \Theta(0) = y(0), \\ \Theta(t_k^+) = \Theta(t_k^-) + \mathfrak{I}_k(t_k,\Theta(t_k),u(t_k)), \;\; k = 1,2,\ldots,m, \end{cases}$$

can be written as

$$\Theta(t) = \begin{cases} Q(t,0)\Theta(0) + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds + \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds, & \text{for } t \in [0,t_1], \\ Q(t,0)\Theta(0) + Q(t,t_1)\mathfrak{I}_1(t_1,\Theta(t_1),u(t_1)) + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds & \\ + \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds, & \text{for } t \in (t_1,t_2], \\ \vdots & \\ Q(t,0)\Theta(0) + \sum_{k=1}^m Q(t,t_k)\mathfrak{I}_k(t_k,\Theta(t_k),u(t_k)) + \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds & \\ + \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds, & \text{for } t \in (t_m,\tau]. \end{cases}$$

Let y be the solution of the inequality (4). Then for every $t \in (t_k, t_{k+1}]$, we can obtain that,

$$\begin{aligned}
& ||y(t) - Q(t,0)y(0) - \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int\limits_0^t Q(t,s)f(s,y(s),u(s))ds \\
- & \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,y(t_k),u(t_k))|| \\
\leq & \ m\epsilon\mathcal{M}\psi + \int\limits_0^t \epsilon\mathcal{M}\varphi(s)ds \\
\leq & \ \epsilon\mathcal{M}\Big(m\psi + \int\limits_0^t \varphi(s)ds\Big) \\
\leq & \ \epsilon\mathcal{M}\Big(m+\eta_\varphi\Big)\Big(\varphi(t)+\psi\Big).
\end{aligned}$$

Therefore for every $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned}
||y(t)-\Theta(t)||^\beta \ = \ & ||y(t) - Q(t,0)\Theta(0) - \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int\limits_0^t Q(t,s)f(s,\Theta(s),u(s))ds \\
& - \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,\Theta(t_k),u(t_k))||^\beta \\
= \ & ||y(t) - Q(t,0)y(0) - \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int\limits_0^t Q(t,s)f(s,\Theta(s),u(s))ds \\
& + \int\limits_0^t Q(t,s)f(s,y(s),u(s))ds - \int\limits_0^t Q(t,s)f(s,y(s),u(s))ds \\
& - \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,\Theta(t_k),u(t_k)) \\
& + \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,y(t_k),u(t_k)) - \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,y(t_k),u(t_k))||^\beta \\
\leq \ & \Big(||y(t) - Q(t,0)y(0) - \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int\limits_0^t Q(t,s)f(s,y(s),u(s))ds \\
& - \sum_{0<k<m} Q(t,t_k)\Im_k(t_k,y(t_k),u(t_k))||\Big)^\beta \\
& + \Big(\int\limits_0^t ||Q(t,s)f(s,y(s),u(s)) - Q(t,s)f(s,\Theta(s),u(s))||\Big)^\beta ds \\
& + \Big(\sum_{0<k<m} ||Q(t,t_k)\Im_k(t_k,y(t_k),u(t_k)) - Q(t,t_k)\Im_k(t_k,\Theta(t_k),u(t_k))||\Big)^\beta \\
\leq \ & \Big(\epsilon\mathcal{M}(m+\eta_\varphi)(\varphi(t)+\psi)\Big)^\beta + \Big(\mathcal{M}\int\limits_0^t \mathcal{L}_f(s)||y(s)-\Theta(s)||ds\Big)^\beta \\
& + \Big(\mathcal{M}\sum_{k=1}^m \mathcal{L}_{\Im_k}||y(t_k)-\Theta(t_k)||\Big)^\beta,
\end{aligned}$$

where

$$\int_0^t ||Q(t,s)f(s,y(s),u(s)) - Q(t,s)f(s,\Theta(s),u(s))||ds \leq \mathcal{M}\int_0^t \mathcal{L}_f(s)||y(s)-\Theta(s)||ds$$

and

$$\sum_{k=1}^{m} ||Q(t,t_k)\mathfrak{I}_k(t_k,y(t_k),u(t_k)) - Q(t,t_k)\mathfrak{I}_k(t_k,\Theta(t_k),u(t_k))|| \leq \mathcal{M}\sum_{k=1}^{m}\mathcal{L}_{\mathfrak{I}_k}||y(t_k)-\Theta(t_k)||.$$

Thus,

$$\begin{aligned} ||y(t)-\Theta(t)|| &\leq 3^{\frac{1}{\beta}-1}\Big[\Big(\epsilon\mathcal{M}(m+\eta_\varphi)(\varphi(t)+\psi)\Big)+\Big(\mathcal{M}\int_0^t \mathcal{L}_f(s)||y(s)-\Theta(s)||ds\Big) \\ &\quad +\mathcal{M}\sum_{k=1}^{m}\mathcal{L}_{\mathfrak{I}_k}||y(t_k)-\Theta(t_k)||\Big], \end{aligned}$$

by using relation

$$(x+y+z)^\gamma \leq 3^{\gamma-1}(x^\gamma+y^\gamma+z^\gamma), \text{where } x,y,z\geq 0, \text{and } \gamma>1.$$

Consider $\mathcal{L}_{\mathfrak{I}} = \max\{\mathcal{L}_{\mathfrak{I}_1},\mathcal{L}_{\mathfrak{I}_2},\ldots,\mathcal{L}_{\mathfrak{I}_m}\}$. Using Grönwall Lemma 1 we get that

$$||y(t)-\Theta(t)|| \leq 3^{\frac{1}{\beta}-1}\Big(\epsilon\mathcal{M}\Big((m+\eta_\varphi)(\varphi(t)+\psi)\Big)\Big)\Big(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}}\Big)^k \exp\Big(3^{\frac{1}{\beta}-1}\mathcal{M}\int_0^t \mathcal{L}_f(s)ds\Big).$$

Hence

$$\begin{aligned} ||y(t)-\Theta(t)||^\beta &\leq 3^{1-\beta}\Big(\epsilon\mathcal{M}\Big(m+\eta_\varphi\Big)(\varphi(t)+\psi)\Big)^\beta \\ &\quad \Big(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}}\Big)^{k\beta}\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{M}\int_0^t \mathcal{L}_f(s)ds\Big)^\beta \\ &\leq 3^{1-\beta}\Big(\epsilon\mathcal{M}\Big(m+\eta_\varphi\Big)\Big)^\beta\Big(\varphi(t)+\psi\Big)^\beta \\ &\quad \Big(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}}\Big)^{k\beta}\exp\Big(3^{\frac{1}{\beta}-1}\beta\mathcal{M}\int_0^t \mathcal{L}_f(s)ds\Big) \\ &\leq \mathcal{K}_{f,\mathcal{M},\varphi,\psi}\epsilon^\beta\Big(\varphi^\beta(t)+\psi^\beta\Big), \end{aligned}$$

using the fact that $(x+y)^r \leq (x^r+y^r),\ x,\ y\geq 0,$ for any $r\in(0,1]$.

Where,

$$\mathcal{K}_{f,\mathcal{M},\varphi,\psi} = 3^{1-\beta}\Big(\mathcal{M}\Big(m+\eta_\varphi\Big)\Big)^\beta\Big(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}}\Big)^{m\beta}\exp\Big(3^{\frac{1}{\beta}-1}\beta\mathcal{M}\int_0^\tau \mathcal{L}_f(s)ds\Big).$$

Hence the system (2) is β–Hyers–Ulam–Rassias stable on compact interval with respect to $(\psi^\beta,\varphi^\beta)$. $\square$

2.3. β–Hyers–Ulam–Rassias Stability on an Unbounded Interval

Here we study β–Hyers–Ulam–Rassias stability on an unbounded interval. For the desired proof we need the following assumptions which can be used in our later work.
$[A_0]$: The operators family $\{Q(t,s) : t \geq s \geq 0\}$ is exponentially stable, that is we can find $\mathcal{M} \geq 1$ and $\kappa < 0$ so that

$$||Q(t,s)|| \leq \mathcal{M}e^{\kappa(t-s)},\ t > s \geq 0.$$

$[A_6]$: $f \in \mathcal{C}(\Re^+ \times \mathcal{S}, \mathcal{S})$ and $\exists$ a function $\mathcal{L}_f \in \mathcal{C}(\Re^+, \mathcal{S})$ satisfying

$$||f(t,\Theta,u) - f(t,\Theta',u)|| \leq \mathcal{L}_f(t)||\Theta - \Theta'||,$$

for every $t \in \Re^+$ and $\Theta, \Theta' \in \mathcal{S}$. Also we assume that

$$\int_0^t \mathcal{L}_f(s)ds \leq \kappa_f t + \zeta_f,$$

for each $t \geq 0$, $\kappa_f, \zeta_f \geq 0$ and $3^{\frac{1}{\beta}-1}\mathcal{M}\kappa_f + \kappa < 0$ for some $\beta \in (0,1)$.
$[A_7]$: $\mathfrak{I}_k : \mathcal{S} \to \mathcal{S}$ and there exists a constant $\mathcal{L}_{\mathfrak{I}_k} > 0$ so that

$$||\mathfrak{I}_k(t,\Theta,u) - \mathfrak{I}_k(t,\Theta',u)|| \leq \mathcal{L}_{\mathfrak{I}_k}||\Theta - \Theta'||,$$

for every $t \in \Re^+$ and $\Theta, \Theta' \in \mathcal{S}$. Furthermore, we assume that

$$\mathcal{L}_{\mathfrak{I}} := 3^{\frac{1}{\beta}-1}\mathcal{M} \sup_{k \in M} \sum_{i=1}^{k} \mathcal{L}_{\mathfrak{I}_i} < \infty.$$

$[A_8]$: A function $\varphi \in \mathcal{PC}(\Re^+, \mathcal{S})$ and a constant $\eta_\varphi > 0$ so that

$$\int_0^t e^{\kappa(t-s)+3^{\frac{1}{\beta}-1}\mathcal{M}\kappa_f t}\varphi(s)ds \leq \eta_\varphi \varphi(t), \text{for each } t \in \Re^+.$$

$[A_9]$: Put

$$\mathcal{M}_1 := \sup_{k \in M} \sum_{i=1}^{k} e^{\kappa(t_k - t_i)+3^{\frac{1}{\beta}-1}\mathcal{M}\kappa_f t_k} + e^{\kappa t_k + 3^{\frac{1}{\beta}-1}\mathcal{M}\kappa_f t_k},$$

moreover for the case $M = N$ we assume that $\mathcal{M}_1 < \infty$.

By considering the inequality (4) and above assumptions we state our second result as follows.

Theorem 3. *Suppose that $[A_0]$, $[A_1]$ and $[A_6] - [A_9]$ are fulfilled. Then the system (2) is β–Hyers–Ulam–Rassias stable with respect to $(\psi^\beta, \varphi^\beta)$ on unbounded interval.*

Proof of Theorem 3. Unique solution of the semilinear nonautonomous impulsive differential system:

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t) + f(t,\Theta(t),u(t)), \ \ t \in \mathcal{I}' \\ \Theta(0) = y(0), \\ \Theta(t_k^+) = \Theta(t_k^-) + \Im_k(t_k,\Theta(t_k),u(t_k)), \ \ k = M, \end{cases}$$

is given by

$$\begin{cases} \Theta(t) = Q(t,0)Q(0) + \sum\limits_{i=1}^{k} Q(t,t_i)\Im_i(t_i,\Theta(t_i),u(t_i)) + \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\ + \int\limits_0^t Q(t,s)f(s,\Theta(s),u(s))ds, \ t \in \mathcal{I}', \ k \in M. \end{cases} \tag{9}$$

Let y satisfy (4). Then for every $t \in (t_k, t_{k+1}]$, $k \in M_0$, we obtain that,

$$||y(t) - Q(t,0)y(0) - \int\limits_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int\limits_0^t Q(t,s)f(s,y(s),u(s))ds - \sum_{i=1}^{k} Q(t,t_i)\Im_i(t_i,y(t_i),u(t_i))||$$

$$\leq \sum_{i=1}^{k} ||Q(t,t_i)|| \, ||h_i|| + \int\limits_0^t ||Q(t,s)|| \, ||h(s)||ds,$$

$$\leq \mathcal{M}\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)}\epsilon\psi + \int\limits_0^t e^{\kappa(t-s)}\epsilon\varphi(s)ds \Big).$$

Thus for each and every $t \in (t_k, t_{k+1}]$ we get that,

$$
\begin{aligned}
& ||y(t) - \Theta(t)||^{\beta} \\
= \; & ||y(t) - Q(t,0)\Theta(0) - \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds \\
& - \sum_{i=0}^{k} Q(t,t_i)\Im_i(t_i,\Theta(t_i),u(t_i))||^{\beta} \\
= \; & ||y(t) - Q(t,0)Q(0) - \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds - \int_0^t Q(t,s)f(s,\Theta(s),u(s))ds \\
& + \int_0^t Q(t,s)f(s,y(s),u(s))ds - \int_0^t Q(t,s)f(s,y(s),u(s))ds - \sum_{i=0}^{k} Q(t,t_i)\Im_i(t_i,\Theta(t_i),u(t_i)) \\
& + \sum_{i=0}^{k} Q(t,t_i)\Im_i(t_i,y(t_i),u(t_i)) - \sum_{i=0}^{k} Q(t,t_i)\Im_i(t_i,y(t_i),u(t_i))||^{\beta} \\
\leq \; & \Big(||y(t) - Q(t,0)y(0) - \int_0^t Q(t,s)\mathcal{B}(s)u(s)ds \\
& - \int_0^t Q(t,s)f(s,y(s),u(s))ds - \sum_{i=0}^{k} Q(t,t_i)\Im_i(t_i,y(t_i),u(t_i))||\Big)^{\beta} \\
& + \Big(\int_0^t ||Q(t,s)f(s,y(s),u(s)) - Q(t,s)f(s,\Theta(s),u(s))||\Big)^{\beta} ds \\
& + \Big(\sum_{i=0}^{k} ||Q(t,t_i)\Im_i(t_i,y(t_i),u(t_i)) - Q(t,t_i)\Im_i(t_i,\Theta(t_i),u(t_i))||\Big)^{\beta} \\
\leq \; & \mathcal{M}\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)}\epsilon\psi + \int_0^t e^{\kappa(t-s)}\epsilon\varphi(s)ds\Big)^{\beta} \\
& + \mathcal{M}\Big(\int_0^t e^{\kappa(t-s)}\mathcal{L}_f(s)||y(s) - \Theta(s)||ds\Big)^{\beta} + \Big(\sum_{i=1}^{k} \mathcal{L}_{\Im_i}\mathcal{M}e^{\kappa(t-t_i)}||y(t_i) - \Theta(t_i)||\Big)^{\beta}.
\end{aligned}
$$

If we set $\bar{y}(t) := e^{-\kappa t}y(t)$, $\bar{\Theta}(t) := e^{-\kappa t}\Theta(t)$, we have

$$
\begin{aligned}
||\bar{y}(t) - \bar{\Theta}(t)||^{\beta} \;\leq\; & \mathcal{M}\epsilon\Big(\sum_{i=1}^{k} e^{-\kappa t_i}\psi + \int_0^t e^{-\kappa s}\varphi(s)ds\Big)^{\beta} + \Big(\int_0^t \mathcal{M}\mathcal{L}_f(s)||\bar{y}(s) - \bar{\Theta}(s)||ds\Big)^{\beta} \\
& + \Big(\sum_{i=1}^{k} \mathcal{L}_{\Im_i}\mathcal{M}||\bar{y}(t_i) - \bar{\Theta}(t_i)||\Big)^{\beta},
\end{aligned}
$$

with the help of

$$\Big(x + y + z\Big)^{\gamma} \leq 3^{\gamma-1}\Big(x^{\gamma} + y^{\gamma} + z^{\gamma}\Big), \text{where } x, y, z \geq 0, \text{and } \gamma > 1,$$

we get that

$$\begin{aligned}||\bar{y}(t)-\bar{\Theta}(t)|| &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\Big(\sum_{i=1}^{k} e^{-\kappa t_i}\psi+\int_0^t e^{-\kappa s}\varphi(s)ds+3^{\frac{1}{\beta}-1}\int_0^t \mathcal{M}\mathcal{L}_f(s)||\bar{y}(s)-\bar{\Theta}(s)||ds \\ &\quad +3^{\frac{1}{\beta}-1}\sum_{i=1}^{k}\mathcal{L}_{\mathfrak{J}_i}\mathcal{M}||\bar{y}(t_i)-\bar{\Theta}(t_i)||.\end{aligned}$$

Using Lemma 1, we obtain

$$||\bar{y}(t)-\bar{\Theta}(t)|| \leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\Big(\sum_{i=1}^{k} e^{-\kappa t_i}\psi+\int_0^t e^{-\kappa s}\varphi(s)ds\Big)\mathcal{L}_{\mathfrak{J}}\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{M}\int_0^t \mathcal{L}_f(s)ds\Big),$$

resubmitting some values we have

$$\begin{aligned}||y(t)-\Theta(t)|| &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)}\psi+\int_0^t e^{\kappa(t-s)}\varphi(s)ds\Big)\mathcal{L}_{\mathfrak{J}}\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{M}\int_0^t \mathcal{L}_f(s)ds\Big) \\ &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{J}}\epsilon\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)+3^{\frac{1}{\beta}-1}\mathcal{M}(\kappa_f t+\zeta_f)}\psi \\ &\quad +e^{\kappa t+3^{\frac{1}{\beta}-1}\mathcal{M}(\kappa_f t+\zeta_f)}+\int_0^t e^{\kappa(t-s)+3^{\frac{1}{\beta}-1}\mathcal{M}(\kappa_f t+\zeta_f)}\varphi(s)ds\Big) \\ &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{J}}\epsilon e^{3^{\frac{1}{\beta}-1}\mathcal{M}\zeta_f}\Big(\mathcal{M}_1+\eta_\varphi\Big)\Big(\varphi(t)+\psi\Big),\end{aligned}$$

which implies,

$$||y(t)-\Theta(t)||^{\beta} \leq \mathcal{K}_{f,\mathcal{M},\varphi,\beta}\epsilon^{\beta}\Big(\varphi^{\beta}(t)+\psi^{\beta}\Big),$$

where

$$\mathcal{K}_{f,\mathcal{M},\varphi,\beta} := 3^{1-\beta}\Big(\mathcal{M}\mathcal{L}_{\mathfrak{J}}\Big)^{\beta}\Big(e^{3^{\frac{1}{\beta}-1}\mathcal{M}\zeta_f}(\mathcal{M}_1+\eta_\varphi)\Big)^{\beta} > 0.$$

Hence the system (2) is β–Hyers–Ulam–Rassias stable on unbounded interval with respect to $(\psi^{\beta},\varphi^{\beta})$. □

2.4. β–Hyers–Ulam–Rassias Stability with Infinite Impulses

Now to discuss β–Hyers–Ulam–Rassias stability for the system (2) with infinite impulses, that is when $M = N$. For this case inequality (4) will become

$$\begin{cases}||\Theta'(t)-\mathcal{H}(t)\Theta(t)-\mathcal{B}(t)u(t)-f(t,\Theta(t),u(t))|| \leq \epsilon\varphi(t), \ \ t\in\mathcal{I}' \\ ||\Theta(0)-\varrho_0|| \leq \epsilon\psi_k, \\ ||\Theta(t_k^+)-Q(t_k^-)-\mathfrak{I}_k(t_k,\Theta(t_k),u(t_k))|| \leq \epsilon\psi_k, \ \ k\in N,\end{cases} \tag{10}$$

where $\varphi(.)$ has the same definition and $\psi := \{\psi_k\}_{k\in N}$ is a nonconstant sequence of nonnegative entries $\psi_k \geq 0$, for each $k \in N$. Then definition (6) can be written as

$$||y(t) - \Theta(t)||^\beta \leq \mathcal{K}_{f,\mathcal{M},\varphi,\beta}\epsilon^\beta(\varphi^\beta(t) + \psi_{k+1}^\beta), t \in \mathcal{I}' \text{and } k \in N.$$

We call it as extended β–Hyers–Ulam–Rassias stability. To prove β–Hyers–Ulam–Rassias stability with infinite impulses, we consider:
$[A_{10}]$: $f \in \mathcal{C}(\Re^+ \times \mathcal{S}, \mathcal{S})$ and $\exists$ a function $\mathcal{L}_f \in \mathcal{C}(\Re^+, \Re^+)$ so that

$$||f(t, \Theta, u) - f(t, \Theta', u)|| \leq \mathcal{L}_f(t)||\Theta - \Theta'||,$$

for every $t \in \Re^+$ and $\Theta, \Theta' \in \mathcal{S}$.
$[A_{11}]$: $\Im_k : \mathcal{S} \to \mathcal{S}$ and there exists a constant $\mathcal{L}_{\Im_k} > 0$ so that

$$||\Im_k(t, \Theta, u) - \Im_k(t, \Theta', u)|| \leq \mathcal{L}_{\Im_k}||\Theta - \Theta'||,$$

for every $k \in N, t \in \Re^+$ and $\Theta, \Theta' \in \mathcal{S}$.

$[A_{12}]$: $\prod_{i=1}^k \Big(1 + 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\Im_i}\Big) \max\Big(\sum\limits_{i=1}^k e^{\kappa(t_k - t_i) + 3^{\frac{1}{\beta}-1}\mathcal{M}\int\limits_0^{t_k}\mathcal{L}_f(s)ds}\Big)\psi_i \leq \eta_\psi\psi_{k+1}$, for each $k \geq 0$,
and
$[A_{13}]$: $\prod_{i=1}^k \Big(1 + 3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\Im_i}\Big) e^{3^{\frac{1}{\beta}-1}\mathcal{M}\int\limits_0^t\mathcal{L}_f(s)ds} \int\limits_0^t e^{\kappa(t-s)}\varphi(s)ds \leq \eta_\varphi\varphi(t)$, for each $k \geq 0$.

Theorem 4. *Suppose that the assumptions* $[A_0]$, $[A_1]$ *and* $[A_{10}] - [A_{13}]$ *are fulfilled. Then the system (2) with* $\mathcal{I} = \Re^+$ *and* $M = N$ *is extended* β*–Hyers–Ulam–Rassias stable.*

Proof of Theorem 4. Consider Θ is the mild solution of the semilinear nonautonomous impulsive differential system:

$$\begin{cases} \Theta'(t) = \mathcal{H}(t)\Theta(t) + \mathcal{B}(t)u(t) + f(t, \Theta(t), u(t)), \ \ t \in \Re', \\ \Theta(0) = y(0), \\ \Theta(t_k^+) = \Theta(t_k^-) + \Im_k(t_k, \Theta(t_k), u(t_k)), \ \ k = N. \end{cases}$$

Let y be the solution of the inequality (10). To prove the required result we follow the method of Theorem 3, for any $t \in (t_k, t_{k+1}]$, we obtain that

$$\begin{aligned} ||y(t) - \Theta(t)||^\beta \ \ \leq \ \ & \Big(\mathcal{M}\Big(\sum_{i=1}^k e^{\kappa(t-t_i)}\epsilon\psi_i + \int_0^t e^{\kappa(t-s)}\epsilon\varphi(s)ds\Big)^\beta + \Big(\int_0^t \mathcal{M}e^{\kappa(t-s)}\mathcal{L}_f(s)||y(s) - \Theta(s)||ds\Big)^\beta \\ & + \Big(\sum_{i=1}^k \mathcal{L}_{\Im_i}\mathcal{M}e^{\kappa(t-t_i)}||y(t_i) - \Theta(t_i)||\Big)^\beta, \end{aligned}$$

which gives that

$$\begin{aligned}
||y(t)-\Theta(t)|| &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)}\psi_i + \int_0^t e^{\kappa(t-s)}\varphi(s)ds\Big) \\
&\quad +3^{\frac{1}{\beta}-1}\int_0^t e^{\kappa(t-s)}\mathcal{M}\mathcal{L}_f(s)||y(s)-\Theta(s)||ds \\
&\quad +3^{\frac{1}{\beta}-1}\sum_{i=1}^{k} e^{\kappa(t-t_i)}\mathcal{L}_{\mathfrak{I}_i}\mathcal{M}||y(t_i)-\Theta(t_i)||.
\end{aligned}$$

Thus,

$$\begin{aligned}
||y(t)-\Theta(t)|| &\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\prod_{i=1}^{k}\Big(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}_i}\Big)\Big(\sum_{i=1}^{k} e^{\kappa(t-t_i)}\psi_i \\
&\quad + \int_0^t e^{\kappa(t-s)}\varphi(s)ds\Big)\exp\Big(3^{\frac{1}{\beta}-1}\mathcal{M}\int_0^t L_f(s)ds\Big) \\
&\leq 3^{\frac{1}{\beta}-1}\mathcal{M}\epsilon\Big(\eta_\psi\psi_{k+1}+\eta_\varphi\varphi(t)\Big).
\end{aligned}$$

At last, we obtain that

$$\begin{aligned}
||y(t)-\Theta(t)||^\beta &\leq 3^{1-\beta}\mathcal{M}^\beta\epsilon^\beta\Big(\eta_\psi\psi_{k+1}+\eta_\varphi\varphi(t)\Big)^\beta \\
&\leq \mathcal{K}_{f,\mathcal{M},\varphi,\beta}\epsilon^\beta\Big(\varphi^\beta(t)+\psi_{k+1}^\beta\Big),
\end{aligned}$$

where

$$\mathcal{K}_{f,\mathcal{M},\varphi,\beta} := 3^{1-\beta}\mathcal{M}^\beta\Big(\eta_\psi^\beta+\eta_\varphi^\beta\Big) > 0.$$

The proof is complete. □

3. Example

Consider the following semilinear impulsive heat equation

$$\begin{cases}
\Theta_t = \triangle\Theta + 1_\omega u(t,y) + f(t,\Theta,u(t,y)), \ \text{for all } t\ [0,\tau]\times\Omega,\ t\neq t_k, \\
\Theta(0,y)=\Theta_0(y), \quad y\in\Omega, \\
\Delta\Theta(k,y)=\frac{1}{3k^2}\Theta(k^-,y),\ k\in M,\ y\in\Omega,
\end{cases} \tag{11}$$

where Ω is the bounded domain in $\mathfrak{R}^{\mathcal{N}}$ $(\mathcal{N}\geq 1)$, $\Theta_0\in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the control function u belongs to $\mathcal{C}([0,\tau];L^2(\Omega))$, $f\in\mathcal{C}([0,\tau]\times\mathfrak{R}\times\mathfrak{R};\mathfrak{R})$ and $\mathfrak{I}_k\in\mathcal{C}(\mathfrak{R}\times\mathfrak{R};\mathfrak{R}),\ k\in M$, so that the assumptions $[A_0]$ and $[A_1]$ holds with $\mathcal{M}=1$, $\kappa=-2<0$. Obviously $[A_6]$ and $[A_7]$ hold with $\kappa_f=0$ and

$$\begin{aligned}\mathcal{L}_{\mathfrak{I}} &= \sup_{k\in M}\prod_{i=1}^{k}\left(1+3^{\frac{1}{\beta}-1}\mathcal{M}\mathcal{L}_{\mathfrak{I}_i}\right)\\ &= \sup_{k\in M}\prod_{i=1}^{k}\left(1+\frac{1}{k^2}\right)\\ &\leq e^{\sup\limits_{k\in M}\sum\limits_{i=1}^{k}\frac{1}{k^2}}\\ &\leq e^{\frac{\pi^2}{6}}.\end{aligned}$$

Also $\zeta_f = 0$. Put $\varphi(t) = e^t$ and $\psi = 1$, then assumption $[A_8]$ holds if $\eta_\varphi = \frac{1}{2}$. Similarly, $[A_9]$ holds with $\mathcal{M}_1 = \frac{e^2}{e^2-1}$. Thus by using Theorem 3, we can say that the equation (5.2) is $\frac{1}{2}$–Hyers–Ulam–Rassias stable with respect to $\left(\sqrt{e^t}, 1\right)$ on $\mathfrak{R}^+$ with $\mathcal{K}_{f,\mathcal{M},\varphi,\beta} = \sqrt{3}e^{\frac{\pi^2}{12}}\left(\frac{1}{2}+\frac{e^2}{e^2-1}\right)^{\frac{1}{2}}$.

4. Conclusions

In the last few decades, many mathematicians showed their interests in the qualitative theory of impulsive differential equations. In particular, to discuss β–Hyers–Ulam–Rassias stability of differential equations, different types of conditions were used in the form of integral inequalities. For the case of semilinear nonautonomous differential system a strong Lipschitz condition of functions were common among them and mostly results were obtained via Grönwall integral inequality. In this article, we present β–Hyers–Ulam–Rassias stability of the semilinear nonautonomous impulsive differential system with the help of evolution family and Grönwall integral inequality.

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