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A New Approach to the Solution of Non-Linear Integral Equations via Various F_{B_e}-Contractions

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Abstract: In this article, we introduce and establish various approaches related to the F-contraction using new sorts of contractions, namely the extended $\mathcal{F}_{B_e}$-contraction, the extended F_{B_e}-expanding contraction, and the extended generalized F_{B_e}-contraction. Thereafter, we propose a simple and efficient solution for non-linear integral equations using the fixed point technique in the setting of a B_e-metric space. Moreover, to address conceptual depth within this approach, we supply illustrative examples where necessary.

Keywords: extended b-metric space; extended $\mathcal{F}_{B_e}$-contraction; extended F_{B_e}-expanding contraction; extended weak generalized F_{B_e}-contraction; non-linear integral equation

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1. Introduction

Widely renowned for the "Fredholm Integral Equation" Erik Ivar Fredholm [1], a mathematician and researcher par excellence, has provided research contributions on various aspects of integral equation theory.

Inspired by his great work, many fixed point researchers have focused their work on solving the Fredholm integral equation [2–5].

There was an amazing publication called F-contraction, which was one of the most influential publication in metric fixed point theory. It was introduced by a fellow named Wardkowski in 2012, and he brought this development to mathematical world with his idealistic touch [6]. It contained topological notions such as Cauchy, completeness, converges, and fixed point.

Definition 1. *Let (X, d) be a metric space. A mapping $\mathcal{H} : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$,*

$$d(\mathcal{H}x, \mathcal{H}y) > 0 \Rightarrow \tau + F(d(\mathcal{H}x, \mathcal{H}y)) \leq F(d(x, y)). \tag{1}$$

F-expanding mappings were introduced in 2017 by Gornicki [6] as below:
Let (X, d) be a metric space. A mapping $\mathcal{H} : X \to X$ is said to be F-expanding if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(x, y) > 0 \Rightarrow F(d(\mathcal{H}x, \mathcal{H}y)) \geq +F(d(x, y)) + \tau \tag{2}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying:

$(F1)$ F is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

$(F2)$ For each sequence of positive numbers $\{\alpha_n\}_{n \in \mathbb{N}}$,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ iff } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We represent by $\mathcal{F}$ the set of all functions satisfying the conditions $(F1)$–$(F3)$. There is an effort, however, to convert fixed point theorems that are in the theory of *topological fixed point theory* into non-linear integral equations and differential equations. This effort is spearheaded major developments in related research areas (see for more info References [6–13]).

Recently, a new kind of generalized metric space was introduced by Kamran et al. [14], as shown below, and named an extended b-metric space (simply, B_e-metric space).

Definition 2. *Let X be a non-empty set and $s : X \times X \to [1, \infty)$. A function $B_e : X \times X \to [0, \infty)$ is called a B_e-metric if, for all $x, y, z \in X$, it satisfies:*

(i) $B_e(x, y) = 0$ *iff* $x = y$;
(ii) $B_e(x, y) = B_e(y, x)$;
(iii) $B_e(x, y) \le s(x, y)[B_e(x, z) + B_e(z, y)]$.

The pair (X, B_e) is called a B_e-metric space.

Example 1. *Let $X = \{-1, 0, 1\}$. Define the function $s : X \times X \to \mathbb{R}^+$ and $B_e : X \times X \to \mathbb{R}^+$ as $s(x, y) = 2 + x + y$.*

$B_e(-1, -1) = B_e(0, 0) = B_e(1, 1) = 0$; $B_e(-1, 0) = B_e(0, -1) = 3$; $B_e(-1, 1) = B_e(1, -1) = 7$; $B_e(0, 1) = B_e(1, 0) = 1$.

First, we prove that B_e is a B_e-metric space. It is clear that (i) and (ii) trivially hold. For (iii), we have

$$B_e(-1, 0) = 3; s(-1, 0)[B_e(-1, 1) + B_e(1, 0)] = 8.$$

Thus,

$$B_e(-1, 0) \le s(-1, 0)[B_e(-1, 1) + B_e(1, 0)].$$

$$B_e(0, 1) = 1; \ \ s(0, 1)[B_e(0, -1) + B_e(-1, 1)] = 30.$$

$$B_e(-1, 1) = 7; \ \ s(-1, 1)[B_e(-1, 0) + B_e(0, 1)] = 8.$$

Hence, for all $x, y, z \in X$, $B_e(x, z) \le s(x, z)[B_e(x, y) + B_e(y, z)]$.
Hence, (X, B_e) is a B_e-metric space.

Definition 3. *Let (X, B_e) be a B_e-metric space and a sequence $\{x_n\}$ in X is said to*

(a) *Converge to $x \in X$ iff if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $B_e(x_n, x) < \epsilon$, for all $n \ge N$. For this particular case, we write $\lim_{n\to\infty} x_n = x$.*
(b) *Cauchy iff for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $B_e(x_m, x_n) < \epsilon$, for all $m, n \ge N$.*

Definition 4. *A B_e-metric space (X, B_e) is complete if every Cauchy sequence in X is convergent.*

Observe that usually a b-metric is not a continuous functional. Analogously, the functional B_e-metric is also not necessarily a continuous function [15–19].

Within the past century, mathematical research has been increasingly drawn towards understanding the link between the Banach contraction principle and non-linear integral equations. The brief and chronological history of these two topics are explored through a developing conceptual model. Since then, many researchers have formulated and developed fixed point approaches of non-linear integral equations in many directions.

Motivated by the above facts, we establish fixed point theorems by using F-contractions in the context of an extended b-metric space since it was very hard to obtain fixed points via the Warkowski [15] approach, which gives a solutions for non-linear integral equations by using the fixed point technique.

2. An Extended $\mathcal{F}_{B_e}$-Contraction

Now, we introduce the following definition:

Definition 5. *Let (X, B_e) be a B_e-metric space. A mapping $\mathcal{H} : X \to X$ is said be an extended $\mathcal{F}_{B_e}$-contraction if there exists $\tau > 0$ such that for all $x, y \in X$,*

$$B_e(\mathcal{H}x, \mathcal{H}y) > 0 \Rightarrow \tau + F_{B_e}(B_e(\mathcal{H}x, \mathcal{H}y)) \leq F_{B_e}(B_e(x, y)), \tag{3}$$

such that for each $x_0 \in X$, $\lim_{n,m\to\infty} s(x_n, x_m) < \frac{1}{k}$, where $k \in (0, 1)$, here $x_n = \mathcal{H}^n x_0$; $n = 1, 2, 3, \dots$ and $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying:

- $(F1)$ *F_{B_e} is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $F_{B_e}(\alpha) < F_{B_e}(\beta)$;*
- $(F2)$ *For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim\limits_{n\to\infty} \alpha_n = 0$ iff $\lim\limits_{n\to\infty} F_{B_e}(\alpha_n) = -\infty$;*
- $(F3)$ *There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F_{B_e}(\alpha) = 0$.*

We denote by F_{B_e} the set of all functions satisfying the conditions $(F1)$–$(F3)$.

Theorem 1. *Let (X, B_e) be a complete B_e-metric space such that B_e is a continuous functional and let $\mathcal{H} : X \to X$ be an extended $\mathcal{F}_{B_e}$-contraction, then $\mathcal{H}$ has a fixed point.*

Proof. In order to show that $\mathcal{H}$ has a fixed point, let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n\in\mathbb{N}\in X}$, by

$$x_0, \mathcal{H}x_0 = x_1, x_2 = \mathcal{H}x_1 = \mathcal{H}(\mathcal{H}x_0) = \mathcal{H}^2(x_0) \dots x_n = \mathcal{H}^n x_0 \dots$$

Denote $\gamma_n = B_e(x_{n+1}, x_n)$, $n = 0, 1, 2, \dots$.

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ then $\mathcal{H}x_{n_0} = x_{n_0}$ and the proof is finished. Suppose now that $x_{n+1} \neq x_n$ for every $n \in \mathbb{N}$ which yields $B_e(x_{n+1}, x_n) > 0$, i.e., $B_e(\mathcal{H}x_n, \mathcal{H}x_{n-1}) > 0$. Thus, by using (3), the following holds for every $n \in \mathbb{N}$:

$$\begin{aligned} F_{B_e}(\gamma_n) &\leq F_{B_e}(\gamma_{n-1}) - \tau \\ &\leq F_{B_e}(\gamma_{n-2}) - 2\tau \\ &\vdots \\ &\leq F_{B_e}(\gamma_0) - n\tau, \end{aligned} \tag{4}$$

which yields, $\lim_{n\to\infty} F_{B_e}(\gamma_n) = -\infty$.

By $F2$,

$$\lim_{n\to\infty} \gamma_n = 0. \tag{5}$$

From $F3$, there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F_{B_e}(\gamma_n) = 0. \tag{6}$$

By Equation (4), the following holds for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} \gamma_n^k F_{B_e}(\gamma_n) - \gamma_n^k F_{B_e}(\gamma_0) &\le \gamma_n^k (F_{B_e}(\gamma_0) - n\tau) - \gamma_n^k F_{B_e}(\gamma_0) \\ &= -\gamma_n^k n\tau \\ &\le 0. \end{aligned} \tag{7}$$

Letting $n \to \infty$ in (7) and using (4) and (5), we obtain

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{8}$$

Now, let us observe that from (8) there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \le 1$ for all $n \ge n_1$. Consequently, we have

$$\gamma_n \le \frac{1}{n^{\frac{1}{k}}} \quad \text{for all } n \ge n_1. \tag{9}$$

In order to prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. By triangle inequality,

$$\begin{aligned} B_e(x_n, x_m) &\le s(x_n, x_m)[B_e(x_n, x_{n+1}) + B_e(x_{n+1}, x_m)] \\ &\le s(x_n, x_m) B_e(x_n, x_{n+1}) + s(x_n, x_m) s(x_{n+1}, x_m)[B_e(x_{n+1}, x_{n+2}) + B_e(x_{n+2}, x_m)] \\ &\le s(x_n, x_m) B_e(x_n, x_{n+1}) + s(x_n, x_m) s(x_{n+1}, x_m) B_e(x_{n+1}, x_{n+2}) + \ldots \\ &\quad + s(x_n, x_m) s(x_{n+1}, x_m) s(x_{n+2}, x_m) \ldots s(x_{m-2}, x_m) s(x_{m-1}, x_m) B_e(x_{m-1}, x_m) \\ &\le s(x_1, x_m) s(x_2, x_m) \ldots s(x_n, x_m) B_e(x_n, x_{n+1}) \\ &\quad + s(x_1, x_m) s(x_2, x_m) \ldots s(x_{n+1}, x_m) B_e(x_{n+1}, x_{n+2}) + \ldots \\ &\quad + s(x_1, x_m) s(x_2, x_m) \ldots s(x_{m-1}, x_m) B_e(x_{m-1}, x_m). \end{aligned} \tag{10}$$

Note that this series

$$\sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m) \text{ converges.}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m) &\le \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \prod_{i=1}^{n} s(x_i, x_m) \\ &< \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} . \frac{1}{k} \\ &= \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}; \text{ which is convergent.} \end{aligned} \tag{11}$$

Let

$$S = \sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m);$$

$$S_n = \sum_{j=1}^{n} B_e(x_j, x_{j+1}) \prod_{i=1}^{j} s(x_i, x_m).$$

Thus, for $m > n$, the above inequality implies

$$B_e(x_n, x_m) \leq S_{m-1} - S_{n-1}.$$

Letting $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, let $x_n \to \rho \in X$.

Case 1. $\mathcal{H}$ is continuous, we have

$$\begin{aligned} B_e(\mathcal{H}\rho, \rho) &= \lim_{n\to\infty} B_e(\mathcal{H}x_n, x_n) \\ &= \lim_{n\to\infty} B_e(x_{n+1}, x_n) \\ &= 0. \end{aligned}$$

Thus, $\mathcal{H}\rho = \rho$. Thus ρ is a fixed point of $\mathcal{H}$.

Case 2. F_{B_e} is continuous, in this case, we consider two following subcases:

Case 2.1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n} = \mathcal{H}\rho$ and $i_n > i_{n-1}$ where $i = 0$. Then, we have $\rho = \lim_{n\to\infty} x_{i_n} = \lim_{n\to\infty} \mathcal{H}\rho = \mathcal{H}\rho$, which yields that ρ is a fixed point of $\mathcal{H}$.

Case 2.2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq \mathcal{H}\rho$ for al $n \geq n_0$. That is $B_e(\mathcal{H}x_n, \mathcal{H}\rho) > 0$ for all $n \geq n_0$.

It follows from (3) that

$$\begin{aligned} \tau + F_{B_e}(B_e(x_{n+1}, \mathcal{H}\rho)) &= \tau + F_{B_e}(B_e(\mathcal{H}x_n, \mathcal{H}\rho)) \\ &\leq F_{B_e}(B_e(x_n, \rho)). \end{aligned}$$

Since F_{B_e} is continuous, taking the limit as $n \to \infty$, then we obtain

$$\tau + F_{B_e}(B_e(\rho, \mathcal{H}\rho)) \leq F_{B_e}(B_e(\rho, \rho))$$

$$\Rightarrow F_{B_e}(B_e(\rho, \rho)) \leq F_{B_e}(B_e(\rho, \rho)) - \tau,$$

which is a contradiction due to $F1$. Therefore, $B_e(\rho, \mathcal{H}\rho) = 0$. Hence, ρ is a fixed point of $\mathcal{H}$.

Thus, from above two cases, we can conclude that $\mathcal{H}$ has a fixed point ρ. Hence, $\mathcal{H}\rho = \rho$.

In order to prove uniqueness, first, let us observe that $\mathcal{H}$ has at most one fixed point. Indeed, if $x_1, x_2 \in X$, $\mathcal{H}x_1 = x_1 \neq x_2 = \mathcal{H}x_2$, then $B_e(x_1, x_2) > 0$, i.e., $B_e(\mathcal{H}x_1, \mathcal{H}x_2) > 0$. From (3), we get

$$\tau \leq F_{B_e}(B_e(\mathcal{H}x_1, \mathcal{H}x_2)) \leq F_{B_e}(B_e(x_1, x_2)),$$

$$\Rightarrow \tau \leq F_{B_e}(B_e(x_1, x_2)) - F_{B_e}(B_e(x_1, x_2)) = 0,$$

which is a contradiction. Hence, $\mathcal{H}$ has a unique fixed point. □

Example 2. *Let $X = \{\frac{1}{5^{n-1}};\ n \in \mathbb{N}\} \cup \{0\}$. Define $B_e : X \times X \to \mathbb{R}^+$ by $B_e(x, y) = (x - y)^2$ and $s : X \times X \to [1, \infty)$ as $s(x, y) = 1 + x + y$. Then, (X, B_e) is a complete B_e-metric space.*

Define $\mathcal{H} : X \to X$ by

$$\mathcal{H}(x) = \begin{cases} \{\frac{1}{5^{2n}}\}, & \text{if } x \in \{\frac{1}{5^{2n-1}};\ n \in \mathbb{N}\} \\ 0, & \text{if } x = 0. \end{cases}$$

Define the function $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = \ln \alpha$ for all $\alpha \in \mathbb{R}^+$ and $\tau > 0$.

Case 1. Let $x = \frac{1}{5^{2n-1}}, y = \frac{1}{5^{2m-1}}$, for $m > n \geq 1$.

Consider

$$\begin{aligned}
F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) &= F\left(B_e\left(\mathcal{H}\frac{1}{5^{2n-1}},\mathcal{H}\frac{1}{5^{2m-1}}\right)\right)\\
&= F_{B_e}\left(B_e\left(\frac{1}{5^{2n}},\frac{1}{5^{2m}}\right)\right)\\
&= F_{B_e}\left(\left(\frac{1}{5^{2n}}-\frac{1}{5^{2m}}\right)^2\right)\\
&= F_{B_e}\left(\left(\frac{5^{2m}-5^{2n}}{5^{2n+2m}}\right)^2\right)\\
&= \ln\left(\frac{5^{2m}-5^{2n}}{5^{2n+2m}}\right)^2\\
&= 2\ln\left(\frac{5^{2m}-5^{2n}}{5^{2n+2m}}\right).
\end{aligned}$$

$$\begin{aligned}
F_{B_e}(B_e(x,y)) &= F\left(B_e\left(\frac{1}{5^{2n-1}},\frac{1}{5^{2m-1}}\right)\right)\\
&= F_{B_e}\left(\left(\frac{1}{5^{2n-1}}-\frac{1}{5^{2m-1}}\right)^2\right)\\
&= F_{B_e}\left(\left(\frac{5^{2m-1}-5^{2n-1}}{5^{2n+2m-2}}\right)^2\right)\\
&= \ln\left(\frac{5^{2m-1}-5^{2n-1}}{5^{2n+2m-2}}\right)^2\\
&= 2\ln\left(\frac{5^{2m-1}-5^{2n-1}}{5^{2n+2m-2}}\right).
\end{aligned}$$

Consider

$$\begin{aligned}
F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) - F_{B_e}(B_e(x,y)) &= 2\left(\ln\frac{5^{2m}-5^{2n}}{5^{2n+2m}} - \ln\frac{5^{2m-1}-5^{2n-1}}{5^{2n+2m-2}}\right)\\
&= 2\left(\ln\left(\frac{5^{2m}-5^{2n}}{5^{2n+2m}}\times\frac{5^{2n+2m-2}}{5^{2m-1}-5^{2n-1}}\right)\right)\\
&= 2\left(\ln\left(\frac{5^{2m}-5^{2n}}{5^{2n+2m}}\times\frac{5^{2n+2m}.5^{-2}}{5^{-1}(5^{2m}-5^{2n})}\right)\right)\\
&= 2(\ln(\frac{1}{5}))\\
&< -3.
\end{aligned}$$

Thus, $\mathcal{H}$ is an extended $\mathcal{F}_{B_e}$ contraction for $\tau = 3$.

Case 2. Let $x = \frac{1}{5^{2n-1}};\ y = 0$.

$\mathcal{H}x = \frac{1}{5^{2n}};\ \mathcal{H}y = 0$.

Consider

$$F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) = F_{B_e}\left(B_e\left(\frac{1}{5^{2n}},0\right)\right)$$
$$= F_{B_e}\left(\frac{1}{5^{4n}}\right)$$
$$= \ln\left(\frac{1}{5^{4n}}\right).$$

Now consider

$$F_{B_e}(B_e(x,y)) = F_{B_e}\left(B_e\left(\frac{1}{5^{2n-1}},0\right)\right)$$
$$= F_{B_e}\left(\frac{1}{5^{4n-2}}\right)$$
$$= \ln\left(\frac{1}{5^{4n-2}}\right).$$

Now take

$$F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) - F_{B_e}(B_e(x,y)) = \ln\left(\frac{1}{5^{4n}}\right) - \ln\left(\frac{1}{5^{4n-2}}\right)$$
$$= \ln\left(\frac{5^{4n-2}}{5^{4n}}\right)$$
$$= \ln\left(\frac{1}{5^{2}}\right)$$
$$= \ln\left(\frac{1}{25}\right)$$
$$< -3.$$

For $\tau = 3$, $\mathcal{H}$ satisfied all the conditions of the above theorem and 0 is the unique fixed point.

Similarly, for $x = 0$ and $\frac{1}{5^{2n-1}}$, the same proof follows as above. Hence, all the conditions of the above theorem are satisfied for all the cases and 0 is the unique fixed point.

Example 3. *Let $X = \{-1,0,1\}$. Define the function $s : X \times X \to [1,\infty)$ by $s(x,y) = 2 + x + y$ and $B_e : X \times X \to \mathbb{R}^+$ as:*

$$B_e(-1,-1) = B_e(0,0) = B_e(1,1) = 0;$$
$$B_e(-1,0) = B_e(0,-1) = 3;$$
$$B_e(-1,1) = B_e(1,-1) = 7;$$
$$B_e(0,1) = B_e(1,0) = 1.$$

It is clear that (X, B_e) is a complete B_e-metric space.

Let $\mathcal{H} : X \to X$ given by $\mathcal{H}0 = 0 = \mathcal{H}1, \mathcal{H}(-1) = 1$. Define $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = -\frac{1}{\alpha} + \alpha$ and $\tau \in (0,2]$.

Case 1. Let $x = 0$. Now, $B_e(\mathcal{H}0,\mathcal{H}1) = B_e(\mathcal{H}0,\mathcal{H}0) = B_e(0,0) = 0$. Therefore, we only need to consider $y = -1$.

Now, $B_e(\mathcal{H}0,\mathcal{H}(-1)) = B_e(0,1) = 1$ and $B_e(0,-1) = 3$.

Consider

$$\begin{aligned} \tau + F_{B_e}(B_e(\mathcal{H}0, \mathcal{H}(-1))) &= \tau - \tfrac{1}{B_e(\mathcal{H}0,\mathcal{H}(-1))} + (B_e(\mathcal{H}0, \mathcal{H}(-1)) \\ &= \tau - 1 + 1 \\ &= \tau \end{aligned}$$

$$\begin{aligned} F_{B_e}(B_e(0, -1)) &= -\tfrac{1}{B_e(0,-1)} + B_e(0, -1) \\ &= -\tfrac{1}{3} + 3 \\ &= 3 - \tfrac{1}{3} \\ &= \tfrac{8}{3}. \end{aligned}$$

Clearly for $\tau \in (0, \frac{1}{2})$,

$$\tau + F_{B_e}(B_e(\mathcal{H}0, \mathcal{H}(-1))) \leq F_{B_e}(B_e(0, -1)).$$

Case 2. Let $x = 1$. Now, $B_e(\mathcal{H}1, \mathcal{H}1) = B_e(\mathcal{H}1, \mathcal{H}0) = 0$. Therefore, we only need to consider $y = -1$. Now, $B_e(\mathcal{H}1, \mathcal{H}(-1)) = B_e(0, 1) = 1$ and $B_e(1, -1) = 7$.
Consider

$$\begin{aligned} \tau + F_{B_e}(B_e(\mathcal{H}1, \mathcal{H}(-1))) &= \tau - \frac{1}{B_e(\mathcal{H}1, \mathcal{H}(-1))} + (B_e(\mathcal{H}1, \mathcal{H}(-1)) \\ &= \tau - 1 + 1 \\ &= \tau. \end{aligned}$$

$$\begin{aligned} F_{B_e}(B_e(1, -1)) &= -\frac{1}{B_e(1, -1)} + B_e(1, -1) \\ &= -\frac{1}{7} + 7 \\ &= 7 - \frac{1}{7} \\ &= \frac{48}{7}. \end{aligned}$$

Clearly for $\tau \in (0, \frac{1}{2})$,

$$\tau + F_{B_e}(B_e(\mathcal{H}1, \mathcal{H}(-1))) \leq F_{B_e}(B_e(1, -1)).$$

For $x = -1$, the proof is similar as above cases. Hence, all the conditions of the Theorem 1 are satisfied and 0 is the unique fixed point. Thus, the above examples illustrate the above theorem.

3. An Extended F_{B_e}-Expanding Contraction

We start this section by introducing following definition.

Definition 6. *Let (X, B_e) be a B_e-metric space. A mapping $\mathcal{H} : X \to X$ is said to be an extended expanding if*

$$\forall \;\; x, y \in X \quad B_e(\mathcal{H}x, \mathcal{H}y) \geq \kappa B_e(x, y); \;\; \textit{where} \;\; \kappa > 1.$$

Theorem 2. *Let (X, B_e) be a complete B_e-metric space such that B_e is a continuous functional. Let $\mathcal{H} : X \to X$ be surjective and extended expanding. Then, $\mathcal{H}$ is bijective and has a unique fixed point.*

Proof. First, we will prove that $\mathcal{H}$ is bijective. For this, we need to prove $\mathcal{H}$ is injective.

Let $x, y \in X$ with $x \neq y$. From the definition of extended expanding,

$$B_e(\mathcal{H}x, \mathcal{H}y) \geq \kappa B_e(x, y) > 0.$$

which yields $\mathcal{H}x \neq \mathcal{H}y$. Hence, $\mathcal{H}$ is bijective.

Since $\mathcal{H}$ is bijective, $\mathcal{H}$ has an inverse on its range. Note that $\mathcal{H}^{-1}$ is a Banach contraction in the setting of an B_e-metric space. In addition, since $\frac{1}{\kappa} < 1$, we can conclude that $\mathcal{H}^{-1}$ has a unique fixed point by using Theorem 3 of Kamran et al. [13]. This completes the proof of the theorem. □

Theorem 3. *Let (X, B_e) be a complete B_e-metric space such that B_e is a continuous functional. If $\mathcal{H} : X \to X$ is surjective then there exists a mapping $\mathcal{H}^* : X \to X$ such that $\mathcal{H} \circ \mathcal{H}^*$ is the identity map on X.*

The proof is omitted as it is easy to prove.

Now, we define a new definition.

Definition 7. *Let (X, B_e) be a complete B_e-metric space. A mapping $\mathcal{H}$ is said to be extended F-expanding if there exists $F \in \mathcal{F}^*$ and $\tau > 0$ such that for all $x, y \in X$,*

$$B_e(x, y) > 0 \Rightarrow F_{B_e}(B_e(\mathcal{H}x, \mathcal{H}y)) \geq F_{B_e}(B_e(x, y)) + \tau \tag{12}$$

where $F_{B_e} : (0, +\infty) \to \mathbb{R}$ is a mapping satisfying:

$(F1)$ F_{B_e} is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that if $\alpha < \beta$ then $F_{B_e}(\alpha) < F_{B_e}(\beta)$;

$(F2)$ For each sequence $\{\alpha_n\} \subset (0, +\infty)$, then

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F_{B_e}(\alpha_n) = -\infty;$$

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F_{B_e}(\alpha) = 0$.

We represent by $\mathcal{F}^*$ the set of all functions satisfying the conditions $(F1)$–$(F3)$.

Theorem 4. *Let (X, B_e) be a complete B_e-metric space such that B_e is a continuous functional. Let $\mathcal{H} : X \to X$ be surjective and extended F-expanding. Then, $\mathcal{H}$ has a unique fixed point.*

Proof. From Theorem 3, there exists a mapping $\mathcal{H}^* : X \to X$ such that $\mathcal{H} \circ \mathcal{H}^*$ is the identity mapping on X.

Let $x, y \in X$ be arbitrary points such that $x \neq y$, and let $\eta = \mathcal{H}^* x$ and $\xi = \mathcal{H}^* y$ (obviously $\eta \neq \xi$) which yields $B_e(\eta, \xi) > 0$.

From the definition of extended F-expanding, we get

$$F_{B_e}(B_e(\mathcal{H}\eta, \mathcal{H}\xi)) \geq F_{B_e}(B_e(\eta, \xi)) + \tau.$$

Since $\mathcal{H}\eta = \mathcal{H}(\mathcal{H}^* x) = x$ and $\mathcal{H}\xi = \mathcal{H}(\mathcal{H}^* y) = y$, then

$$F_{B_e}(B_e(x, y)) \geq F_{B_e}(B_e(\mathcal{H}^* x, \mathcal{H}^* y)) + \tau.$$

Therefore, $\mathcal{H}^* : X \to X$ is an extended F-contraction. By Theorem 1, $\mathcal{H}^*$ has a unique fixed point $\delta \in X$.

Now consider

$$\begin{aligned} \mathcal{H}\delta &= \mathcal{H}(\mathcal{H}^* \delta) \\ &= \delta \end{aligned} \tag{13}$$

Hence, δ is also a fixed point of $\mathcal{H}$.

In order to get uniqueness, let us suppose that $\mathcal{H}$ has at most two fixed points. If $\delta_1, \delta_2 \in X$ and $\mathcal{H}\delta_1 = \delta_1 \neq \delta_2 = \mathcal{H}\delta_2$, then $B_e(\delta_1, \delta_2) > 0$ which yields

$$F_{B_e}(B_e(\mathcal{H}\delta_1, \mathcal{H}\delta_2)) \geq F_{B_e}(B_e(\delta_1 \delta_2)) + \tau$$

$$0 = F_{B_e}(B_e(\mathcal{H}\delta_1, \mathcal{H}\delta_2)) - F_{B_e}(B_e(\delta_1 \delta_2)) \geq \tau > 0,$$

which is a contradiction. Thus, $\delta_1 = \delta_2$. Therefore, the fixed point of $\mathcal{H}$ is unique. □

Remark 1. *If $\mathcal{H}$ is not surjective, the above theorem is false.*

For example, let $X = [0,1]$. Define $B_e(x,y) : X \times X \to \mathbb{R}^+$ and $s : X \times X \to [1, \infty)$ as $B_e(x,y) = (x-y)^2$, $s(x,y) = x + y + 1$.

Then, B_e is a complete B_e-metric space on X. Define $\mathcal{H} : X \to X$ by $\mathcal{H}x = 2x + 1$ for all $x \in X$. Then,

$$\begin{aligned} B_e(\mathcal{H}x, \mathcal{H}y) &= B_e(2x+1, 2y+1) \\ &= (2x - 2y)^2 \\ &= 4(x-y)^2 \\ &> B_e(x,y). \end{aligned} \tag{14}$$

Thus, $\mathcal{H}$ satisfies all the conditions of the theorem but $\mathcal{H}$ has no fixed point.

If $s(x,y) = 1$, then the above theorem will reduce to Theorem 2.1 of Jaroslaw Gornicki [7]. Thus, we can conclude that our theorem is a standard generalization of Theorem 2.1 of Jaroslaw Gornicki [7].

4. An Extended Generalized F_{B_e}-Contraction

Definition 8. *Let (X, B_e) be a B_e-metric space. A map $\mathcal{H} : X \to X$ is said to be an extended generalized F_{B_e}-contraction on (X, B_e) if there exists $F \in \mathcal{F}^*$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $B_e(\mathcal{H}x, \mathcal{H}y) > 0$, the following holds:*

$$\tau + F_{B_e}(B_e(\mathcal{H}x, \mathcal{H}y)) \leq F_{B_e}\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1 + B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1 + B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x) + B_e(y,\mathcal{H}y)}{2}\right\}\right),$$

and for each $x_0 \in X$, $\lim_{n,m\to\infty} s(x_n, x_m) < \frac{1}{k}$, where $k \in (0,1)$. Here $x_n = \mathcal{H}^n x_0$; $n = 1, 2, 3, \ldots$.

Remark 2.

1. *Every $\mathcal{F}$-contraction is an extended generalized F_{B_e}-contraction.*
2. *Let $\mathcal{H}$ be an extended generalized F_{B_e}-contraction and from the definition of extended generalized F_{B_e}-contractions we have for all $x, y \in X$, $\mathcal{H}x \neq \mathcal{H}y$, which gives $B_e(\mathcal{H}x, \mathcal{H}y) > 0$. Thus,*

$$\begin{aligned} F_{B_e}(B_e(\mathcal{H}x, \mathcal{H}y)) &< \tau + F_{B_e}(B_e(\mathcal{H}x, \mathcal{H}y)) \\ &\leq F_{B_e}\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1 + B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1 + B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x) + B_e(y,\mathcal{H}y)}{2}\right\}\right). \end{aligned} \tag{15}$$

Then, by (F1), we get

$$B_e(\mathcal{H}x, \mathcal{H}y) \leq \max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1 + B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1 + B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x) + B_e(y,\mathcal{H}y)}{2}\right\};$$

$$\forall_{x,y \in X},\ \mathcal{H}x \neq \mathcal{H}y.$$

Counter example for Remark: The following example shows that the inverse implication of the remark does not hold. Let $X = [0, \infty)$ define $B_e : X \times X \to \mathbb{R}$ by $B_e(x, y) = (x - y)^2$ and $s : X \times X \to [1, \infty)$ by $s(x, y) = x + y + 1$. Then, B_e is an B_e-metric. Define $\mathcal{H} : X \to X$ as

$$\mathcal{H}x = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ \frac{1}{4}, & \text{if } x \geq 1. \end{cases}$$

Clearly $\mathcal{H}$ is not continuous.

Thus, $\mathcal{H}$ is not an $\mathcal{F}$-contraction. For $x \in [0, 1)$ and $y = 1$ we have $B_e(\mathcal{H}x, \mathcal{H}1) = B_e(0, \frac{1}{4}) = \frac{1}{16} > 0$ and

$$\begin{aligned} \max\left\{B_e(x,1), \frac{B_e(x,\mathcal{H}x)}{1+B_e(x,\mathcal{H}x)}, \frac{B_e(1,\mathcal{H}1)}{1+B_e(1,\mathcal{H}1)}, \frac{B_e(x,\mathcal{H}x)+B_e(1,\mathcal{H}1)}{2}\right\} &\geq B_e(1,\mathcal{H}1) \\ &= B_e(1,\frac{1}{4}) \\ &= \frac{9}{16} \\ &> \frac{1}{16} \\ &= B_e(\mathcal{H}x,\mathcal{H}1). \end{aligned} \tag{16}$$

Define the function $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = \ln \alpha, \forall \alpha \in \mathbb{R}^+ \& \ \tau > 0$. Then consider

$$\begin{aligned} F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}1)) - F_{B_e}(B_e(1,\mathcal{H}1)) &= F_{B_e}(\frac{1}{16}) - F_{B_e}(\frac{9}{16}) \\ &= \ln(\frac{1}{16}) - \ln(\frac{9}{16}) \\ &= \ln\left((\frac{1}{16}) \times (\frac{16}{9})\right) \\ &= \ln\frac{1}{9} \\ &< -2. \end{aligned} \tag{17}$$

Thus, $\mathcal{H}$ is an extended generalized F_{B_e}-contraction for $\tau = 2$.

Theorem 5. *Let (X, B_e) be a B_e-metric space such that B_e is a continuous functional and $\mathcal{H} : X \to X$ be an extended generalized F_{B_e}-contraction. Then, $\mathcal{H}$ has a unique fixed point.*

Proof. Let $x \in X$ be arbitrary and fixed. We define $x_{n+1} = \mathcal{H}x_n; \ \forall n \in \mathbb{N} \cup \{0\}$, where $x_0 = x$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, then $\mathcal{H}x_{n_0} = x_{n_0}$. This concludes that x_{n_0} is a fixed point of $\mathcal{H}$.

Let us suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Which gives $B_e(x_{n+1}, x_n) > 0$. It follows from extended generalized F_{B_e}-contraction that for each $n \in \mathbb{N}$.

$$\begin{aligned}
F_{B_e}(B_e(x_{n+1},x_n)) &= F_{B_e}(B_e(\mathcal{H}x_n,\mathcal{H}x_{n-1}))\\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,x_{n-1}),\frac{B_e(x_n,x_{n+1})}{1+B_e(x_n,x_{n+1})},\frac{B_e(x_{n-1},x_n)}{1+B_e(x_{n-1},x_n)},\right.\right.\\
&\qquad\left.\left.\frac{B_e(x_n,x_{n+1})+B_e(x_{n-1},x_n)}{2}\right\}\right)-\tau\\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,x_{n-1}),B_e(x_n,x_{n+1}),B_e(x_{n-1},x_n),\right.\right.\\
&\qquad\left.\left.\frac{B_e(x_n,x_{n+1})+B_e(x_{n-1},x_n)}{2}\right\}\right)-\tau\\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,x_{n-1}),B_e(x_n,x_{n+1})\right\}\right)-\tau.
\end{aligned} \tag{18}$$

If $B_e(x_{n+1},x_n) = B_e(x_n,x_{n+1})$ then $F_{B_e}(B_e(x_{n+1},x_n)) \leq F_{B_e}(B_e(x_n,x_{n+1}))-\tau$, which is a contradiction due to $F1$.

Thus,

$$F_{B_e}(B_e(x_{n+1},x_n)) \leq F_{B_e}(B_e(x_n,x_{n-1}))-\tau; \quad \forall n\in\mathbb{N}\cup\{0\}. \tag{19}$$

Similarly,

$$F_{B_e}(B_e(x_n,x_{n-1})) \leq F_{B_e}(B_e(x_{n-1},x_{n-2}))-\tau; \quad \forall n\in\mathbb{N}\cup\{0\}. \tag{20}$$

By using (20)&(21), we have

$$F_{B_e}(B_e(x_{n+1},x_n)) \leq F_{B_e}(B_e(x_{n-1},x_{n-2}))-2\tau. \tag{21}$$

By repeating same scenario, we get

$$F_{B_e}(B_e(x_{n+1},x_n)) \leq F_{B_e}(B_e(x_1,x_0))-n\tau; \quad \forall\, n\in\mathbb{N}\cup\{0\}. \tag{22}$$

Taking the limit as $n\to\infty$ in (23), we get

$$\lim_{n\to\infty} F_{B_e}(B_e(x_{n+1},x_n)) = -\infty. \tag{23}$$

By using $(F2)$, we get

$$\lim_{n\to\infty} B_e(x_{n+1},x_n) = 0. \tag{24}$$

From $(F3)$, there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}\left((B_e(x_{n+1},x_n))^k F_{B_e}(B_e(x_{n+1},x_n))\right) = 0. \tag{25}$$

Now consider

$$\begin{aligned}
(B_e(x_{n+1},x_n))^k(F_{B_e}(B_e(x_{n+1},x_n))-F_{B_e}(B_e(x_1,x_0))) &\leq -(B_e(x_{n+1},x_n))^k n\tau\\
&\leq 0\ ;\forall\, n\in\mathbb{N}.
\end{aligned} \tag{26}$$

By using (25)&(26) and taking the limit as $n\to\infty$ in (27), we get

$$\lim_{n\to\infty}\left(n(B_e(x_{n+1},x_n))^k\right) = 0. \tag{27}$$

Then, there exists $n_1 \in \mathbb{N}$ such that $n(B_e(x_{n+1}, x_n))^k \leq 1\,;\ \ \forall\, n \geq n_1$, which yields

$$B_e(x_{n+1}, x_n) \leq \frac{1}{n^{\frac{1}{k}}}\,;\ \ \forall\, n \geq n_1. \tag{28}$$

In order to prove that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. By using (29) and the triangle inequality, we get

$$\begin{aligned}
B_e(x_n, x_m) &\leq s(x_n, x_m)[B_e(x_n, x_{n+1}) + B_e(x_{n+1}, x_m)] \\
&\leq s(x_n, x_m)B_e(x_n, x_{n+1}) + s(x_n, x_m)s(x_{n+1}, x_m)[B_e(x_{n+1}, x_{n+2}) + B_e(x_{n+2}, x_m)] \\
&\leq s(x_n, x_m)B_e(x_n, x_{n+1}) + s(x_n, x_m)s(x_{n+1}, x_m)B_e(x_{n+1}, x_{n+2}) + \ldots \\
&\quad + s(x_n, x_m)s(x_{n+1}, x_m)s(x_{n+2}, x_m)\ldots s(x_{m-2}, x_m)s(x_{m-1}, x_m)B_e(x_{m-1}, x_m) \\
&\leq s(x_1, x_m)s(x_2, x_m)\ldots s(x_n, x_m)B_e(x_n, x_{n+1}) \\
&\quad + s(x_1, x_m)s(x_2, x_m)\ldots s(x_{n+1}, x_m)B_e(x_{n+1}, x_{n+2}) + \ldots \\
&\quad + s(x_1, x_m)s(x_2, x_m)\ldots s(x_{m-1}, x_m)B_e(x_{m-1}, x_m).
\end{aligned} \tag{29}$$

Note that this series

$$\sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m) \ \text{ converges.}$$

Since

$$\begin{aligned}
\sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m) &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \prod_{i=1}^{n} s(x_i, x_m) \\
&< \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} . \frac{1}{k} \\
&= \frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}};\ \ \text{which is convergent.}
\end{aligned} \tag{30}$$

Let

$$S = \sum_{n=1}^{\infty} B_e(x_n, x_{n+1}) \prod_{i=1}^{n} s(x_i, x_m);$$

$$S_n = \sum_{j=1}^{n} B_e(x_j, x_{j+1}) \prod_{i=1}^{j} s(x_i, x_m).$$

Thus, for $m > n$ above inequality implies

$$B_e(x_n, x_m) \leq S_{m-1} - S_{n-1}.$$

Letting $n \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $\kappa \in X$ such that $\{x_n\} \to \kappa$.

We shall prove that κ is a fixed point of $\mathcal{H}$ by two following cases:

Case 1. $\mathcal{H}$ is continuous, we have

$$\begin{aligned}
B_e(\kappa, \mathcal{H}\kappa) &= \lim_{n\to\infty} B_e(x_n, \mathcal{H}x_n) \\
&= \lim_{n\to\infty} B_e(x_n, x_{n+1}) \\
&= 0.
\end{aligned} \tag{31}$$

This proves that κ is a fixed point of $\mathcal{H}$.

Case 2. F_{B_e} is continuous. In this case, we consider two following sub-cases:

Case 2.1. For each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $x_{i_n+1} = \mathcal{H}\kappa$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then, we have

$$\kappa = \lim_{n\to\infty} x_{i_n+1} = \lim_{n\to\infty} \mathcal{H}\kappa = \mathcal{H}\kappa.$$

This proves that κ is a fixed point of $\mathcal{H}$.

Case 2.2. There exists $n_0 \in \mathbb{N}$ such that $x_{n+1} \neq \mathcal{H}\kappa$; $\forall\, n \geq n_0$.

$$i.e, \;\; B_e(\mathcal{H}x_n, \mathcal{H}\kappa) > 0 \,;\, \forall\, n \geq n_0.$$

It follows from extended generalized F_{B_e}-contraction and $F1$,

$$\begin{aligned}
\tau + F_{B_e}(B_e(x_{n+1}, \mathcal{H}\kappa)) &= \tau + F_{B_e}(B_e(\mathcal{H}x_n, \mathcal{H}\kappa)) \\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,\kappa), \frac{B_e(x_n,\mathcal{H}x_n)}{1+B_e(x_n,\mathcal{H}x_n)}, \frac{B_e(\kappa,\mathcal{H}\kappa)}{1+B_e(\kappa,\mathcal{H}\kappa)},\right.\right. \\
&\qquad\left.\left.\frac{B_e(x_n,\mathcal{H}x_n)+B_e(\kappa,\mathcal{H}\kappa)}{2}\right\}\right) \\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,\kappa), B_e(x_n,x_{n+1}), B_e(\kappa,\mathcal{H}\kappa),\right.\right. \\
&\qquad\left.\left.\frac{B_e(x_n,\mathcal{H}x_n)+B_e(\kappa,\mathcal{H}\kappa)}{2}\right\}\right) \\
&\leq F_{B_e}\left(\max\left\{B_e(x_n,\kappa), B_e(x_n,x_{n+1}), B_e(\kappa,\mathcal{H}\kappa),\right.\right. \\
&\qquad\left.\left.\frac{B_e(x_n,x_{n+1})+B_e(\kappa,\mathcal{H}\kappa)}{2}\right\}\right).
\end{aligned} \tag{32}$$

If $B_e(\kappa, \mathcal{H}\kappa) > 0$ then $\lim_{n\to\infty} B_e(x_n, \kappa) = \lim_{n\to\infty} B_e(\kappa, x_{n+1}) = 0$.
Then, there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we have

$$\max\left\{B_e(x_n,\kappa), B_e(x_n,x_{n+1}), B_e(\kappa,\mathcal{H}\kappa), \frac{B_e(x_n,x_{n+1})+B_e(\kappa,\mathcal{H}\kappa)}{2}\right\} = B_e(\kappa,\mathcal{H}\kappa).$$

From (33), we get

$$\tau + F_{B_e}(B_e(x_{n+1}, \mathcal{H}\kappa)) \leq F_{B_e}(B_e(\kappa, \mathcal{H}\kappa)); \;\; \forall\, n \geq \max\{n_0, n_1\}. \tag{33}$$

Since F_{B_e} is continuous, taking the limit as $n \to \infty$ in (34) , we obtain

$$\tau + F_{B_e}(B_e(\kappa, \mathcal{H}\kappa)) \leq F_{B_e}(B_e(\kappa, \mathcal{H}\kappa)),$$

which is a contradiction. Hence, $B_e(\kappa, \mathcal{H}\kappa) = 0$. Therefore, κ is a fixed point of $\mathcal{H}$.

By the above two cases, $\mathcal{H}$ has a fixed point κ.

To prove uniqueness, let κ, κ^* be two fixed points of $\mathcal{H}$, such that $\kappa \neq \kappa^*$.

Thus, $B_e(\kappa, \kappa^*) > 0$ which implies $B_e(\mathcal{H}\kappa, \mathcal{H}\kappa^*) > 0$.

From extended generalized F-contraction,

$$\begin{aligned}\tau + F_{B_e}(B_e(\kappa,\kappa^*)) &= \tau + F_{B_e}(B_e(\mathcal{H}\kappa,\mathcal{H}\kappa^*)) \\ &\leq F_{B_e}\left(\max\left\{B_e(\kappa,\kappa^*), \frac{B_e(\kappa,\mathcal{H}\kappa)}{1+B_e(\kappa,\mathcal{H}\kappa)}, \frac{B_e(\kappa^*,\mathcal{H}\kappa^*)}{1+B_e(\kappa^*,\mathcal{H}\kappa^*)},\right.\right. \\ &\qquad\qquad \left.\left.\frac{B_e(\kappa,\mathcal{H}\kappa)+B_e(\kappa^*,\mathcal{H}\kappa^*)}{2}\right\}\right) \qquad (34) \\ &= F_{B_e}\left(\max\left\{B_e(\kappa,\kappa^*), \frac{B_e(\kappa,\kappa)}{1+B_e(\kappa,\kappa)}, \frac{B_e(\kappa^*,\kappa^*)}{1+B_e(\kappa^*,\kappa^*)}\right\}\right) \\ &= F_{B_e}(B_e(\kappa,\kappa^*)).\end{aligned}$$

which implies, $\tau \leq F_{B_e}(B_e(\kappa,\kappa^*)) - F_{B_e}(B_e(\kappa,\kappa^*)) = 0$. This is a contradiction.

Thus, $B_e(\kappa,\kappa^*) = 0$, which yields $\kappa = \kappa^*$. Hence, the fixed point of $\mathcal{H}$ is unique. □

Example 4. *Let* $X = \left\{\frac{1}{2^{n-1}}; n \in \mathbb{N}\right\} \cup \{0\}$. *Define* $B_e : X \times X \to \mathbb{R}^+$ *by* $B_e(x,y) = (x-y)^2$ *and* $s : X \times X \to [1,\infty)$ *as* $s(x,y) = x+y+1$. *Then,* B_e *is a complete* B_e*-metric on* X.

Define $\mathcal{H} : X \to X$ by

$$\mathcal{H}(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x \in \left\{\frac{1}{2^{n-1}}; n \in \mathbb{N}\right\}; \\ 0, & \text{if } x \in X. \end{cases}$$

Define the function $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = \ln\alpha$ for all $\alpha \in \mathbb{R}^+$ and $\tau > 0$.

Case 1. For $m > n \geq 1$. Let $x = \frac{1}{2^{n-1}}$ and $y = \frac{1}{2^{m-1}}$.

Now take $n = 1$ and $m = 2$.

Consider

$$\begin{aligned} F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) &= F_{B_e}(B_e(\mathcal{H}1,\mathcal{H}\tfrac{1}{2})) \\ &= F_{B_e}(B_e(\tfrac{1}{2},\tfrac{1}{4})) \\ &= F_{B_e}(\tfrac{1}{16}) \\ &= \ln\tfrac{1}{16} \\ &= -2.7725. \end{aligned}$$

Additionally,

$$\begin{aligned} &F_{B_e}\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1+B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1+B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x)+B_e(y,\mathcal{H}y)}{2}\right\}\right) \\ &= F_{B_e}\left(\max\left\{B_e(1,\frac{1}{2}), \frac{B_e(1,\frac{1}{2})}{1+B_e(1,\frac{1}{2})}, \frac{B_e(\frac{1}{2},\frac{1}{2})}{1+B_e(\frac{1}{2},\frac{1}{2})}, \frac{B_e(1,\frac{1}{2})+B_e(\frac{1}{2},\frac{1}{4})}{2}\right\}\right) \\ &= F_{B_e}\left(\max\left\{\frac{1}{4},\frac{1}{5},\frac{1}{17},\frac{5}{16}\right\}\right) \\ &= F_{B_e}(\frac{5}{16}) \\ &= \ln\frac{5}{16} \\ &= -1.1631. \end{aligned}$$

Consider

$$F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) - F\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1+B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1+B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x)+B_e(y,\mathcal{H}y)}{2}\right\}\right)$$
$$= -2.7725 + 1.1631$$
$$= -1.6094$$
$$< -1.$$

Thus, $\mathcal{H}$ is an extended generalized F_{B_e}-contraction for $\tau = 1$.

Case 2. Let $x = \frac{1}{2}$ and $y = 0$.
Consider $F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) = F_{B_e}(B_e(\frac{1}{4},0)) = F_{B_e}(\frac{1}{16}) = \ln\frac{1}{16} = -2.77$.
Now,

$$F_{B_e}\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1+B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1+B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}y)+B_e(y,\mathcal{H}x)}{2}\right\}\right)$$
$$= F_{B_e}\left(\max\left\{B_e(\frac{1}{2},0), \frac{B_e(\frac{1}{2},\frac{1}{4})}{1+B_e(\frac{1}{2},\frac{1}{4})}, \frac{B_e(0,0)}{1+B_e(0,0)}, \frac{B_e(\frac{1}{2},\frac{1}{4})+B_e(0,0)}{2}\right\}\right)$$
$$= F_{B_e}\left(\max\left\{\frac{1}{4}, \frac{1}{17}, 0, \frac{1}{32}\right\}\right)$$
$$= F_{B_e}(\frac{1}{4})$$
$$= \ln\frac{1}{4}$$
$$= -1.38.$$

Now consider

$$F_{B_e}(B_e(\mathcal{H}x,\mathcal{H}y)) - F\left(\max\left\{B_e(x,y), \frac{B_e(x,\mathcal{H}x)}{1+B_e(x,\mathcal{H}x)}, \frac{B_e(y,\mathcal{H}y)}{1+B_e(y,\mathcal{H}y)}, \frac{B_e(x,\mathcal{H}x)+B_e(y,\mathcal{H}y)}{2}\right\}\right)$$
$$= -2.77 + 1.38$$
$$= -1.39$$
$$< -1.$$

Thus, $\mathcal{H}$ is an extended generalized F_{B_e}-contraction for $\tau = 1$.

Hence, we can conclude that all the conditions of above theorem are satisfied in all cases and 0 is the unique fixed point.

5. Applications to Existence of Solutions of Non-linear Integral Equation

As applications, we use Theorem 1 and Theorem 5 to study the existence problem of unique solutions of non-linear integral equations.

Theorem 6. *Let X be the set of all continuous real valued functions defined on $[a,b]$. i.e., $X = \mathbb{C}([a,b],\mathbb{R})$.*

Define $B_e : X \times X \to \mathbb{R}$ by $B_e(U,V) = sup\ |U(t) - V(t)|^2,\ \ t \in [a,b]$ with $s(U,V) = |U(t)| + |V(t)| + 1$, where $s : X \times X \to [1,\infty)$.

Note that (X, B_e) is a complete B_e-metric space.

Consider the Fredholm integral equation as

$$U(t) = \int_a^b \mathcal{H}(t,p,U(p))dp + F_{B_e}(t) \ \ \forall \ \ t,p \in [a,b], \tag{35}$$

where $F_{B_e} : [a,b] \to \mathbb{R}$ and $\mathcal{H} : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Define $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ by $\mathcal{H}(U(t)) = \int_a^b \mathcal{H}(t,p,U(p))dp + F_{B_e}(t) \ \ \forall \ \ t,p \in [a,b]$; where $F_{B_e} : [a,b] \to \mathbb{R}$ and $\mathcal{H} : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Further assume that the following condition holds:

$$|\mathcal{H}(t,p,U(p)) - \mathcal{H}(t,p,V(p))| \leq e^{-\frac{\tau}{2}} |U(p) - V(p)|$$

for each $t,p \in [a,b], U,V \in X$ and $\tau > 0$. Then, the integral Equation (35) has a solution. We will prove that the operator $\mathcal{H}$ satisfies the conditions of Theorem 1.

For any $U(t), V(t) \in X$. Consider

$$\begin{aligned}
|\mathcal{H}U(t) - \mathcal{H}V(t)|^2 &= \left(\int_a^b |\mathcal{H}(t,p,U(p)) - \mathcal{H}(t,p,V(p))| \right)^2 dp \\
&\leq \left(\int_a^b e^{-\frac{\tau}{2}} |U(p) - V(p)| \right)^2 dp \\
&\leq e^{-\tau} \left(\int_a^b |U(p) - V(p)| \right)^2 dp \\
&\leq e^{-\tau} B_e(U(t), V(t)),
\end{aligned}$$

which implies $B_e(\mathcal{H}U(t), \mathcal{H}V(t)) \leq e^{-\tau} B_e(U(t), V(t))$.

Applying logarithms on both sides, we get

$$\begin{aligned}
&\ln(B_e(\mathcal{H}U(t), \mathcal{H}V(t))) \leq \ln(e^{-\tau} B_e(U(t), V(t))); \\
\Rightarrow &\ln(B_e(\mathcal{H}U(t), \mathcal{H}V(t))) \leq \ln(e^{-\tau}) + \ln(B_e(U(t), V(t))); \\
\Rightarrow &\ln(B_e(\mathcal{H}U(t), \mathcal{H}V(t))) \leq -\tau + \ln(B_e(U(t), V(t))).
\end{aligned}$$

Thus,

$$\tau + \ln(B_e(\mathcal{H}U(t), \mathcal{H}V(t))) \leq \ln(B_e(U(t), V(t))). \tag{36}$$

Let us define $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = \ln(\alpha), \ \ \alpha > 0$. Then, from (36), we get

$$\tau + F_{B_e}(B_e(\mathcal{H}U(t), \mathcal{H}V(t))) \leq F(B_e(U(t), V(t))).$$

Thus all the conditions of the Theorem 1 are satisfied. Thus, the operator $\mathcal{H}$ has a unique fixed point. Hence, the Fredholm integral equation has a solution.

Theorem 7. *Let us consider the non-linear integral equation.*

$$U(t) = F_{B_e}(t) + \int_0^t k(t,p) g(p, U(p)) dp, \tag{37}$$

where the unknown function $U(t)$ takes real values.

Let $X = \mathbb{C}([0,\beta])$ be the space of all real continuous functions defined on $[0,\beta]$.

Define $B_e : X \times X \to \mathbb{R}$ by $B_e(U,V) = \max_{t \in [0,\beta]} |U(t) - V(t)|^2$ and $s : X \times X \to [1,\infty)$ by $s(U,V) = |U(t)| + |V(t)| + 1$.

Clearly, (X, B_e) is a complete B_e-metric space.

Define a mapping $\mathcal{H} : X \to X$ by $\mathcal{H}U(t) = F_{B_e}(t) + \int_0^t k(t,p)g(p,U(p))dp; \ \forall t \in [0,\beta]$.
Furthermore, we assume the following conditions:

1. $g \in \mathbb{C}([0,\beta] \times (-\infty, -\infty))$ and $k \in \mathbb{C}([0,\beta] \times [0,\beta])$ such that $k(t,p) \geq 0$.
2. $g(t,.) : (-\infty, +\infty) \to (-\infty, +\infty)$ is increasing for all $t \in [0,\beta]$.
3. There exists $\tau \in [1, +\infty)$ such that for all $U, V \in X$, $t \in [0,\beta]$, $|g(t,U) - g(t,V)|^2 \leq \tau e^{-\tau} \mathcal{M}(U,V)$,
   where, $\mathcal{M}(U,V) = \max\left\{|U-V|^2, \frac{|U-\mathcal{H}U|^2}{1+|U-\mathcal{H}U|^2}, \frac{|V-\mathcal{H}V|^2}{1+|V-\mathcal{H}V|^2}, \frac{|U-\mathcal{H}U|^2+|V-\mathcal{H}V|^2}{2}\right\}$.
4. $\max_{t,p\in[0,\beta]} |k(t,s)|^2 \leq 1$; For $U \in X$, we define a norm $||U||_\tau = \max_{t\in[0,\beta]} |U(t)|e^{-\tau t}$, where $t \geq 1$ is chosen arbitrarily.

It is easy to check that $||.||_\tau$ is equivalent to the maximum norm $||.||$ in X, and X be endowed with the B_{e_τ} defined by

$$\begin{aligned} B_{e_\tau}(U,V) &= ||U-V||_\tau \\ &= \max_{t\in[0,\beta]} \{|U(t)-V(t)|^2 e^{-\tau t}\}; \ U,V \in X \ \text{ and } \ e^{t\tau} \geq 1. \end{aligned} \tag{38}$$

Then, (X, B_{e_τ}) is a complete B_e-metric space.

Now, we will prove that the non-linear integral Equation (37) has a unique solution. For any $U, V \in \mathbb{C}([0,\beta]), t \in [0,\beta]$ we have

$$\begin{aligned} |\mathcal{H}U(t) - \mathcal{H}V(t)|^2 &= |\int_0^t k(t,p)[g(p,U(p)) - g(p,V(p))]dp|^2 \\ &\leq \int_0^t |k(t,p)|^2 ||g(p,U(p)) - g(p,V(p))|^2 dp \\ &\leq \int_0^t ||g(p,U(p)) - g(p,V(p))|^2 dp \\ &\leq \int_0^t \tau e^{-\tau} \mathcal{M}(U(p),V(p))dp \\ &= \tau e^{-\tau} \int_0^t e^{p\tau} \max\left\{ |U(p)-V(p)|^2 e^{-p\tau}, \frac{|U(p)-\mathcal{H}U(p)|^2 e^{-2p\tau}}{1+|U(p)-\mathcal{H}U(p)|^2 e^{-p\tau}}, \right. \\ &\qquad \frac{|V(p)-\mathcal{H}V(p)|^2 e^{-2p\tau}}{1+|V(p)-\mathcal{H}V(p)|^2 e^{-p\tau}}, \\ &\qquad \left. \frac{|U(p)-\mathcal{H}U(p)|^2 + |V(p)-\mathcal{H}V(p)|^2}{2} e^{-p\tau} \right\} dp \\ &\leq \tau e^{-\tau} \int_0^t e^{s\tau} \max\left\{ B_{e_\tau}(U,V), \frac{B_{e_\tau}(U,\mathcal{H}U)}{1+B_{e_\tau}(U,\mathcal{H}U)}, \frac{B_{e_\tau}(V,\mathcal{H}V)}{1+B_{e_\tau}(V,\mathcal{H}V)}, \right. \\ &\qquad \left. \frac{B_{e_\tau}(U,\mathcal{H}U) + B_{e_\tau}(V,\mathcal{H}V)}{2} \right\} dp \\ &= \tau e^{-\tau} \mathcal{M}(U,V) \int_0^t e^{s\tau} dp \\ &\leq \tau e^{-\tau} \mathcal{M}(U,V) \frac{e^{t\tau}}{\tau} \\ &\leq e^{-\tau} \mathcal{M}(U,V) e^{t\tau} \\ &\leq e^{-(1-t)\tau} \mathcal{M}(U,V) \end{aligned} \tag{39}$$

which implies

$$|\mathcal{H}U(t) - \mathcal{H}V(t)|^2 e^{-t\tau} \leq e^{-\tau}\mathcal{M}(U,V),$$

which yields

$$\begin{aligned} B_{e_\tau}(\mathcal{H}U,\mathcal{H}V) = & \max_{t\in[0,\beta]}\{|\mathcal{H}U(t) - \mathcal{H}V(t)|^2 e^{-t\tau}\} \\ \leq & \, e^{-\tau}\mathcal{M}(U,V). \end{aligned} \tag{40}$$

Applying logarithms on both sides, we get

$$\tau + \ln B_{e_\tau}(\mathcal{H}U,\mathcal{H}V) \leq \ln \mathcal{M}(U,V); \;\; \forall \;\; U,V \in X. \tag{41}$$

Define $F_{B_e} : \mathbb{R}^+ \to \mathbb{R}$ by $F_{B_e}(\alpha) = \ln \alpha, \;\; \alpha > 0$. Then, from (41) we get

$$\tau + F_{B_e}(B_{e_\tau}(\mathcal{H}U,\mathcal{H}V)) \leq F_{B_e}(\mathcal{M}(U,V));$$

where $\mathcal{M}(U,V) = \max\left\{B_{e_\tau}(U,V), \frac{B_{e_\tau}(U,\mathcal{H}U)}{1+B_{e_\tau}(U,\mathcal{H}U)}, \frac{B_{e_\tau}(V,\mathcal{H}V)}{1+B_{e_\tau}(V,\mathcal{H}V)}, \frac{B_{e_\tau}(U,\mathcal{H}U)+B_{e_\tau}(V,\mathcal{H}V)}{2}\right\}$.

Thus, $\mathcal{H}$ is an extended generalized F-contraction. By Theorem 5, $\mathcal{H}$ has a unique fixed point. Hence, it is the unique solution of the non-linear integral equation.

6. Conclusions

The research topic of *fixed point theory and applications*, with an extended approach being the latest, has continued for decades.

An extended b-metric space was introduced in 2017 by Kamran et al. [14]. Since then, very few researchers established fixed point theorems using F-contractions in an extended b-metric space since it was very hard to obtain fixed points via the Warkowski [15] approach. In this article, we first introduce various topics called the extended $\mathcal{F}_{B_e}$-contraction, the extended F_{B_e}-expanding contraction, and the extended generalized F_{B_e}-contraction. Thereafter, we presented various fixed point theorems related to F-contractions, which gives a solutions for a non-linear integral equation by using the fixed point technique. Our results are important as they open new research avenues for non-linear analysis and its applications.

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