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Analytical Solution of Linear Fractional Systems with Variable Coefficients Involving Riemann–Liouville and Caputo Derivatives

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Abstract: This paper deals with the initial value problem for linear systems of fractional differential equations (FDEs) with variable coefficients involving Riemann–Liouville and Caputo derivatives. Some basic properties of fractional derivatives and antiderivatives, including their non-symmetry w.r.t. each other, are discussed. The technique of the generalized Peano–Baker series is used to obtain the state-transition matrix. Explicit solutions are derived both in the homogeneous and inhomogeneous case. The theoretical results are supported by examples.

Keywords: fractional differential equation; Riemann–Liouville derivative; Caputo derivative; state transition matrix; generalized Peano–Baker series

1. Introduction

Fractional differential equations (FDEs) provide a powerful tool to describe memory effect and hereditary properties of various materials and processes [1–7]. While linear systems of FDEs represent a fairly well investigated field of research, relatively few papers deal with linear FDEs involving variable coefficients. Meanwhile, models of many real-life systems and processes are described in terms of linear FDEs with variable coefficients, e.g., linearized aircraft models, linearized models of population restricted growth, models related to the distribution of parameters in the charge transfer and the diffusion of the batteries, etc. Explicit solutions to linear systems of differential equations provide a basis to solve control problems. Analytical solutions of the linear systems of fractional differential equations with constant coefficients were derived in the papers [8,9] and then applied to solving control problems and differential games in [10–13].

However, only a few papers are devoted to solutions of the systems of FDEs with variable coefficients and their control. In [14], explicit solutions for the linear systems of initialized [15] FDEs are obtained in terms of generalized Peano–Baker series [16].

This paper deals with the initial value problem for linear systems of FDEs with variable coefficients involving Riemann–Liouville and Caputo derivatives. The technique of the generalized Peano–Baker series is used to obtain the state-transition matrix. Explicit solutions are derived both in the homogeneous and inhomogeneous case. The theoretical results are supported by examples.

2. Preliminaries

Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by I some interval of the real line, $I \subset \mathbb{R}$. In what follows we will assume that $I = [t_0, T]$ for some $T > t_0$ and denote $\mathring{I} = (t_0, T)$. Suppose

$f : I \to \mathbb{R}^n$ is an absolutely continuous function. Let us recall that the Riemann–Liouville (left-sided) fractional integral and derivative of order α, $0 < \alpha < 1$, are defined as follows:

$$
\begin{aligned}
{}_{t_0}J_t^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \\
{}_{t_0}D_t^{\alpha} f(t) &= \frac{d}{dt}\, {}_{t_0}J_t^{1-\alpha} f(t),\ t \in \mathring{I}.
\end{aligned}
$$

The Riemann–Liouville fractional derivative of a constant does not equal zero. Moreover, it becomes infinite as t approaches t_0. That is why the regularized Caputo derivative was introduced, which is free of these shortcomings.

The Caputo (regularized) derivative of fractional order α, $0 < \alpha < 1$, can be introduced by the following formula:

$$
{}_{t_0}D_t^{(\alpha)} f(t) = {}_{t_0}J_t^{1-\alpha} \frac{d}{dt} f(t),\ t \in \mathring{I}. \tag{1}
$$

The following properties of the fractional integrals and derivatives [1,17–19] will be used in the sequel.

Lemma 1. *If $\alpha, \beta > 0$, and $f(t)$ is such that the derivatives and integrals below exist, the following equalities hold true:*

$$
{}_{t_0}D_t^{\alpha}\, {}_{t_0}J_t^{\alpha} f(t) = f(t), \tag{2}
$$

$$
{}_{t_0}D_t^{(\alpha)}\, {}_{t_0}J_t^{\alpha} f(t) = f(t), \tag{3}
$$

$$
{}_{t_0}J_t^{\alpha}\, {}_{t_0}J_t^{\beta} f(t) = {}_{t_0}J_t^{\alpha+\beta} f(t). \tag{4}
$$

If, moreover, $\alpha < 1$, then

$$
{}_{t_0}D_t^{(\alpha)} f(t) = {}_{t_0}D_t^{\alpha} f(t) - f(t_0) \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5}
$$

Lemma 2. *For $\beta > 0$*

$$
{}_{t_0}J_t^{\alpha} (t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-t_0)^{\beta+\alpha-1}, \tag{6}
$$

$$
{}_{t_0}D_t^{\alpha} (t-t_0)^{\beta-1} = \begin{cases} 0, & \beta \in \{\alpha - m + 1, \dots, \alpha\} \\ \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-t_0)^{\beta-\alpha-1}, & \text{otherwise} \end{cases} \tag{7}
$$

$$
{}_{t_0}D_t^{(\alpha)} (t-t_0)^{\beta-1} = \begin{cases} 0, & \beta \in \{1, 2 \dots, m\} \\ \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-t_0)^{\beta-\alpha-1}, & \beta > m \\ \text{non-existent} & \text{otherwise} \end{cases} \tag{8}
$$

where $m = \lceil \alpha \rceil$ is the least integer greater than or equal to α.

In particular, from Equations (6), (7) and (10) it follows that

$$
{}_{t_0}D_t^{\alpha} \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} = 0, \tag{9}
$$

$$
{}_{t_0}D_t^{(\alpha)} 1 = 0, \tag{10}
$$

$$
{}_{t_0}J_t^{1-\alpha} \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} = 1. \tag{11}
$$

Let us formulate some preliminary results on Lebesgue integration before we proceed.

Theorem 1 ([20]). *Suppose that $X, Y \subseteq \mathbb{R}$ are intervals. Suppose also that the function $f : X \times Y \to \mathbb{R}$ satisfies the following conditions:*

(a) *For every fixed $y \in Y$, the function $f(\cdot, y)$ is measurable on X.*
(b) *The partial derivative $\frac{\partial}{\partial y} f(x, y)$ exists for every interior point $(x, y) \in X \times Y$.*
(c) *There exists a non-negative integrable function g such that $\left|\frac{\partial}{\partial y} f(x, y)\right| \le g(x)$ for every interior point $(x, y) \in X \times Y$.*
(d) *There exists $y_0 \in Y$ such that $f(x, y_0)$ is integrable on X.*

Then for every $y \in Y$, the Lebesgue integral

$$\int_X f(x, y)dx$$

exists. Furthermore, the function $F : Y \to \mathbb{R}$, defined by

$$F(y) = \int_X f(x, y)dx$$

for every $y \in Y$, is differentiable at every interior point of Y, and the derivative $F(y)$ satisfies

$$F'(y) = \int_X \frac{\partial}{\partial y} f(x, y)dx.$$

Corollary 1. *If $X = (y_0, y)$, and hypotheses of Theorem 1 are fulfilled, the following equality holds true for all $y \in Y$*

$$\frac{d}{dy} \int_{y_0}^{y} f(x, y)dx = \int_{y_0}^{y} \frac{\partial}{\partial y} f(x, y)dx + f(y - 0, y). \tag{12}$$

This corollary can be obtained by differentiating the function $G(y, \gamma) = \int_{y_0}^{\gamma} f(x, y)dx$, then applying Theorem 1 and the chain rule.

3. Homogeneous System of Linear FDEs with Variable Coefficients Involving Riemann–Liouville Derivatives

Let us consider the following initial value problem:

$$\begin{aligned} {}_{t_0}D_t^{\alpha} x(t) &= A(t)x(t),\ t \in \mathring{I}, \\ {}_{t_0}J_t^{1-\alpha} x(t)\big|_{t=t_0} &= x_0, \end{aligned} \tag{13}$$

hereafter it is assumed that $x(t)$ is a vector function taking values in $\mathbb{R}^n$ and the matrix function $A(t)$, $A : I \to \mathbb{R}^{n \times n}$ is continuous on I.

Definition 1. *The state-transition matrix of the system given by Equation (13) is defined as follows:*

$$\Phi(t, t_0) = \sum_{k=0}^{\infty} {}_{t_0}J_t^{k \circ \alpha} A(t), \tag{14}$$

where

$$ {}_{t_0}J_t^{0\circ\alpha}A(t) = \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\mathbb{1}, $$
$$ {}_{t_0}J_t^{(k+1)\circ\alpha}A(t) = {}_{t_0}J_t^{\alpha}(A(t)\,{}_{t_0}J_t^{k\circ\alpha}A(t)),\ k=0,1,\ldots $$

Hereafter $\mathbb{1}$ *stands for an identity matrix.*

We will refer to the series on the right-hand side of Equation (14) as the generalized Peano–Baker series [14,16].

Assumption 1. *The generalized Peano–Baker series in the right-hand side of Equation* (14) *converges uniformly.*

In view of Lemma 1 and of Equations (9) and (11), the following lemma holds true.

Lemma 3. *Under Assumption 1 the state-transition matrix* $\Phi(t,t_0)$ *satisfies the following initial value problem*

$$ {}_{t_0}D_t^{\alpha}\Phi(t,t_0) = A(t)\Phi(t,t_0),\ {}_{t_0}J_t^{1-\alpha}\Phi(t,t_0)\big|_{t=t_0} = \mathbb{1}. $$

Lemma 3 implies the following

Theorem 2. *Under Assumption 1 solution to the initial value problem described by Equation* (13) *is given by the following expression:*

$$ x(t) = \Phi(t,t_0)x_0. \tag{15} $$

Remark 1. *If* $A(t)$ *is a constant matrix, i.e.,* $A(t)\equiv A$*, then in view of Equation* (6) *one gets*

$$ {}_{t_0}J_t^{k\circ\alpha}A = \frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}A^k $$

and

$$ \Phi(t,t_0) = e_\alpha^{(t-t_0)A} = t^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k(t-t_0)^{\alpha k}}{\Gamma[(k+1)\alpha]} = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}), $$

where $E_{\alpha,\alpha}(At^{\alpha})$ *is a matrix Mittag-Leffler function and* $e_\alpha^{(t-t_0)A}$ *is the matrix* α*-exponential function [18].*

Equation (15) *takes on the form*

$$ x(t) = e_\alpha^{(t-t_0)A}x_0, $$

which is consistent with the formulas, obtained for the systems of fractional differential equations with constant coefficients [10,18].

Example

Let us consider the following system

$$ \begin{gathered} {}_0D_t^{\alpha}x(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} x(t), \\ {}_0J_t^{1-\alpha}x(t)\big|_{t=0} = x_0. \end{gathered} \tag{16} $$

Direct calculation yields

$$\Phi(t,0) = \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}. \tag{17}$$

It can be readily seen that

$$\begin{aligned} {}_0D_t^{\alpha}\Phi(t,0) &= \begin{pmatrix} 0 & \frac{t^{\alpha}}{\Gamma(\alpha)} \\ 0 & 0 \end{pmatrix} = A(t)\Phi(t,0), \\ {}_0J_t^{1-\alpha}x(t)\big|_{t=0}\Phi(t,0) &= \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \\ 0 & 1 \end{pmatrix}\Bigg|_{t=0} = \mathbb{1}, \end{aligned}$$

hence Lemma 3 holds true.

Suppose that

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, the solution of the initial value problem given by Equation (31) can be written down as follows:

$$x(t) = \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}.$$

4. Inhomogeneous System of Linear FDEs with Variable Coefficients Involving Riemann–Liouville Derivatives

Let us consider partial Riemann-Liouville fractional integral and derivative of order α $(0 < \alpha < 1)$ with respect to t of a function $k(t,s)$ of two variables $(t,s) \in I \times I$, $k : I \times I \to \mathbb{R}$, defined by

$$ {}^{\;t}_{t_0}J_t^{\alpha}k(t,s) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}k(\tau,s)d\tau, \tag{18}$$

$$ {}^{\;t}_{t_0}D_t^{\alpha}k(t,s) = \frac{\partial}{\partial t}{}^{\;t}_{t_0}J_t^{1-\alpha}k(t,s) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_{t_0}^{t}(t-\tau)^{-\alpha}k(\tau,s)d\tau. \tag{19}$$

The following crucial result in the theory of fractional differential equations was first obtained by Prof. Podlubny [1]. Lemma 4 provides its slightly modified version suitable for dealing with singular kernels.

Lemma 4. *Let the function $\varphi(t,s)$, $\varphi : I \times I \to \mathbb{R}$ be such that the following hypotheses are fulfilled*

- *(a) For every fixed $t \in I$, the function $\tilde{\varphi}(t,s) = {}^{\,t}_{s}J_t^{1-\alpha}\varphi(t,s)$ is measurable on I and integrable on I w.r.t. s for some $t^* \in I$.*
- *(b) The partial derivative ${}^{\,t}_{s}D_t^{\alpha}\varphi(t,s)$ exists for every interior point $(t,s) \in \mathring{I} \times \mathring{I}$.*
- *(c) There exists a non-negative integrable function g such that $\left|{}^{\,t}_{s}D_t^{\alpha}\varphi(t,s)\right| \leq g(s)$ for every interior point $(t,s) \in \mathring{I} \times \mathring{I}$.*

Then

$$ {}_{t_0}D_t^{\alpha}\int_{t_0}^{t}\varphi(t,s)ds = \int_{t_0}^{t}{}^{\,t}_{s}D_t^{\alpha}\varphi(t,s)ds + \lim_{s\to t-0}{}^{\,t}_{s}J_t^{1-\alpha}\varphi(t,s),\ t \in \mathring{I}.$$

Proof. According to Equation (9), we have

$$ {}_{t_0}D_t^\alpha \int_{t_0}^t \varphi(t,s)ds = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^t \frac{d\tau}{(t-\tau)^{-\alpha}} \int_{t_0}^\tau \varphi(t,s)ds. $$

Now applying consequently Fubini's theorem [20] and Corollary 1, in view of Equations (18) and (28), we obtain

$$ \begin{aligned} {}_{t_0}D_t^\alpha \int_{t_0}^t \varphi(t,s)ds &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^t \frac{d\tau}{(t-\tau)^\alpha} \int_{t_0}^\tau \varphi(\tau,s)ds \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^t ds \int_s^t \frac{\varphi(\tau,s)d\tau}{(t-\tau)^\alpha} \\ &= \frac{d}{dt} \int_{t_0}^t {}_s^tJ_t^{1-\alpha} \varphi(t,s)ds \\ &= \int_{t_0}^t \frac{\partial}{\partial t} {}_s^tJ_t^{1-\alpha} \varphi(t,s)ds + \lim_{s\to t-0} {}_s^tJ_t^{1-\alpha} \varphi(t,s) \\ &= \int_{t_0}^t {}_s^tD_t^\alpha \varphi(t,s)ds + \lim_{s\to t-0} {}_s^tJ_t^{1-\alpha} \varphi(t,s). \end{aligned} $$

□

Consider the inhomogeneous linear initial value problem

$$ \begin{aligned} {}_{t_0}D_t^\alpha x(t) &= A(t)x(t) + u(t),\ t \in \mathring{I}, \\ {}_{t_0}J_t^{1-\alpha} x(t)\big|_{t=t_0} &= x_0 \end{aligned} \tag{20} $$

For the sake of simplicity we assume $u : I \to \mathbb{R}^n$ to be continuous on I.

Theorem 3. *Provided that Assumption 1 is fulfilled, solution to the initial value problem given by Equation (20) can be written down as follows:*

$$ x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\tau)u(\tau)d\tau. \tag{21} $$

Proof. Let us apply the fractional differentiation operator ${}_{t_0}D_t^\alpha$ to the both sides of Equation (21). In view of Lemmas 3 and 4 one gets

$$ \begin{aligned} {}_{t_0}D_t^\alpha x(t) &= A(t)\Phi(t,t_0)x_0 + \int_{t_0}^t {}_\tau D_t^\alpha \Phi(t,\tau)u(\tau)d\tau + \lim_{\tau\to t-0} {}_\tau J_t^{1-\alpha} \Phi(t,\tau)u(\tau) \\ &= A(t)\Phi(t,t_0)x_0 + A(t)\int_{t_0}^t \Phi(t,\tau)u(\tau)d\tau + \lim_{\tau\to t-0} {}_\tau J_t^{1-\alpha} \Phi(t,\tau)u(\tau) \\ &= A(t)x(t) + \lim_{\tau\to t-0}\left[\left(\mathbb{1} + {}_\tau J_t^{1-\alpha} \sum_{k=1}^\infty {}_\tau J_t^{k\circ\alpha} A(t)\right)u(\tau)\right] \\ &= A(t)x(t) + \lim_{\tau\to t-0}\left[\left(\mathbb{1} + \sum_{k=1}^\infty {}_\tau J_t^{1-\alpha} {}_\tau J_t^\alpha \left(A(t) {}_\tau J_t^{(k-1)\circ\alpha} A(t)\right)\right)u(\tau)\right]. \end{aligned} $$

Now taking into account the semigroup property of the fractional integrals described by Equation (4), one can rewrite the latter expression as follows:

$$
\begin{aligned}
{}_{t_0}D_t^{\alpha} x(t) &= A(t)x(t) + \lim_{\tau \to t-0} \left[\left(\mathbb{1} + \sum_{k=1}^{\infty} \int_{\tau}^{t} (A(s) {}_{\tau}J_s^{(k-1)\circ\alpha} A(s)) ds \right) u(\tau) \right], \\
&= A(t)x(t) + u(t),
\end{aligned}
$$

which completes the proof. □

Example

Let us consider the following system

$$
\begin{aligned}
{}_0D_t^{\alpha} x(t) &= \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} x(t) + u(t), \\
{}_0J_t^{1-\alpha} x(t)\big|_{t=0} &= x_0.
\end{aligned} \tag{22}
$$

Suppose that

$$
x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u(t) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ t > 0.
$$

Then, in view of Equation (17) the solution of the initial value problem given by Equation (22) can be written down as follows:

$$
x(t) = \begin{pmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)} (t-\tau)^{2\alpha} \\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau.
$$

Finally, we arrive at the explicit closed-form solution

$$
\begin{aligned}
x_1(t) &= \frac{\alpha}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
x_2(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}
$$

5. Homogeneous System of Linear FDEs with Variable Coefficients Involving Caputo Derivatives

We now examine homogeneous linear FDEs with variable coefficients involving Caputo derivatives. Let us consider the following initial value problem:

$$
\begin{aligned}
{}_{t_0}D_t^{(\alpha)} x(t) &= A(t)x(t), \ t \in \mathring{I}, \\
x(t_0) &= \tilde{x}_0,
\end{aligned} \tag{23}
$$

where the matrix function $A(t)$ is continuous on I.

Definition 2. *The state-transition matrix of the system described by Equation (23) is defined as follows:*

$$
\Psi(t, t_0) = \sum_{k=0}^{\infty} {}_{t_0}\tilde{J}_t^{k\circ\alpha} A(t), \tag{24}
$$

where

$$
{}_{t_0}\tilde{J}_t^{0\circ\alpha} A(t) = \mathbb{1},
$$
$$
{}_{t_0}\tilde{J}_t^{(k+1)\circ\alpha} A(t) = {}_{t_0}J_t^{\alpha}(A(t)\,{}_{t_0}\tilde{J}_t^{k\circ\alpha} A(t)),\ k = 0, 1, \ldots
$$

Again, we will refer to the series on the right-hand side of Equation (24) as the generalized Peano–Baker series [14,16].

Assumption 2. *The generalized Peano–Baker series in the right-hand side of Equation (24) converges uniformly.*

In view of Lemma 1 and of Equations (9) and (11), the following lemma holds true.

Lemma 5. *Under Assumption 2 the state-transition matrix $\Psi(t, t_0)$ satisfies the following initial value problem*

$$
{}_{t_0}D_t^{(\alpha)} \Psi(t, t_0) = A(t)\Psi(t, t_0),\ \Psi(t_0, t_0) = \mathbb{1}.
$$

Lemma 5 implies the following

Theorem 4. *Under Assumption 2 solution to the initial value problem described by Equation (23) is given by the following expression:*

$$
x(t) = \Psi(t, t_0)\tilde{x}_0. \tag{25}
$$

Remark 2. *If $A(t)$ is a constant matrix, i.e., $A(t) \equiv A$, then in view of Equation (6) one gets*

$$
{}_{t_0}\tilde{J}_t^{k\circ\alpha} A = \frac{(t - t_0)^{k\alpha}}{\Gamma(k\alpha + 1)} A^k
$$

and

$$
\Psi(t, t_0) = E_\alpha((t - t_0)^\alpha A) = \sum_{k=0}^{\infty} \frac{A^k (t - t_0)^{\alpha k}}{\Gamma[k\alpha + 1]},
$$

where $E_\alpha(t^\alpha A) = E_{\alpha,1}(t^\alpha A)$.

Equation (25) takes on the form

$$
x(t) = E_\alpha((t - t_0)^\alpha A)\tilde{x}_0,
$$

which is consistent with the formulas, obtained for the systems of fractional differential equations with constant coefficients [10,18].

Example

Let us consider the following system

$$
\begin{gathered}
{}_0D_t^{(\alpha)} x(t) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} x(t), \\
x(0) = \tilde{x}_0.
\end{gathered} \tag{26}
$$

Direct calculation yields

$$
\Psi(t, 0) = \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 0 & 1 \end{pmatrix}. \tag{27}
$$

It can be readily seen that

$$
{}_0D_t^{(\alpha)}\Psi(t,0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = A(t)\Psi(t,0),
$$
$$
\Psi(0,0) = \mathbb{1},
$$

hence Lemma 5 holds true.

Suppose that

$$
\tilde{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

Then, the solution of the initial value problem given by Equation (27) can be written down as follows:

$$
x(t) = \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 1 \end{pmatrix}.
$$

6. Inhomogeneous System of Linear FDEs with Variable Coefficients Involving Caputo Derivatives

Let us first formulate a lemma about Caputo differentiation under the integral sign.

Let us consider partial Caputo derivative of order α $(0 < \alpha < 1)$ with respect to t of a function $k(t,s)$ of two variables $(t,s) \in I \times I$, $k : I \times I \to \mathbb{R}$, defined by

$$
{}_{t_0}^{\,t}D_t^{(\alpha)}k(t,s) = {}_{t_0}^{\,t}J_t^{1-\alpha}\frac{\partial}{\partial t}k(t,s) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\frac{\partial}{\partial \tau}k(\tau,s)d\tau. \tag{28}
$$

Lemma 6. *Let the function $\varphi(t,s)$, $\varphi : I \times I \to \mathbb{R}$ satisfy hypotheses of Lemma 4. Then*

$$
{}_{t_0}D_t^{(\alpha)}\int_{t_0}^{t}\varphi(t,s)ds = \int_{t_0}^{t}{}_{s}^{\,t}D_t^{\alpha}\varphi(t,s)ds + \lim_{s\to t-0}{}_{s}^{\,t}J_t^{1-\alpha}\varphi(t,s),\ t \in \mathring{I}.
$$

Proof. According to Equation (5), we have

$$
\begin{aligned}
{}_{t_0}D_t^{(\alpha)}\int_{t_0}^{t}\varphi(t,s)ds &= {}_{t_0}D_t^{\alpha}\int_{t_0}^{t}\varphi(t,s)ds - \int_{t_0}^{t}\varphi(t,s)ds\Big]_{t=t_0}\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\
&= {}_{t_0}D_t^{\alpha}\int_{t_0}^{t}\varphi(t,s)ds.
\end{aligned}
$$

Thus, the statement of the lemma directly follows from Lemma 4. □

Consider the inhomogeneous linear initial value problem

$$
\begin{aligned}
{}_{t_0}D_t^{(\alpha)}x(t) &= A(t)x(t) + u(t),\ t \in \mathring{I}, \\
x(t_0) &= x_0.
\end{aligned} \tag{29}
$$

Again, we assume $u : I \to \mathbb{R}^n$ to be continuous on I.

Theorem 5. *Provided that Assumption 2 is fulfilled, solution to the initial value problem given by Equation (29) can be written down as follows:*

$$
x(t) = \Psi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)u(\tau)d\tau. \tag{30}
$$

Proof. Let us apply the fractional differentiation operator ${}_{t_0}D_t^{(\alpha)}$ to both sides of Equation (30). In view of Lemmas 3, 5 and 6 one gets

$$\begin{aligned}
{}_{t_0}D_t^{(\alpha)}x(t) &= A(t)\Psi(t,t_0)x_0 + \int_{t_0}^{t} {}_{\tau}D_t^{\alpha}\Phi(t,\tau)u(\tau)d\tau + \lim_{\tau\to t-0} {}_{\tau}J_t^{1-\alpha}\Phi(t,\tau)u(\tau) \\
&= A(t)\Psi(t,t_0)x_0 + A(t)\int_{t_0}^{t}\Phi(t,\tau)u(\tau)d\tau + \lim_{\tau\to t-0} {}_{\tau}J_t^{1-\alpha}\Phi(t,\tau)u(\tau) \\
&= A(t)x(t) + \lim_{\tau\to t-0}\left[\left(\mathbb{1} + {}_{\tau}J_t^{1-\alpha}\sum_{k=1}^{\infty} {}_{\tau}J_t^{k\circ\alpha}A(t)\right)u(\tau)\right] \\
&= A(t)x(t) + \lim_{\tau\to t-0}\left[\left(\mathbb{1} + \sum_{k=1}^{\infty} {}_{\tau}J_t^{1-\alpha}\,{}_{\tau}J_t^{\alpha}\left(A(t)\,{}_{\tau}J_t^{(k-1)\circ\alpha}A(t)\right)\right)u(\tau)\right].
\end{aligned}$$

Now taking into account the semigroup property of the fractional integrals described by Equation (4), one can rewrite the latter expression as follows:

$$\begin{aligned}
{}_{t_0}D_t^{(\alpha)}x(t) &= A(t)x(t) + \lim_{\tau\to t-0}\left[\left(\mathbb{1} + \sum_{k=1}^{\infty}\int_{\tau}^{t}\left(A(s)\,{}_{\tau}J_s^{(k-1)\circ\alpha}A(s)\right)ds\right)u(\tau)\right] \\
&= A(t)x(t) + u(t),
\end{aligned}$$

which completes the proof. □

Remark 3. *It should be noted that for* $A(t) \equiv A = const$ *and* $\alpha = 1$ *one gets* $\Phi(t,t_0) = \Psi(t,t_0) = e^{A(t-t_0)}$ *and Expressions* (21) *and* (30) *yield the well-known explicit formula*

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}u(\tau)d\tau$$

for the solution of the integer-order Cauchy problem

$$\begin{aligned}
\dot{x} &= Ax + u, \\
x(t_0) &= x_0.
\end{aligned}$$

Example

Let us consider the following system

$$\begin{aligned}
{}_0D_t^{(\alpha)}x(t) &= \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} x(t) + u(t), \\
x(0) &= x_0.
\end{aligned} \tag{31}$$

Suppose that

$$x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u(t) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; t > 0.$$

Then, according to Theorem 5 and in view of Equations (17) and (27), the solution of the initial value problem given by Equation (31) can be written down as follows:

$$x(t) = \begin{pmatrix} 1 & \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} & \frac{\alpha}{\Gamma(2\alpha+1)}(t-\tau)^{2\alpha} \\ 0 & \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\tau$$
$$= \begin{pmatrix} \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ 1 \end{pmatrix}.$$

Finally, we arrive at the explicit closed-form solution

$$x_1(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$
$$x_2(t) = 1.$$

7. Conclusions

The paper deals with the linear systems of fractional differential equations with variable coefficients. The state transition matrix is represented in terms of the generalized Peano–Baker series, provided that the latter is convergent. Explicit solutions of the initial value problems in both homogeneous and inhomogeneous cases are derived. Systems with both the Riemann–Liouville- and Caputo-type operators are studied. Several examples are given to illustrate the use of the derived formulas. Future research will be devoted to applications of these results in solving control problems for systems with fractional dynamics.

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