symmetry

MDPI

Article

Asymmetric Putnam-Fuglede Theorem for (n,k)-Quasi-$*$-Paranormal Operators

Ahmed Bachir [1,*] and Abdelkader Segres [2]

1 Department of Mathematics, King Khalid University, P. O. Box 9004, Abha, Saudi Arabia; abishr@kku.edu.sa
2 Department of Mathematics, Mascara University, Mascara 29000, Algeria; segres03@yahoo.fr
* Correspondence: bachir_ahmed@hotmail.com

Received: 7 October 2018; Accepted: 29 November 2018; Published: 8 January 2019

Abstract: $T \in \mathcal{B}(\mathcal{H})$ is said to be (n,k)-quasi-$*$-paranormal operator if, for non-negative integers k and n, $\|T^*(T^kx)\|^{(1+n)} \leq \|T^{(1+n)}(T^kx)\|\|T^kx\|^n$; for all $x \in \mathcal{H}$. In this paper, the asymmetric Putnam-Fuglede theorem for the pair (A,B) of power-bounded operators is proved when (i) A and B^* are n-$*$-paranormal operators (ii) A is a (n,k)-quasi-$*$-paranormal operator with reduced kernel and B^* is n-$*$-paranormal operator. The class of (n,k)-quasi-$*$-paranormal operators properly contains the classes of n-$*$-paranormal operators, $(1,k)$-quasi-$*$-paranormal operators and k-quasi-$*$-class A operators. As a consequence, it is showed that if T is a completely non-normal (n,k)-quasi-$*$-paranormal operator for $k=0,1$ such that the defect operator D_T is Hilbert-Schmidt class, then $T \in C_{10}$.

Keywords: Putnam-Fuglede theorem; hyponormal operator; (n,k)-quasi-$*$-paranormal operator; paranormal operator; contraction; stable operator

MSC: 47B47; 47A30; 47B20

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on $\mathcal{H}$. Spectrum, point spectrum, residual spectrum, continuous spectrum, and approximate spectrum of an operator T will be denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$, $\sigma_a(T)$, respectively. The kernel and the range of an operator T will be denoted by $\ker T$ and ran(T) respectively.

For any operator $T \in \mathcal{B}(\mathcal{H})$, let $|T| = (T^*T)^{1/2}$, and consider the following standard definitions: normal if $T^*T = TT^*$ and T is hyponormal if $|T^*|^2 \leq |T|^2$ (i.e., equivalently, if $\|T^*x\| \leq \|Tx\|$ for every $x \in \mathcal{H}$).

An operator T is said to be $*$-paranormal iff $\|T^*x\|^2 \leq \|T^2x\|\|x\|$, for all $x \in \mathcal{H}$, or equivalently, $T \in \mathcal{B}(\mathcal{H})$ is $*$-paranormal iff $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$, for all $\lambda > 0$. The class of $*$-paranormal operators was introduced in [1]. Another well-known generalization of $*$-paranormal operators are (n,k)-quasi-$*$-paranormal operators defined as follows: T is said to be (n,k)-quasi-$*$-paranormal operator if

$$\|T^*(T^kx)\|^{(1+n)} \leq \|T^{(1+n)}(T^kx)\|\|T^kx\|^n$$

for all $x \in \mathcal{H}$ and for non-negative integers k and n.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be paranormal [2] iff

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \quad \text{for all } x \in \mathcal{H}.$$

The familiar Putnam-Fuglede theorem asserts that if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ are normal operators and $AX = XB$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB^*$ (see [3]). A simple example of two unilateral shifts shows that this theorem cannot be extended to the class of hyponormal operators. Let us write the Putnam-Fuglede theorem in an asymmetric form: if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$ are normal operators and $AX = XB^*$ for some $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB$.

Many authors extended this theorem for different non-normal classes of operators (see [2,4–12]). In this paper, we shall generalize this theorem to certain (n,k)-quasi-$*$-paranormal operators.

The organization of the paper is as follows; in Section 2, we give some properties for (n,k)-quasi-$*$-paranormal operators needed in the sequel. In Section 3, we present our main theorems to prove that the asymmetric Putnam-Fuglede theorem holds for some power-bounded operators A, B in the following cases:

(i) A and B^* are n-$*$-paranormal operators
(ii) A is a (n,k)-quasi-$*$-paranormal operator with reduced kernel and B^* are n-$*$-paranormal operator;
(iii) A is a $n-*$-paranormal operator and B^* are (n,k)-quasi-$*$-paranormal operator with reduced kernel (an operator T with reduced kernel means that its kernel is invariant under T^*).

These results extend those recently given in [9,13,14] and as applications of our main theorems, we obtain the following:

1. if T is a (n,k)-quasi-$*$-paranormal operator with reduced kernel (resp. n-$*$-paranormal operator or a n- quasi-$*$-class A with reduced kernel), then T has a part in the class C_{00} on a stable subspace $\mathcal{H}_0$ and a compression quasi-affine transform to an isometry on the orthogonal complement of $\mathcal{H}_0$.
2. Next, we prove that if T is completely non-normal (n,k)-quasi-$*$-paranormal operator; for $k = 0,1$ and verifying the defect operator D_T is a Hilbert-Schmidt class, then $T \in C_{10}$.

This generalizes the results given by Takahashi and Uchiyama [15] for completely non-normal hyponormal contraction operators and those given by Duggal, Jeonb, Kim [13] for the case of completely non-normal $*$-paranormal contraction operators.

Let us recall some facts about the construction of the limit isometric operator or the g-asymptotic limit associated with a power-bounded operator T (see [16]).

Definition 1. *A Banach limit or a generalized limit is a bounded linear functional $glim$ on $l^\infty(\mathbb{N})$ (the Banach space of bounded complex sequences) which preserves the ordinary notion of convergence. That is if $lim x_n = x$ then $glim(x_n) = x$.*

Banach limit may be characterized as those continuous functional which satisfy the following conditions:

- $glim$ is positive, i.e., if $x_n \geq 0$ for all $n \in \mathbb{N}$ then $glim(x_n) \geq 0$;
- $glim(1) = 1$, where $(1) = (1,1,1,....)$;
- $glim$ is shift-invariant, i.e., $glim(x_n) = glim(x_{n+1})$.

(see [17]) for further details.

In the sequel we fix a generalized Banach limit $glim$ on $l^\infty(\mathbb{N})$ for a power-bounded operator T; $sup_n \|T^n x\| \leq \infty$, on the Hilbert space $\mathcal{H}$. The following map is a bounded sesquilinear form

$$\phi_T(x,y) = glim_n \langle T^n x, T^n y \rangle; \quad x, y \in \mathcal{H} \tag{1}$$

Since $\{\|T^n\| : n \geq 1\}$ is bounded, then $glim\|T^n x\| = 0$ if and only if $inf_n \|T^n x\| = 0$ and so, this holds if and only if $lim_n \|T^n x\| = 0$.

We denote by $\mathcal{H}_0$ the kernel of ϕ_T, i.e.,

$$\mathcal{H}_0 = \{x \in \mathcal{H} : lim_n \|T^n x\| = 0\} \tag{2}$$

$\mathcal{H}_0$ is said the stable subspace for T. It is clear that $\mathcal{H}_0$ is an invariant subspace for any operator in the commutant of T, i.e., $\mathcal{H}_0$ is an hyperinvariant subspace. We recall the following definitions:

(i) A power-bounded operator T is said to be of class $C_{1.}$ if the sequence $\{\|T^n x\| : n \in N\}$ does not converge to 0 for any non-zero vector x i.e., $\mathcal{H}_0(T) = \{0\}$.
(ii) T is said to be strongly stable if $\mathcal{H}_0(T) = \mathcal{H}$ and we write $T \in C_{0.}$;
(iii) T is of class $C_{.j} : j = 0, 1$ if T^* is of class $C_{j.}; j = 0, 1$;
(iv) T is of class $C_{ij} : i, j = 0, 1$ if $T \in C_{i.} \cap C_{.j}$.

It follows from Equation (1) of the sesquilinear application ϕ_T that there exists a positive operator $A_{T,g} \in \mathcal{B}(\mathcal{H})$ such that the equation $\phi_T(x, y) = \langle A_{T,g} x, y\rangle$ holds for all vectors $x, y \in \mathcal{H}$. The operator $A_{T,g}$ is said the g-asymptotic limit of T which is usually depends on the particular choice of the generalized limit g. It is well known that ker $A_{T,g} = \mathcal{H}_0$ holds for every Banach limit g and

$$glim_n \|T^n x\|^2 = \|A^{\frac{1}{2}}_{T,g} x\|^2 = \|A^{\frac{1}{2}}_{T,g} T x\|^2 \tag{3}$$

Furthermore, there exists an isometry V on $\overline{\operatorname{ran}(A)}$ such that

$$VA^{\frac{1}{2}} = A^{\frac{1}{2}} T. \tag{4}$$

The concept of asymptotic limit and their generalizations play an important role in the hyperinvariant subspace problem [16,18]. Since T^* is a power-bounded operator whenever T is, let A_* be the strong limit of $\{T^n T^{*n} : n \geq 1\}$ and let V^* be the associated isometry on $\overline{\operatorname{ran}(A^*)}$ so that all the preceding properties hold for T^*, A^*, V^*.

Definition 2. *Let $T \in \mathcal{B}(\mathcal{H})$, then*

(i) the joint point spectrum, denoted by $\sigma_{jp}(T)$ is the set

$$\sigma_{jp}(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ and } T^* x = \overline{\lambda} x\}.$$

(ii) the joint approximate point spectrum, denoted by $\sigma_{ja}(T)$ is the set of scalars λ for which there exists a normalized sequence $\{x_n\} \subset \mathcal{H}$ verifying

$$(T - \lambda) x_n \to 0 \text{ and } (T - \lambda)^* x_n \to 0.$$

Notice that in general, $\sigma_{jp}(T) \subset \sigma_p(T)$; however, the equality holds for the following operator classes: p-hyponormal or log-hyponormal, absolute-$*$-k-paranormal.

Definition 3. *$T \in \mathcal{B}(\mathcal{H})$ is said to be (n, k)-quasi-$*$-paranormal operators if, for non-negative integers k and n,*

$$\|T^*(T^k x)\|^{(1+n)} \leq \|T^{(1+n)}(T^k x)\| \|T^k x\|^n \text{ for all } x \in \mathcal{H}. \tag{5}$$

If $k = 0$, it is clear that T is n-$*$-paranormal operator [8] and if $n = 1$, then T is $*$-paranormal [1]. Also, if $n = 0$, T is k quasi-hyponormal [8] and if $n = 1$, T is k-quasi-$*$-hyponormal operator [19].

2. Properties of (n, k)-Quasi-$*$-Hyponormal Operators

Lemma 1. *[19] If $T \in \mathcal{B}(\mathcal{H})$, then T is (n, k)-quasi-$*$-paranormal operator if and only if*

$$T^{*k} T^{*(n+1)} T^{(n+1)} T^k - (n+1) t^n T^{*k} T T^* T^k + n t^{(n+1)} T^{*k} T^k \geq 0 \tag{6}$$

for all $t > 0$.

Lemma 2. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is a normal operator, then*

$$\|Tx\|^n \leq \|T^{(n)}x\|\|x\|^n \tag{7}$$

for all $x \in \mathcal{H}$ and non-negative integer n

Proof. We recall from [20] that if A is a positive operator on Hilbert space then

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \tag{8}$$

for all $r > 1$ and any unit vector x.

Let T be a normal operator and $n \geq 1$, then

$$\|Tx\|^{2n} = \langle T^*Tx, x\rangle^n \tag{9}$$

By the above inequality

$$\|Tx\|^{2n} \leq \langle (T^*T)^n x, x\rangle = \langle T^{*n}T^n x, x\rangle = \|T^n x\|^2 \tag{10}$$

for all $n \geq 1$ and unit vector x. Hence, for $x = y/\|y\|, y \neq 0$, we get our result. □

Lemma 3. *Let T be (n,k)-quasi-$*$-paranormal operator. If $\overline{ran\, T^k} \neq \mathcal{H}$, then T has the following decomposition:*

$$T = \begin{bmatrix} T_1 & S \\ 0 & T_2 \end{bmatrix}.$$

on $\mathcal{H} = \overline{ran\, T^k} \oplus (ran\, T^k)^\perp$, where T_1 is n-$*$-paranormal operator, $T_2^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. If $ran\, T^k \neq \mathcal{H}$, then $\mathcal{H}$ has the following non-trivial decomposition: $\mathcal{H} = \overline{ran\, T^k} \oplus (ran\, T^k)^\perp$. Also, it is clear that $\overline{ran\, T^k}$ is an invariant subspace for T such that $T_1 = T|_{\overline{ran\, T^k}}$ is n-$*$-paranormal operator and $T_2^k = 0$; where $T_2 = T_{|(ran\, T^k)^\perp}$. Hence, T has the triangular matrix form cited above. □

Proposition 1. *Let $T \in \mathcal{B}(\mathcal{H})$.*

1. *If T is a power-bounded n-$*$-paranormal operator and there is an invariant subspace $\mathcal{M}$ for which the restriction $T|_\mathcal{M} = N$ of T on $\mathcal{M}$ is a normal operator, then $\mathcal{M}$ reduces T and $N = U \oplus 0$ where U is unitary.*
2. *If T is a power-bounded (n,k)-quasi-$*$-paranormal operator and there is an invariant subspace $\mathcal{M}$ for which the restriction $T|_\mathcal{M} = N$ of T on $\mathcal{M}$ is an injective normal operator then $\mathcal{M}$ reduces T and N is a unitary operator. In particular, if $\mathcal{M} = \overline{RanT^k}$ and T_1 as in the previous Lemma, is normal operator, then $\overline{RanT^k}$ reduces T.*

Proof.

1. Let T be a power-bounded n-$*$-paranormal operator and let us consider an invariant subspace $\mathcal{M} \subset \mathcal{H}$ for T such that $T|_\mathcal{M} = N$ is normal. The operator T has the following matrix decomposition

$$T = \begin{bmatrix} N & R \\ 0 & * \end{bmatrix}$$

On the one hand, we have that N is a power-bounded normal operator, since N is normaloid it follows that N is a contraction. It is well known that $N = U \oplus N_0$ where U is unitary and N_0

is of class C_{00} for possible $N_0 = 0$. On the other hand, Since the operator T is n-$*$-paranormal, it follows that

$$\|R^*x\|^2 + \|Nx\|^2 = \|T^*x\|^2 \leq \|T^n x\|^{\frac{2}{n}} = \|N^n x\|^{\frac{2}{n}}, \tag{11}$$

for all unit vector $x \in \mathcal{M}$.

Since the kernel of N reduces N, hence $N = N_1 \oplus 0$.

If $x \in kerN$ then from (7) we get $x \in kerR^*$. Thus

$$R^* = 0 \quad on \quad kerN \tag{12}$$

For each unit vector $x \in \mathcal{M} \ominus kerN$ we have, $(\|R^*x\|^2 + 1)^n \leq \frac{1}{\|Nx\|^{2n}}\|N^n x\|^2$

From Lemma 1, we get for all $k \geq 1$, that

$$(\|R^*x\|^2 + 1)^{nk} \leq \frac{1}{\|Nx\|^{2nk}}\|N^n x\|^{2k} \leq \frac{1}{\|Nx\|^{2nk}}\|N^{nk} x\|^2 \tag{13}$$

If $N_0 \neq 0$ then

$$(\|R^*x\|^2 + 1)^{nk} \leq \frac{1}{(\|Ux\|^2 + \|N_0 x\|^2)^{nk}}(\|U^{nk}x\|^2 + \|N_0^{nk}x\|^2) \tag{14}$$

Since U is unitary and $\|x\| = 1$ then

$$(\|R^*x\|^2 + 1)^{nk} \leq \frac{1}{(1 + \|N_0 x\|^2)^{nk}}(1 + \|N_0^{nk}x\|^2) \tag{15}$$

Since $N_0 \in C_{00}$ and $1 + \|N_0 x\|^2 > 1$ then for $k \to \infty$ we get $(\|R^*x\|^2 + 1)^{nk} \to 0$. However, $\|R^*x\|^2 + 1 \geq 1$ which is a contradiction. Hence, $N_0 = 0$ and then $N = U$ on $\mathcal{M} \ominus kerN$.

By substituting $N_0 = 0$ and $k = 1$ in the inequality (11), we get $(\|R^*x\|^2 + 1)^n = 1$. Therefore, $R^*x = 0$ on $\mathcal{M} \ominus kerN$ and by (8), we have $R^* = 0 = R$.

2. As in the case (1), let us consider an invariant subspace $\mathcal{M} \subset \mathcal{H}$ for T such that $T|_{\mathcal{M}} = N$ is normal. The operator T has the following matrix

$$T = \begin{bmatrix} N & R \\ 0 & * \end{bmatrix},$$

on the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, where N is an injective normal operator and then $\overline{ranN^k} = \mathcal{M}$. Hence, $\mathcal{M} \subseteq \overline{ranN^k}$. Indeed, we have $T^k x = N^k x$ for all $x \in \mathcal{M}$. Since N is normal then $ranN^k = ranN$ and by the assumption that N is injective, we get $\overline{ranN^k} = \mathcal{M}$.

Since T is (n,k)-quasi-$*$-paranormal operator, then

$$\|T^*(T^k x)\|^2 \leq \|T^{(n+1)}T^k x\|^{\frac{2}{(n+1)}}\|T^k x\|^{\frac{2n}{(n+1)}} \tag{16}$$

Put $y = T^k x = N^k x \in \mathcal{M} = \overline{ranN^k}$ for all $x \in \mathcal{M}$, we get,

$$(\|R^*y\|^2 + \|Ny\|^2)^{n+1} \leq \|N^{n+1}y\|^2\|y\|^{2n}, \tag{17}$$

Since N is a contraction and $y = N^k x$ we get $\|y\|^{2n} \leq 1$. Hence,

$$(\|R^*y\|^2 + \|Ny\|^2)^{n+1} \leq \|N^{n+1}y\|^2, \tag{18}$$

for all $y \in \mathcal{M} = \overline{ranN^k}$ we get the desired result, by following the same steps as in the proof of the previous assertion (1).
□

Lemma 4. *If $T \in \mathcal{B}(\mathcal{H})$ is (n,k)-quasi-$*$-paranormal operator, then T° is (n,k)-quasi-$*$-paranormal operator.*

Proof. The proof is a consequence of the Lemma 1 and the properties of the isometric *-homomorphism ϕ of the Berberian technique. □

Corollary 1. *Let T be a power-bounded (n,k)-quasi-$*$-paranormal operator.*

1. *If $k = 0$ (i.e., T is n-$*$-paranormal) and $\lambda \in \sigma_p(T)$, then $\ker(T-\lambda)$ reduces T. Also, if $k > 0$ and $\lambda \in \sigma_p(T) - \{0\}$, then $\ker(T-\lambda)$ reduces T.*
2. *If T is n-$*$-paranormal and $Tx = \lambda x$ such that $x \neq 0$, then $T^*x = \overline{\lambda}x$ and $\sigma_p(T) = \sigma_{jp}(T)$. The same result holds in case $k > 0$ and $\lambda \neq 0$ and $\sigma_p(T) - \{0\} = \sigma_{jp}(T) - \{0\}$.*
3. *If $\lambda \neq \mu$, then $\ker\,(T-\lambda) \perp \ker\,(T-\mu)$.*
4. *$T = N \oplus A$ on the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, where $\mathcal{M}$ is the subspace spanned by the eigenspaces of T, N is a normal operator and A is a power-bounded (n,k)-quasi-$*$-paranormal operator with $\sigma_r(A^*) = \emptyset$.*

   Moreover,
$$\sigma(A^*) \subseteq \sigma_p(A^*) \cup \sigma_c(A^*) \subseteq \sigma_a(A^*).$$
5. *If T is n-$*$-paranormal, then $\sigma_a(T) = \sigma_{ja}(T)$; also, and if $k > 0$, then $\sigma_a(T) - \{0\} = \sigma_{ja}(T) - \{0\}$.*

Proof.

1. The result follows immediately from Proposition 1 by taking $\mathcal{M} = ker(T-\lambda)$ and $N = \lambda I$ which is normal.
2. It follows from item (1).
3. If $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$, then
$$\lambda\langle x,y\rangle = \langle Tx,y\rangle = \langle x,T^*y\rangle = \langle x,\overline{\mu}y\rangle = \mu\langle x,y\rangle$$
   implies $\langle x,y\rangle = 0$.
4. From items (1), (2), (3) and according to the decomposition $\mathcal{H} = \mathcal{M} \oplus (\mathcal{M})^{\perp}$, where $\mathcal{M}$ is the subspace spanned by the eigenspaces, the operator T can be written
$$T = \begin{bmatrix} N & 0 \\ 0 & A \end{bmatrix},$$
   where $N = T|_{\mathcal{M}}$ is a normal operator and A is a (n,k)-quasi-$*$-paranormal operator. Since $\sigma_p(A) = \emptyset$ it yields $\ker(A-\lambda) = \{0\}$, for all $\lambda \in \mathbb{C}$ and so
$$\ker(A-\lambda)^{\perp} = \{0\}^{\perp} \text{ i.e., } \overline{\operatorname{ran}(A^* - \overline{\lambda})} = \mathcal{K},$$
   where $\mathcal{K}$ is the initial space of A, i.e., $\mathcal{K} = \left(\oplus_{\lambda_i \in \sigma_p(T)} ker(T-\lambda_i)\right)^{\perp}$.
   Therefore, the residual spectrum of A is empty. From the decomposition of the spectrum, we get
$$\sigma(A^*) = \sigma_p(A^*) \cup \sigma_c(A^*) \subseteq \sigma_a(A^*).$$
5. the last statement follows from Lemma 1 and the assertion (1).

□

Lemma 5. *If T is (n,k)-quasi-$*$-paranormal and $\mathcal{M}$ is an invariant subspace for T, Then $T|_{\mathcal{M}}$ is (n,k)-quasi-$*$-paranormal.*

Proof. According to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, then T can be written

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

where $A = T|_{\mathcal{M}}$. Since T is (n,k)-quasi-$*$-paranormal, then

$$\|T^*(T^k x)\|^2 \leq \|T^{(n+1)} T^k x\|^{\frac{2}{(n+1)}} \|T^k x\|^{\frac{2n}{(n+1)}} \tag{19}$$

and for all $x \in \mathcal{M}$ we have that $T^k x = A^k x$ and

$$\begin{aligned} \|A^*(A^k x)\|^2 &\leq \|T^*(T^k x)\|^2 \\ &\leq \|T^{(n+1)} T^k x\|^{\frac{2}{(n+1)}} \|T^k x\|^{\frac{2n}{(n+1)}} \quad \text{(from inequality (19))} \\ &= \|A^{(n+1)} A^k x\|^{\frac{2}{(n+1)}} \|A^k x\|^{\frac{2n}{(n+1)}} \end{aligned}$$

Hence, A is (n,k)-quasi-$*$-paranormal. □

3. Main Theorems

We are ready to show our main theorems.

Definition 4. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the (PF) property if $TX = XV^*$ for any operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and any isometry $V \in \mathcal{B}(\mathcal{K})$ implies $T^*X = XV$.*

Lemma 6. *[21] Let $A \in \mathcal{B}(\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent:*

1. *A, B satisfy Fuglede-Putnam theorem;*
2. *if $AX = XB$ for any operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $\overline{ran(X)}$ reduces A, $(kerX)^{\perp}$ reduces B and $A|_{\overline{ran(X)}}$, $B|_{(kerX)^{\perp}}$ are unitarily equivalent normal operators.*

The following result was given by Duggal-Kubrusl [22] in the contractive case and by Pagacz [9] in the general case but our proof seems more direct, simpler and gives more explicit decomposition than Pagacz's proof.

Proposition 2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a power-bounded operator. A has the PF property if and only if $A = U \oplus C$ where U is unitary and C is of class $C_{.0}$.*

Proof. Since T is a power-bounded operator then there is a g-asymptotic limit A_* associated with the operator T^* which is a positive operator and has the form $A_* = 0 \oplus A_1$ on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}$ where $\mathcal{H}_0 = kerA_*$ is the stable subspace of T^*.

Furthermore, there exists an isometry V on $\overline{\mathrm{ran}A_*} = \mathcal{H}_0{}^{\perp}$ (the asymptotic isometry associated with A^*), satisfy Equation (4), i.e., $VX = XT^*$, where $Xh = R^{\frac{1}{2}}h$, for all $h \in \mathcal{H}$. Hence,

$$TX = XV^* \tag{20}$$

It follows from the previous Lemma, for $T = A, B = V^*$, that if T has the PF property (i.e., T, V^* satisfy Fuglede-Putnam theorem) then $\overline{\mathrm{ran}X} = \mathcal{H}_0{}^{\perp}$ reduces T and $T|_{\overline{\mathrm{ran}(X)}}$, V^* are unitarily equivalent normal operators (we have $(kerX)^{\perp} = \mathcal{H}_0{}^{\perp}$). Which means that $T|_{\overline{\mathrm{ran}(X)}}$ is a unitary operator. Since $\mathcal{H}_0$ is the stable subspace of T^* and $T^*|_{\mathcal{H}_0}$ is of class $C_{.0}$ then $T|_{\mathcal{H}_0}$ is of class $C_{.0}$. The reverse implications it follows immediately from the previous Lemma. □

Proposition 3. *(P. Pagacz [9]) Every power-bounded n-$*$-paranormal operator has the PF property.*

Theorem 1. *Let $A \in \mathcal{B}(\mathcal{K}), B \in \mathcal{B}(\mathcal{H})$ be power-bounded n-$*$-paranormal operators. If $AX = XB^*$ for any $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $A^*X = XB$.*

Proof. Since B is n-$*$-paranormal operator then by the Propositions 2 and 3, $B = U \oplus B_0$ on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}$ where $\mathcal{H}_0$ is the stable subspace of B^*. Setting $X = [X_1, X_2] \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}, K)$.

It follows from $AX = XB^*$ that

$$AX_1 = X_1 B_0^* \tag{21}$$

$$AX_2 = X_2 U^* \tag{22}$$

Since A is n-$*$-paranormal operator and U is unitary then by the Propositions 3, we get

$$A^* X_2 = X_2 U \tag{23}$$

We have $\overline{\operatorname{ran}(X_1)}$ is invariant for A and $\ker X_1$ is invariant for B_0^*. Hence, the operators A, X_1 and B_0 can be written:

$$A = \begin{bmatrix} A_1 & S \\ 0 & A_2 \end{bmatrix}, \; X_1 = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$B_0 = \begin{bmatrix} B_{01} & R \\ 0 & B_{02} \end{bmatrix}$$

From Lemma 5, A_1 is a (power) n-$*$-paranormal operator, B_{01}^* is of class $C_{0.}$ From the previous decompositions and Equation (21), we get $A_1 Y = Y B_{01}^*$, where Y is an injective operator with dense range.

Thus,

$$A_1^n Y h = Y B_{01}^{*n} h$$

Hence, $\|A_1^n Y h\| = \|Y B_{01}^{*n} h\| \leq \|Y\| \|B_{01}^{*n} h\| \to 0$ (strongly). Since A_1 is a (power) n-$*$-paranormal operator then, by Propositions 2 and 3, we deduce that A_1 is not of class $C_{0.}$. Hence, $Y = 0$. Therefore, $X_1 = 0$. Thus, from Equation (23), we get

$$A^* X = [0, A^* X_2] = [0, X_2 U] = [0, X_2](B_0 \oplus U) = XB. \tag{24}$$

□

Theorem 2. *Let $\mathcal{A}$ be power-bounded (n,k)-quasi-$*$-paranormal operator with reduced kernel and $\mathcal{B}$ be power-bounded n-$*$-paranormal operator. If $\mathcal{A}\mathcal{X} = \mathcal{X}\mathcal{B}^*$ for some $\mathcal{X} \in \mathcal{B}(\mathcal{H})$, then $\mathcal{A}^*\mathcal{X} = \mathcal{X}\mathcal{B}$ holds for all non-negative integers n and $k > 0$.*

Proof. If $\sigma_p(\mathcal{A}) - \{0\} \neq \emptyset \neq \sigma_p(\mathcal{B}) - \{0\}$ and $\mathcal{A}$ is reduced by its kernel, then by Corollary 1, we can write the operators $\mathcal{A}, \mathcal{B}$ as follows

$$\mathcal{A} = \begin{bmatrix} N & 0 \\ 0 & A \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} M & 0 \\ 0 & B \end{bmatrix}$$

according to the decomposition.

$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp} = \mathcal{K} \oplus \mathcal{K}^{\perp}$, where N, M are normal operators and $\mathcal{M}$, $\mathcal{K}$ are the subspaces spanned by the eigenspaces of $\mathcal{A}$ and $\mathcal{B}$ respectively, with $\sigma_r(\mathcal{A}^*) = \sigma_r(\mathcal{B}^*) = \emptyset$.
Moreover, if $\mathcal{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X \end{bmatrix}$, then from $\mathcal{A}\mathcal{X} = \mathcal{X}\mathcal{B}^*$ it follows that

$$\begin{cases} NX_1 = X_1 M^* \\ NX_2 = X_2 B^* \\ AX_3 = X_3 M^* \\ AX = XB^* \end{cases}$$

To prove the adjoint version of this system it is enough to prove the earlier equation because the first three equations are particular cases of it. Instead consider the following decomposition:

$$\begin{array}{rcl} \mathcal{H}_1 & = & \overline{\operatorname{ran}(X)} \oplus \overline{\operatorname{ran}(X)}^{\perp} \quad \text{with} \quad \mathcal{H}_1 = \mathcal{M}^{\perp} \\ \mathcal{H}_2 & = & \overline{\operatorname{ran}(X^*)} \oplus \ker X \ \text{ with } \mathcal{H}_2 = \mathcal{K}^{\perp} \end{array}$$

From the equation $AX = XB^*$ we deduce that $\overline{\operatorname{ran}(X)}$ is invariant for A and $\ker X$ is invariant for B^*. Hence, the operators A, X and B can be written:

$$A = \begin{bmatrix} A_1 & S \\ 0 & A_2 \end{bmatrix} \in B(\mathcal{H}_1),\ X = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \in B\left(\mathcal{H}_2, \mathcal{H}_1\right)$$

and

$$B = \begin{bmatrix} B_1 & R \\ 0 & B_2 \end{bmatrix} \in B(\mathcal{H}_2).$$

From Lemma 5, A_1 is a (power) (n,k)-quasi-$*$-paranormal operator, B_1 is a (power-bounded) n-$*$-paranormal operators and from Corollary 1, $\sigma_r\left(A_1^*\right) = \sigma_r\left(B_1^*\right) = \emptyset$.

Also, $AX = XB^*$ implies $A_1 Y = Y B_1^*$ where Y is injective with dense range.

From Lemma 5, A_1 and B_1 have the following matrices decompositions:

$$A_1 = \begin{bmatrix} A_{11} & R \\ 0 & A_{12} \end{bmatrix} \text{ and } B_1 = B_{11} \oplus 0$$

according to the decomposition

$$\overline{\operatorname{ran}(X)} = \overline{\operatorname{ran}(A_1^k)} \oplus \overline{\operatorname{ran}(A_1^k)}^{\perp}$$

and

$$\overline{\operatorname{ran}(X^*)} = \overline{\operatorname{ran}(B_1^k)} \oplus \overline{\operatorname{ran}(B_1^k)}^{\perp}$$

where A_{11}, B_{11}^* are power-bounded n-$*$-paranormal operators and $A_{22}^k = 0$.

It is clear that $A_1 Y = Y B_1^*$ implies that $A_1^k Y = Y B_1^{*k}$ for any positive integer k and therefore $\overline{Y(\overline{\operatorname{ran}(B^k)})} = \overline{\operatorname{ran} A_1^k}$. So Y has the following matrix

$$Y = \begin{bmatrix} Y_1 & D \\ 0 & Y_2 \end{bmatrix}$$

where $Y_1 : \overline{ran(B_1^k)} \to \overline{ran(A_1^k)}$ is injective with dense range. Also, it follows from the equation $A_1 Y = Y B_1^*$ that $A_{11}^k Y_1 = Y_1 B_{11}^{*k}$. Since A_{11} and B_{11}^* are power-bounded n-$*$-paranormal operators, then from Theorem 1 we have $A_{11}^{*k} Y_1 = Y_1 B_{11}^k$, and because of Y is injective with dense range, we get A_{11} is an injective normal operator unitary equivalent to B_{11}. From Proposition 1, we get that $\overline{ran(A_1^k)}$ reduces A_1. Hence $R = 0$ and from $A_1 Y = Y B_1^*$ it follows that $Y_2^* A_{12}^* = 0$. Since Y has dense range then $A_{12} = 0$. Therefore $A_1 = A_{11} \oplus 0$ which is a (n)-$*$-paranormal operator. Finally, we deduce that $A_1^* Y = YB_1$ and then $A^* X = XB$ and the proof is complete. $\square$

Remark 1. *By the same method, we can prove the dual version of Theorem 2. Indeed, let $\mathcal{A}$ be a power-bounded n-$*$-paranormal operator and $\mathcal{B}$ be a power-bounded (n,k)-quasi-$*$-paranormal operator with reduced kernel. If $\mathcal{A}\mathcal{X} = \mathcal{X}\mathcal{B}^*$ for some $\mathcal{X} \in \mathcal{B}(\mathcal{H})$, then $\mathcal{A}^*\mathcal{X} = \mathcal{X}\mathcal{B}$ holds for all non-negative integers n and $k > 0$.*

Definition 5. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be*

(i) *k-quasi-$*$-class A if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$ for non-negative integer k;*
(ii) *(n,k)-quasi-paranormal operator if*

$$\|T(T^k x)\|^{(1+n)} \leq \|T^{(1+n)}(T^k x)\| \|T^k x\|^n$$

for all $x \in \mathcal{H}$ and for non-negative integers n, k.

Lemma 7. *We have the following proper inclusions:*

(i) *(k-quasi-$*$-class A) $\subset$ (k-quasi-$*$-paranormal);*
(ii) *the class (n,k)-quasi-$*$-paranormal operator is normaloid, for $k = 0, 1$.*

Proof. For (i) see [19].

We give a proof of (ii) which seems direct and simpler than given in [19] Istratescu and Istratescu [23] have proved that n-paranormal operators are normaloid. Thus, for proving (ii) it suffices to show that the class (n,k)-quasi-$*$-paranormal operators; for $k = 0, 1$ is a subset of n-paranormal one.

$$\begin{aligned}
\|T(Tx)\|^{2(n+1)} &= \langle T^2x, T^2x\rangle^{n+1} \\
&= \langle T^*T^2x, Tx\rangle^{n+1} \\
&\leq \|T^*T^2x\|^{n+1}\|Tx\|^{n+1} \\
&\leq |T^{n+1}T^2x\|\|T^2\|^n\|Tx\|^{n+1} \quad (T \text{ is } (n,1)\text{-quasi-*-paranormal}) \\
&= |T^{n+2}Tx\|\|T^2\|^n\|Tx\|^{n+1}
\end{aligned}$$

for all $x \in \mathcal{H}$.

Hence $(n, 1)$-quasi-*-paranormal $\subset$ $(n + 1)$-paranormal.

The case $k = 0$ is similar. $\square$

As a consequence, we get

Corollary 2. *The asymmetric Fuglede-Putnam theorem holds for the pair of power-bounded operators $(\mathcal{A}, \mathcal{B})$ in each of the following cases:*

1. *$\mathcal{A}$ is k-quasi-$*$-class A operator with reduced kernel and $\mathcal{B}$ is n-$*$-paranormal operator;*
2. *$\mathcal{A}$ is n-$*$-paranormal operator and $\mathcal{B}$ is k-quasi-$*$-class A operator with reduced kernel;*
3. *$\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ are k-quasi-$*$-class A operators with 0 not in their approximate spectrum.*

As an application of Theorems 1, 2, Corollary 2 and Pagacz's Theorem [9], we get the following:

Corollary 3. *Let T be a power-bounded operator, then T has the Wold-type decomposition (i.e., T is a direct sum of a unitary operator and an operator of class $C_{.0}$) in each of the following cases:*

(i) *n-$*$-paranormal operator;*
(ii) *(n,k)-quasi-$*$-paranormal operator with reduced kernel;*
(iii) *k-quasi-$*$-class A operator with reduced kernel.*

We note that (i) was proved by Duggal in case $n = 1$ [13] and extended by Pagacz for $n \geq 1$ [9]. The result (iii) generalizes that of Hoxha and Braha [24] which was proved in the contraction operator case.

4. Application

Definition 6. *A non-zero transform $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is said to be a quasi-invertible if it is injective and has dense range. $T \in \mathcal{B}(\mathcal{H})$ is said to be a quasi-affine transform of $R \in \mathcal{B}(\mathcal{K})$ if there exists a quasi-invertible $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ intertwining R to T, i.e., $TX = XR$.*

Proposition 4. *If a power-bounded operator is of class $C_{1.}$, then it is a quasi-affine transform of an isometry.*

Proof. If T is a power-bounded operator on $\mathcal{H}$ of class $C_{1.}$, then it follows by the above remarks that $\ker A_{T,g} = \ker A_{T,g}^{\frac{1}{2}} = \mathcal{H}_0 = \{0\}$. Since A is a positive operator, then

$$\overline{\operatorname{ran}(A_{T,g})} = \overline{\operatorname{ran}(A_{T,g}^{\frac{1}{2}})} = \mathcal{H}.$$

From Equation (3), T is a quasi-affine transform of an isometry V on $\overline{\operatorname{ran}(A_{T,g})}$. □

We note here that the previous was given by Duggal, Kubrusly [13] in the contractive case.

We give the Kerchy's Lemma [16] which was first proven by Sz-Nagy and Foias [18] for contractions and by Kerchy for power-bounded operators.

Lemma 8. *(Kerchy) If T is a power-bounded operator on $\mathcal{H}$, then T has the following matrix form:*

$$T = \begin{bmatrix} T_0 & D \\ 0 & T_1 \end{bmatrix}$$

on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus {\mathcal{H}_0}^{\perp}$, where $\mathcal{H}_0$ is the stable subspace of T, $T_0 \in C_{0.}$ and $T_1 \in C_{1.}$.

Remark 2. *Since the spectral radius of a power-bounded operators is not greater then 1, then the power-bounded normaloid operators are contractions. Hence, by the Lemma 7, (n, k)-quasi-$*$-paranormal operators (in particular k-quasi-$*$-class A operator if $k = 0, 1$) and k-paranormal operators are contractions.*

A contraction T on a separable Hilbert space $\mathcal{H}$ is said to be a completely non-unitary if it has no non-trivial unitary direct summand. T is said to be of class C_0, written $T \in C_0$ if $\psi(f) = f(T) = 0$; for some non-zero function f, where ψ is a weak*weak continuous homomorphism from the Hardy space $\mathcal{H}^{\infty}(D)$ on the open unit disc D to the weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T, that is an extension of the usual functional calculus. This is the $\mathcal{H}^{\infty}$-functional calculus developed by Sz-Nagy and Foias [18]. It is well known that each contraction of class C_0 is of class C_{00} and the converse is given by Takahashi and Uchiyama (Theorem 1, [15]), under the assumption that the defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ is of Hilbert-Schmidt class.

As a consequence of our main results, we have that if T is a power-bounded and completely non-unitary n-$*$-paranormal operator (resp. be a (n, k)-quasi-$*$-paranormal operator or a k-quasi-$*$-class A operator with reduced kernels), then T has part (its restriction on the invariant subspace $\mathcal{H}_0$) in C_0 and its compression on ${\mathcal{H}_0}^{\perp}$ is quasi-affine transform of an isometry.

Proposition 5. *Let T be a power-bounded and completely non-unitary n-$*$-paranormal operator (resp. be a (n, k)-quasi-$*$-paranormal operator or a k-quasi-$*$-class A operator with reduced kernels). Then T has the following triangular matrix*

$$T = \begin{bmatrix} T_0 & D \\ 0 & T_1 \end{bmatrix} \tag{25}$$

on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}$, where $\mathcal{H}_0$ is the stable subspace of T and

(i) $T_0 \in C_{00}$;
(ii) $T_1 \in C_{10}$;
(iii) *T_1 is quasi-affine transform of an isometry.*

Proof. Since T is completely non-unitary, then it follows from Corollary 3 that T is of class $C_{.0}$. Since the $C_{.0}$ property is invariant under the restriction to an invariant subspace, therefore by the Kerchy's Lemma, we get the desired triangular matrix form (25) of T on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}$ and the assertions (i) and (ii) follow immediately. (iii) follows from Kerchy's Lemma and Proposition 4. □

Proposition 6. *Let T be a power-bounded and completely non-unitary n-$*$-paranormal operator (resp. be a (n,k)-quasi-$*$-paranormal operator or a k-quasi-$*$-class A operator with reduced kernels). If T is a contraction with the above matrix form (25) such that the defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ is of Hilbert-Schmidt class. Then, $T_0 \in C_0$ and $\sigma_p(T)$ is at most countable.*

Furthermore, the following assertions are equivalent:

(i) $T \in C_{0.}$;
(ii) $T \in C_0$;
(iii) $ind(T) = 0$.

Proof. Since T_0 is a contraction such that the defect operator D_{T_0} is of Hilbert-Schmidt class, i.e., $tr(I - T_0^*T_0) < \infty$, by Theorem 1 in [15], T_0 is of class C_0.

Since the point spectrum of a completely non-unitary does not intersect with the unite circle and $\sigma_p(T_1)$ is empty, then $\sigma_p(T)$ lies in $\sigma(T_0)$. However, $T_0 \in C_0$, that is the spectrum of T_0 does not fill the unit disc. Hence, $\sigma_p(T)$ is at most countable. □

Remark 3. *The assertions (i), (ii) and (iii) above are proven in [15] for all contraction in $C_{.0}$ such that the defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ is of Hilbert-Schmidt class.*

Proposition 7. *If T is a power-bounded n-$*$-paranormal operators (resp. the $(n,1)$-quasi-$*$-paranormal operators with reduced kernel) such that its spectrum lies in the unit circle $\mathbb{T}$, then T is a unitary operator.*

Proof. We have that our classes cited in the Proposition are invariant under multiplication by non-zero scalar and are contractions normaloid by Remark 3 and Lemma 5. Therefore, by following the proof given by Duggal [13], we obtain the desired result. □

It is well known that a contraction normal operator is a direct sum of a unitary operator and un operator of class C_{00}. So the natural question is what happen for a non-normal operators? Takahashi and Uchiyama [15] proved that a completely non-normal hyponormal operator such that the defect operator D_T is of Hilbert-Schmidt class, is of class C_{10} and Duggal, Jeonb, Kim [13] extended this result under the same assumptions to the case $*$-paranormal operators.
In the following, we generalize this result in more general classes.

Theorem 3. *If T is a completely non-normal n-$*$-paranormal operators (resp. the $(n,1)$-quasi-$*$-paranormal operators with reduced kernel) such that the defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ is of Hilbert-Schmidt class, then $T \in C_{10}$.*

Proof. From Proposition 4, T has the following triangular matrix

$$T = \begin{bmatrix} T_0 & D \\ 0 & T_1 \end{bmatrix} \tag{26}$$

on the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0{}^{\perp}$ where $\mathcal{H}_0$ is the stable subspace of T, $T_0 \in C_{00}$ and $T_1 \in C_{10}$. Therefore, by Proposition 5, D_{T_0} is of Hilbert-Schmidt class and $T_0 \in C_0$ with the form

$$T_0 = \begin{bmatrix} A & D \\ 0 & B \end{bmatrix} \tag{27}$$

where $\sigma(A) = \sigma_p(A) \subset \mathbb{D}$ and $\sigma(B) \subset \mathbb{T}$ (where $\mathbb{D}$ is the open unit disc). From Corollary 3 and the fact that T_0 is completely non-normal, it follows that $\sigma_p(T_0)$ is empty and yields $\sigma(T_0) = \sigma(B) \subset \mathbb{T}$. Therefore, by the Proposition 7, T_0 is unitary; a contradiction. This shows that T_0 is absent. Finally, we conclude that $T = T_1 \in C_{10}$. $\square$

5. Discussions and Further Studies

The following Putnam-Fuglede theorem is very well known:

Theorem 4. *(Putnam-Fuglede Theorem) [4,5].*
Assume that $A, B \in B(\mathcal{H})$ are normal operators. If $AX = XB$ for some $X \in B(\mathcal{H})$, then $A^*X = XB^*$.

There are many generalizations of this theorem to several classes of operators (see [3–5,7,8,10,16,21,25–27]) etc. In 1978, S.K Berberian [28] showed that the Putnam-Fuglede theorem holds when A and B* are hyponormal and X is a Hilbert-Schmidt operator. Radjapalipour [3] showed that Putnam-Fuglede theorem remains valid for hyponormal operators. In 2002, Uchiyama and Tanahashi [25] proved that Putnam-Fuglede theorem still holds for p-hyponormal and log-hyponormal operators. Bachir and Lombarkia [5] gave the extension of Putnam-Fuglede Theorem for w-hyponormal and class (Y). Recently, Mecheri and Uchiyama [7] extended Putnam-Fuglede to class $\mathcal{A}$ operators. In this paper, we generalize the Putnam-Fuglede theorem to a large class of operators, say (n,k)-quasi-*- paranormal operators. These results extend those given in [8,14,17,20]. As application of our main theorems, we obtain:

1. Characterization of (n,k)-quasi-*- paranormal operators with reduced kernel.
2. Characterization of completely non-normal $(n,k) - quasi - *-$paranormal operators. These generalizes the results given by
   (i) Tanahashi and Uchiyama [15] for completely non-normal hyponormal contraction operator.
   (ii) Duggal, Jeon, and Kim [13] completely non-normal *-paranormal contraction operator.

Author Contributions: Conceptualization, A.B. and A.S.; Investigation, A.B. and A.S.

Acknowledgments: At the end of this paper we would like to thank the referee for his useful remarks. The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project grant number (G.R.P- 26-38).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Arora, S.C.; Thukral, J.K. On a class of operators. *Kyngpook Math. J.* **1986**, *21*, 381–386.
2. Istrăţescu, V.I.; Saitô, T.; Yoshino, T. On a class of operators. *Tôhoku Math. J.* **1966**, *18*, 410–413.
3. Putnam, C.R. On the normal operators on Hilbert space. *Am. J. Math.* **1951**, *73*, 357–362.
4. Bachir, A. An Extension of Fuglede-Putnam Theorem for Certain Posinormal Operators. *Int. J. Contemp. Math. Sci.* **2013**, *8*, 827–832.
5. Bachir, A.; Lombarkia, F. Fuglede-Putnam's theorem for w-hyponormal operators. *Math. Inequal. Appl.* **2012**, *15*, 777–786.
6. Moore, R.L.; Rogers, D.D.; Trent, T.T. A note on intertwining M -hyponormal operators. *Proc. Am. Math. Soc.* **1981**, *83*, 514–516.

7. Mecheri, S.; Uchiyama, A. An extension of the Fuglede-Putnam's theorem to class A operators. *Math. Inequal. Appl.* **2010**, *13*, 57–61.
8. Mecheri, S. On quasi-*-paranormal operators. *Ann. Funct. Anal.* **2012**, *3*, 861791.
9. Pagacz, P. The Putnam-Fuglede property for paranormal and *-paranormal operators. *Opusc. Math.* **2013**, *33*, 565–574.
10. Radjabalipour, M. An extension of Putnam-Fuglede Theorem for Hyponormal Operators. *Math. Z.* **1987**, *194*, 117–120.
11. Uchiyama, A.; Tanahashi, K. Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators. *Glasg. Math. J.* **2002**, *44*, 397–410
12. Yoshino, T. Remark on the generalized Putnam–Fuglede theorem. *Proc. Am. Math. Soc.* **1985**, *95*, 571–572.
13. Duggal, B.P.; Jeon, I.H.; Kim, I.H. On $*$-paranormal contractions and properties for $*$-class A operators. *Linear Algebr. Appl.* **2012**, *436*, 954–962.
14. Gao, F.; Li, X. On $*$-class A contractions. *J. Inequal. Appl.* **2013**, *2013*, 239.
15. Takahashi, K.; Uchiyama, M. Every C_{00} contractions with Hilbert-Schmidt defect operator is of class C_0. *J. Oper. Theory* **1983**, *12*, 331–335.
16. Kérchy, L. Isometric asymptotes of power bounded operators. *Indiana Univ. Math. J.* **1989**, *38*, 173–188.
17. Conway, J. *A Course in Functional Analysis*; Springer: New York, NY, USA, 1985.
18. Sz-Nagy, B.; Foias, C. *Harmonic Analysis of Operators on Hilbert Space*; North-Holland: Amsterdam, The Netherlands, 1970.
19. Zeng, Q.P.; Zhong, H.J. On (n, k)-quasi-*-paranormal operators. *Bull. Malays Math. Sci. Soc.* **2017**, *40*, 1363–1376, doi:10.1007/s40840-015-0119-z.
20. Furuta, T. *Invitation to Linear Operators, From Matrices to Bounded Linear Operators on a Hilbert Space*; Taylor Francis: London, UK, 2001.
21. Takahashi, K. On the converse of the Fuglede-Putnam theorem. *Acta Sci. Math.* **1981**, *43*, 123–125
22. Duggal, B.P.; Kubrusly, C.S. Contractions with C.0 direct summands. *Adv. Math. Sci. Appl.* **2001**, *11*, 593–601.
23. Istrăţescu, I.; Istrăţescu, V. On some classes of operators I. *Proc. Jpn. Acad.* **1967**, *43*, 605–606.
24. Hoxhha, I.; Braha, N.L. The k-quasi-*-class A contractions have property PF. *J. Inequal. Appl.* **2014**, *2014*, 433.
25. Rashid, M.H.M.; Noorani, M.S.M. On relaxation normality in the Fulgede-Putnam's theorem for a quasi-class A operators. *Tamkang J. Math.* **2009**, *40*, 307–312.
26. Furuta, T. An extenstion of the Fuglede theorem to subnormal operators using a Hilbert-Schmidt norm inequality. *Proc. Am. Math. Soc.* **1981**, *81*, 240–242.
27. Mecheri, S. A generalization of Fuglede-Putnam theorem. *J. Pure Math.* **2004**, *21*, 31–38.
28. Berberian, S.K. Approximate proper vectors. *Proc. Am. Math. Soc.* **1962**, *13*, 111–114.