symmetry

MDPI

Article

Edge Even Graceful Labeling of Polar Grid Graphs

Salama Nagy Daoud [1,2]

[1] Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah 41411, Saudi Arabia; salamadaoud@gmail.com

[2] Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El Kom 32511, Egypt

Received: 6 December 2018; Accepted: 23 December 2018; Published: 2 January 2019

Abstract: Edge Even Graceful Labelingwas first defined byElsonbaty and Daoud in 2017. An edge even graceful labeling of a simple graph G with p vertices and q edges is a bijection f from the edges of the graph to the set $\{2, 4, \ldots, 2q\}$ such that, when each vertex is assigned the sum of all edges incident to it mod$2r$ where $r = \max\{p, q\}$, the resulting vertex labels are distinct. In this paper we proved necessary and sufficient conditions for the polar grid graph to be edge even graceful graph.

Keywords: graceful labeling; edge graceful labeling; edge even graceful labeling; polar grid graph

1. Introduction

The field of Graph Theory plays an important role in various areas of pure and applied sciences. One of the important areas in graph theory is Graph Labeling of a graph G which is an assignment of integers either to the vertices or edges or both subject to certain conditions. Graph labeling is a very powerful tool that eventually makes things in different fields very ease to be handled in mathematical way. Nowadays graph labeling has much attention from different brilliant researches ingraph theory which has rigorous applications in many disciplines, e.g., communication networks, coding theory, x-raycrystallography, radar, astronomy, circuit design, communication network addressing, data base management and graph decomposition problems. More interesting applications of graph labeling can be found in [1–10].

Let $G = (V(G), E(G))$ with $p = |V(G)|$ and $q = |E(G)|$ be a simple, connected, finite, undirected graph.

A function f is called a graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. This type of graph labeling first introducedby Rosa in 1967 [11] as a $\beta-$ valuation, later on Solomon W. Golomb [12] called as graceful labeling.

A function f is called an odd graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ is injective and the induced function $f^* : E(G) \to \{1, 3, \ldots, 2q-1\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. This type of graph labeling first introducedby Gnanajothi in 1991 [13]. For more results on this type of labeling see [14,15].

A function f is called an edge graceful labeling of a graph G if $f : E(G) \to \{1, 2, \ldots, q\}$ is bijective and the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, p-1\}$ defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \bmod p$ is bijective. This type of graph labeling first introducedby Lo in 1985 [16]. For more results on this labeling see [17,18].

A function f is called an edge odd graceful labeling of a graph G if $f : E(G) \to \{1, 3, \ldots, 2q-1\}$ is bijective and the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \bmod 2q$ is injective. This type of graph labeling first introducedby Solairaju and Chithra in 2009 [19]. See also Daoud [20].

A function f is called an edge even graceful labeling of a graph G if $f : E(G) \to \{2, 4, \ldots, 2q\}$ is bijective and the induced function $f^* : V(G) \to \{0, 2, 4, \ldots, 2q-2\}$ defined as $f^*(u) = \sum_{e=uv \in E(G)} f(e) \bmod 2r$, where $r = \max\{p, q\}$ is injective. This type of graph labeling first introduced by Elsonbaty and Daoud in 2017 [21].

For a summary of the results on these five types of graceful labels as well as all known labels so far, see [22].

2. Polar Grid Graph $P_{m,n}$

The polargrid graph $P_{m,n}$ is the graph consists of n copies of circles C_m which will be numbered from the inner most circle to the outer circle as $C_m^{(1)}, C_m^{(2)}, \ldots, C_m^{(n-1)}, C_m^{(n)}$ and m copies of paths P_{n+1} intersected at the center vertex v_0 which will be numbered as $P_{n+1}^{(1)}, P_{n+1}^{(2)}, \ldots, P_{n+1}^{(m-1)}, P_{n+1}^{(m)}$. See Figure 1.

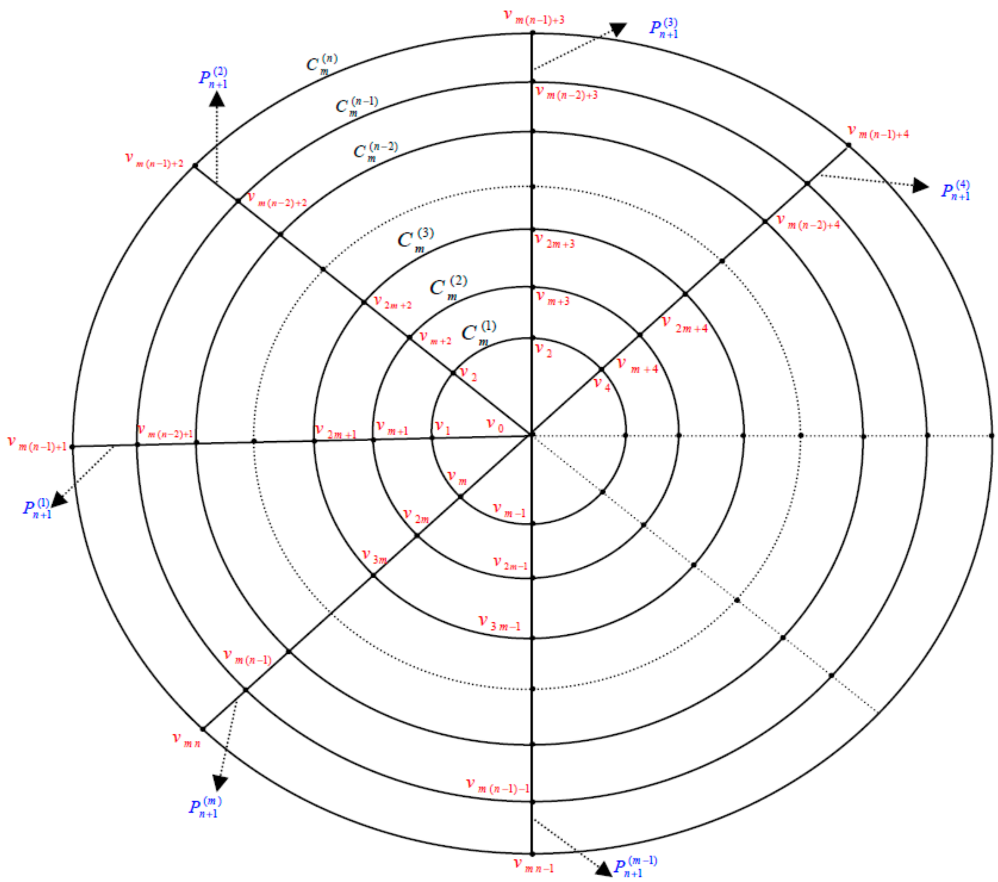


Figure 1. Polar grid graph $P_{m,n}$.

Theorem 1. *If m and n are even positive integes such that $m \geq 4$ and $n \geq 2$, then the polar grid graph $P_{m,n}$ is an edge even graceful graph.*

Proof. Using standard notation $p = |V(P_{m,n})| = mn + 1$, $q = |E(P_{m,n})| = 2mn$ and $r = \max\{p, q\} = 2mn$ Let the polar grid graph $P_{m,n}$ be labeled as in Figure 2. Let $f : E(G) \to \{2, 4, \ldots, 2q\}$. □

First we label the edges of paths $P_{n+1}^{(k)}, \leq k \leq m$ begin with the edges of the path $P_{n+1}^{(1)}$ to the edges of the path $P_{n+1}^{(m)}$ as follows: Move clockwise to label the edges $v_0v_1, v_0v_2, \ldots, v_0v_{m-1}, v_0v_m$ by $2, 4, \ldots, 2m-2, 2m$, then move anticlockwise to label the edges

$v_1v_{m+1}, v_mv_{2m}, v_{m-1}v_{2m-1}, \ldots, v_3v_{m+3}, v_2v_{m+2}$ by $2m+2, 2m+4, 2m+6, \ldots, 4m-2, 4m$, then move clockwise to label the edges $v_{m+1}v_{2m+1}, v_{m+2}v_{2m+2}, v_{m+3}v_{2m+3}, \ldots, v_{2m-1}v_{3m-1}, v_mv_{3m}$ by $4m+2, 4m+4, 4m+6, \ldots, 6m-2, 6m$ and so on. Finally move anticlockwise to label the edges $v_{m(n-2)+1}v_{m(n-1)+1}, v_{m(n-1)-1}v_{mn}, v_{m(n-1)-1}v_{mn-1}, \ldots, v_{m(n-2)+2}v_{m(n-1)+2}$ by $2m(n-1)+2, 2m(n-1)+4, 2m(n-1)+6, \ldots, 2mn-2, 2mn$. Second we label the edges of the circles $C_m^{(k)}, 1 \le k \le n$ begin with the edges of the inner most circle $C_m^{(1)}$ to the edges of the circle $C_m^{(\frac{n}{2})}$, then the edges of the outer circle $C_m^{(n)}$. Finally the edges of circles $C_m^{(\frac{n}{2}+1)}, C_m^{(\frac{n}{2}+2)}, \ldots, C_m^{(n-1)}$ respectively as follows: $f(v_{m(k-1)+i}v_{m(k-1)+i+1}) = 2mn + 2m(k-1) + 2i$, $f(v_{km}v_{m(k-1)+1}) = 2mn + 2mk$, $1 \le i \le m-1, 1 \le k \le \frac{n}{2}$; $f(v_{m(n-1)+i}v_{m(n-1)+i+1}) = 3mn + 2i$, $1 \le i \le m-1$; $f(v_{km}v_{m(k-1)+1}) = 2mn + (2k+1)m$, $\frac{n}{2}+1 \le k \le n-1$.

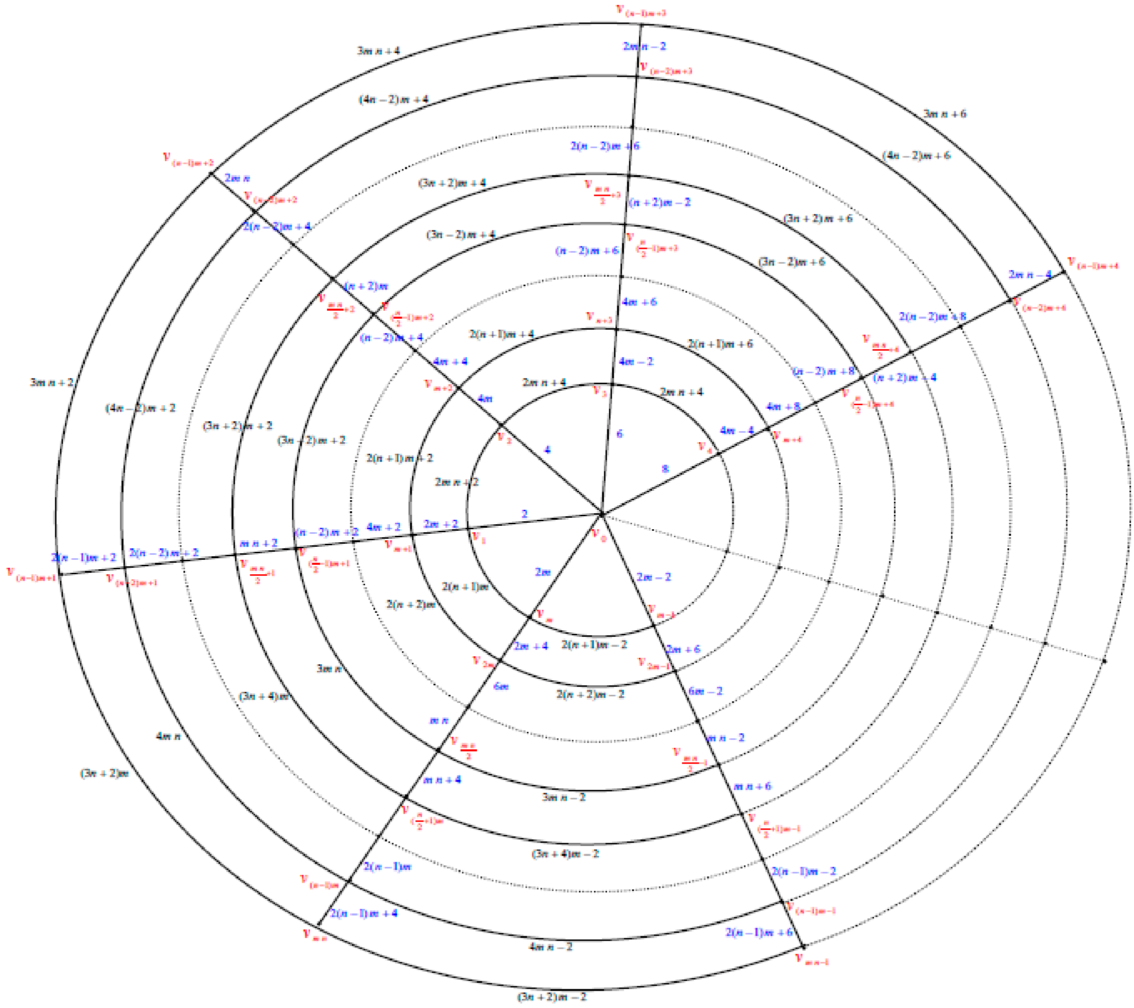


Figure 2. Labeling ofthe polar grid graph $P_{m,n}$ when n is even, $n \ge 2$.

Now the corresponding labels of vertices mod$4mn$ are assigned as follows:

Case (1) $m \equiv 4k \bmod 4n$, $1 \le k \le n-1$ and $m = 2 \bmod 4n$.

The labels of the vertices of the inner most circle $C_m^{(1)}$ to the circle $C_m^{(\frac{n}{2})}$ are given by $f^*(v_{(k-1)m+i}) \equiv 4m(2k-1) + 4i$, $1 \le k \le \frac{n}{2}, 1 \le i \le m$, the labels of the vertices of the outre circle $C_m^{(n)}$ are given by $f^*(v_{(n-1)m+i}) \equiv 2i+2$, $1 \le i \le m$ and the labels of the vertices of the circles $C_m^{(\frac{n}{2}+1)}, \ldots, C_m^{(n-1)}$ are given by $f^*(v_{(k-1)m+i}) \equiv 8m(k-\frac{n}{2}) + 4i + 2$, $1 \le i \le m$, $\frac{n}{2}+1 \le k \le n-1$.

The label of the center vertex v_0 is assigned as follows: when $m = 4k \bmod 4n$, $1 \le k \le n-1$, $f^*(v_0) \equiv \frac{m}{2}(2m+2) = m^2 + m$, since $m = 4k \bmod 4n$ then $m = 4nh + 4k$, thus $f^*(v_0) = m(4k+1)$ and when $m = 2 \bmod 4n$, we have $f^*(v_0) = 3m$.

Case (2) $m \equiv (8k-2) \bmod 4n$, $1 \leq k \leq \frac{n}{2}$. In this case the vertex $v_{km-(\frac{m+2}{4})}$ in the circle $C_m^{(k)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{km-(\frac{m+2}{4})}v_{km-(\frac{m-2}{4})}$ and $v_{km-(\frac{m-2}{4})}v_{km-(\frac{m-6}{4})}$. That is $f(v_{km-(\frac{m+2}{4})}v_{km-(\frac{m-2}{4})}) = 2mn + m(2k-1) + \frac{m+2}{2}$ and $f(v_{km-(\frac{m-2}{4})}v_{km-(\frac{m-6}{4})}) = 2mn + m(2k-1) + \frac{m-2}{2}$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{km-(\frac{m+2}{4})}) \equiv m(8k-1)+2$, $f^*(v_{km-(\frac{m-2}{4})}) \equiv m(8k-1)+4$, $f^*(v_{km-(\frac{m-6}{4})}) \equiv m(8k-1)+6$ and the label of the center vertex v_0 is assigened as $f^*(v_0) \equiv m(8k-1)$.The rest vertices are labeled as in case(1).

Case (3) $m \equiv (8k+2) \bmod 4n$, $1 \leq k \leq \frac{n}{2}-1$. In this case the vertex $v_{km-(\frac{m+2}{2})}$ in the circle $C_m^{(\frac{n}{2}+k)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{km-(\frac{m+2}{2})}v_{km-(\frac{m}{2})}$ and $v_{km-(\frac{m}{2})}v_{km-(\frac{m-2}{2})}$. That is $f(v_{km-(\frac{m+2}{2})}v_{km-(\frac{m}{2})}) = 2mn + m(2k+1) + \frac{m+2}{2}$ and $f(v_{km-(\frac{m}{2})}v_{km-(\frac{m-2}{2})}) = 2mn + m(2k+1) + \frac{m-2}{2}$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{km-(\frac{m+2}{2})}) \equiv m(8k+3)+2$, $f^*(v_{km-(\frac{m}{2})}) \equiv m(8k+3)+4$, $f^*(v_{km-(\frac{m-2}{2})}) \equiv m(8k+3)+6$ and the label of the center vertex v_0 is assigened $f^*(v_0) \equiv m(8k-29)$ as. The rest vertices are labeled as in case(1).

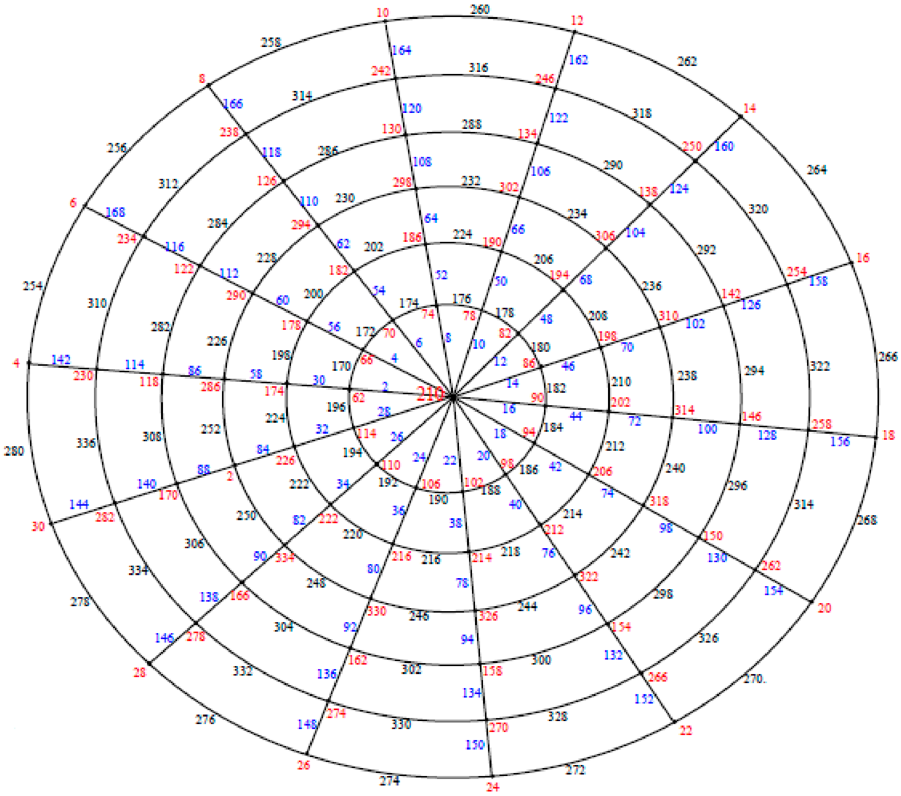
Case (4) $m \equiv 0 \bmod 4n$. In this case the vertex $v_{mn-(\frac{m+2}{2})}$ in the outer circle will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{mn-(\frac{m+4}{2})}v_{mn-(\frac{m+2}{2})}$ and $v_{mn}v_{m(n-1)+1}$. That is $f(v_{mn-(\frac{m+4}{2})}v_{mn-(\frac{m+2}{2})}) = m(3n+2)$ and $f(v_{mn}v_{m(n-1)+1}) = 3mn + m - 4$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{mn-(\frac{m+4}{2})}) \equiv 2m + 2$, $f^*(v_{mn-(\frac{m+2}{2})}) \equiv 2m+4$, $f^*(v_{mn}) \equiv m-2$ and $f^*(v_{m(n-1)+1}) \equiv 4mn - m$ and the label of the center vertex v_0 is assigened as $f^*(v_0) \equiv m$. The rest vertices are labeled as in case (1).

Illustration. *The edge even graceful labeling of the polar grid graphs $P_{14,6}$, $P_{16,6}$, $P_{18,6}$, $P_{24,6}$ and $P_{26,6}$ respectively are shown in Figure 3.*

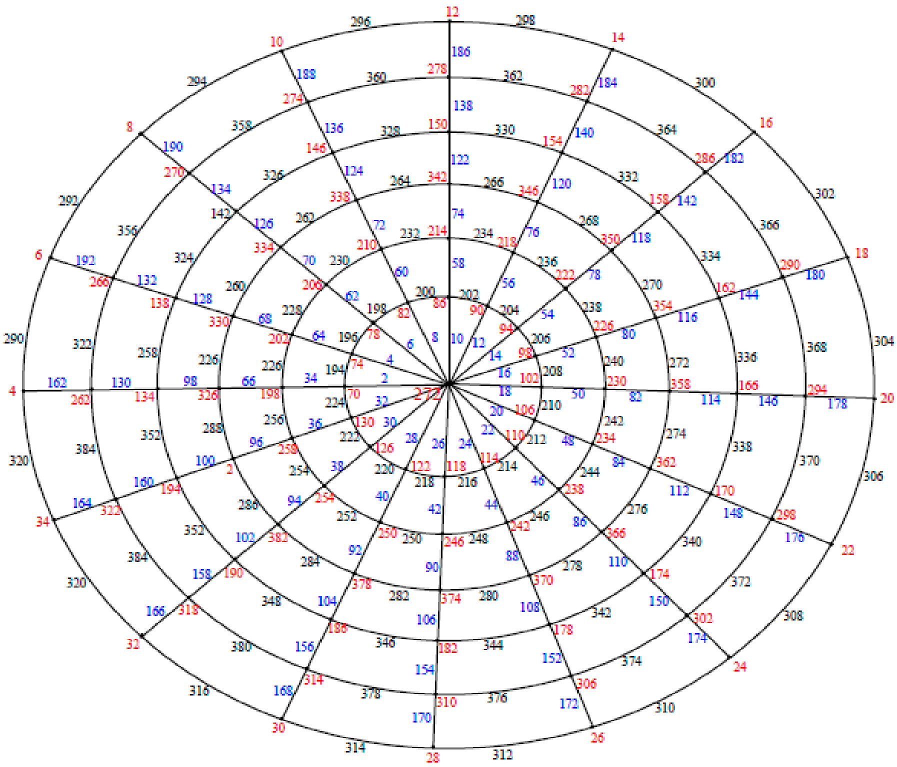
Remark 1. *In case $m = 2$ and n is even, $n > 2$.*

Let the label of edges of the polar grid graph be as in Figure 4. Thus we have the label of the corresponding vertices are as follows:

$f^*(v_1) \equiv 4n+12$; $f^*(v_i) \equiv 16i-2$, $2 \leq i \leq \frac{n}{2}$; $f^*(v_{\frac{n}{2}+i}) \equiv 16i+6$, $1 \leq i \leq \frac{n}{2}-1$;
$f^*(v_n) \equiv 4$; $f^*(v'_i) \equiv 16i+2$, $1 \leq i \leq \frac{n}{2}-1$; $f^*(v'_{\frac{n}{2}+i}) \equiv 2$; $f^*(v'_{\frac{n}{2}+i}) \equiv 16i+10$, $1 \leq i \leq \frac{n}{2}-2$;
$f^*(v'_{n-1}) \equiv 4n-4$; $f^*(v'_n) \equiv 4n+8$ and $f^*(v_0) \equiv 4n+4$.

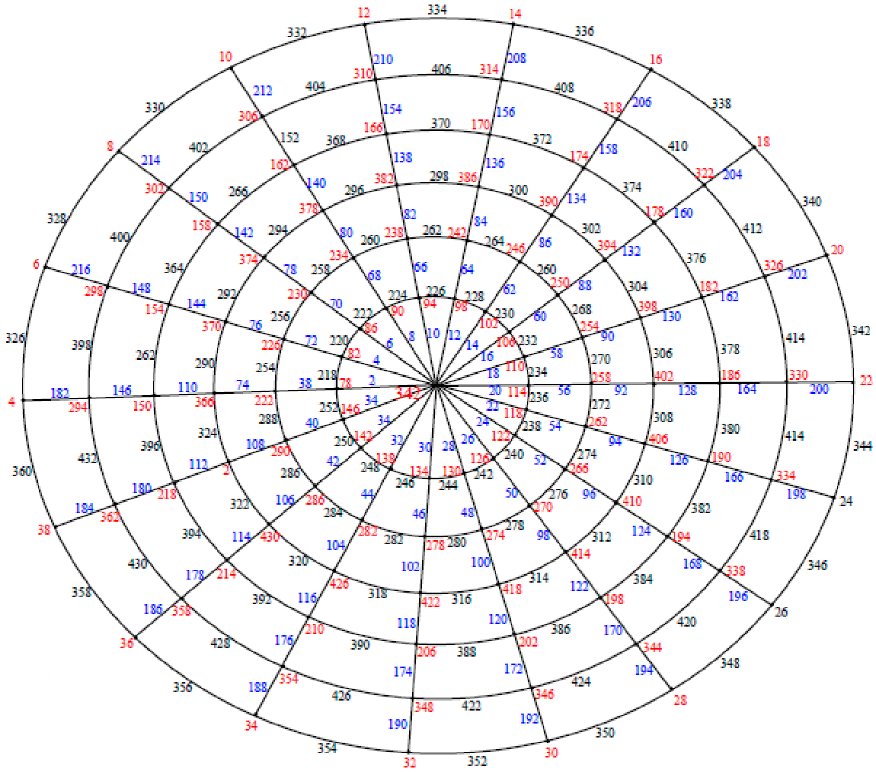


(**a**) $P_{14,6}$

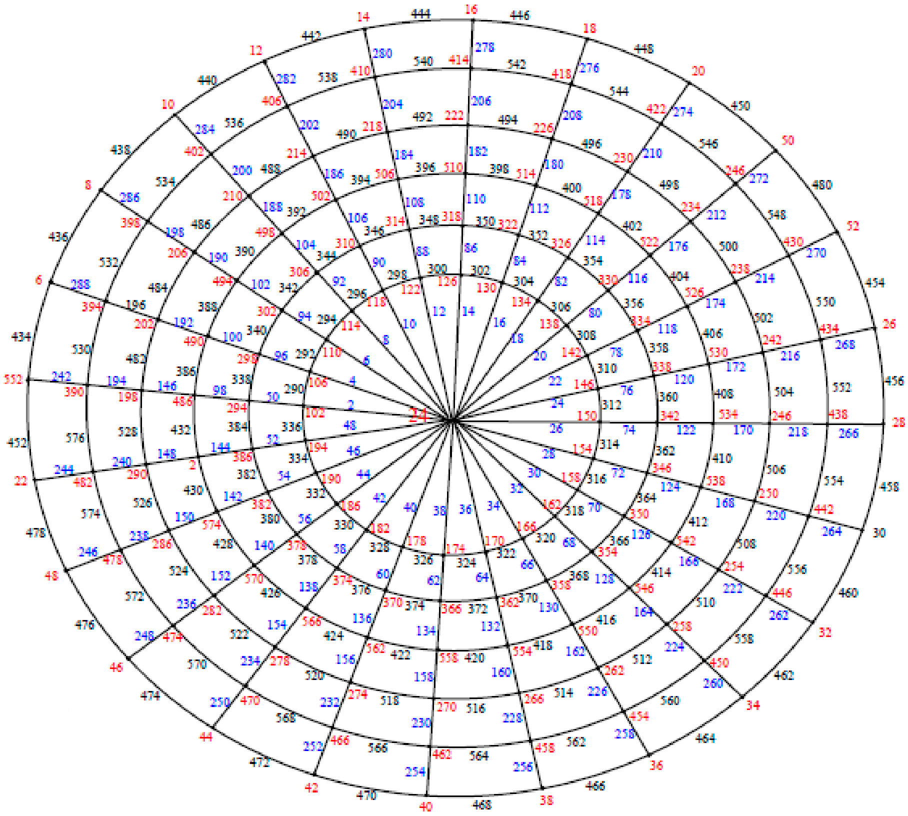


(**b**) $P_{16,6}$

Figure 3. *Cont.*

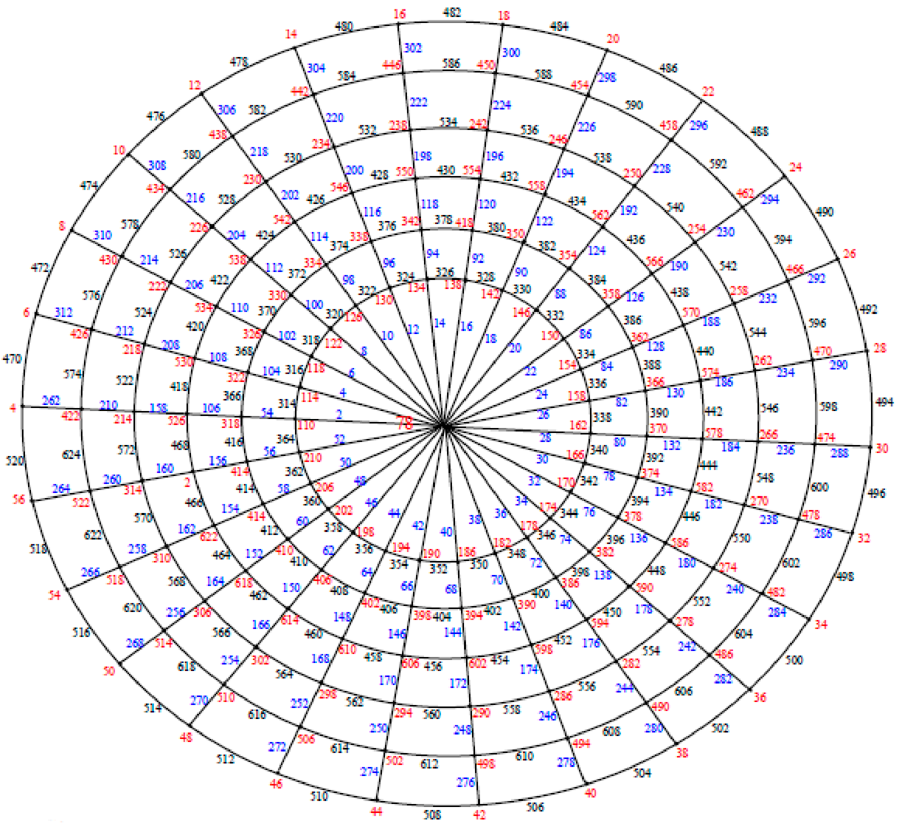


(c) $P_{18,6}$



(d) $P_{24,6}$

Figure 3. *Cont.*



(e) $P_{26,6}$

Figure 3. The edge even graceful labeling of the polar grid graphs $P_{14,6}$, $P_{16,6}$, $P_{18,6}$, $P_{24,6}$ and $P_{26,6}$.

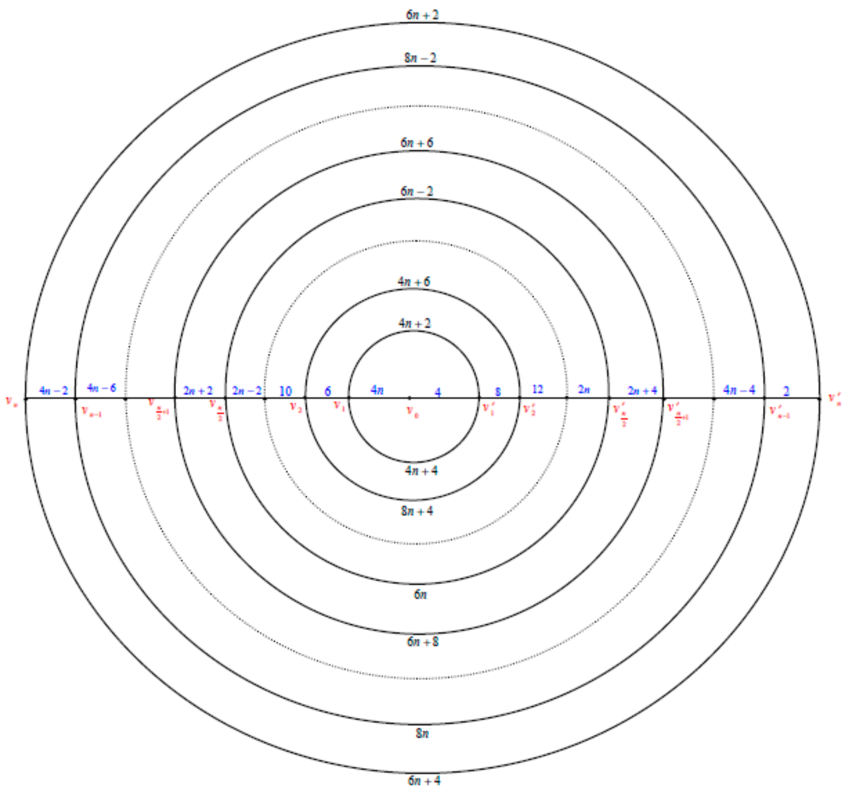


Figure 4. Labeling of the polar grid graph $P_{2,n}$, n is even integer greater than 2.

Note that $P_{2,2}$ is an edge even graceful graph but not follow this rule. See Figure 5.

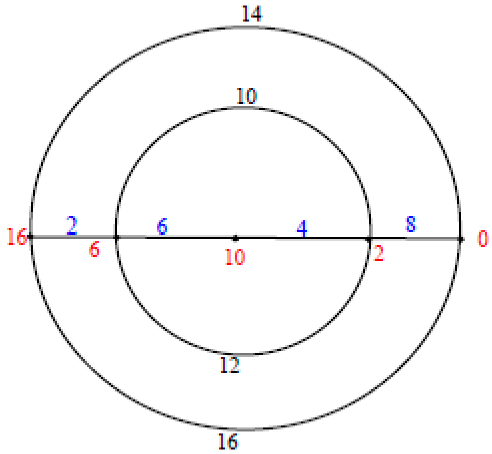


Figure 5. The polar grid graph $P_{2,2}$.

Theorem 2. *If m is an odd positive integer greater than 1 and n an is even positive integer greater than or equal 2, then the polar grid graph $P_{m,n}$ is an edge even graceful graph.*

Proof. Let the edges of the polar grid graph $P_{m,n}$ be labeled as in Figure 2. □

Now the corresponding labels of vertices mod$4mn$ are assigned as follows: There are four cases

Case (1): $m \equiv (4k-1)\text{mod}4n,\ 1 \le k \le n$

The labels of the vertices of the inner most circle $C_m^{(1)}$ to the circle $C_m^{(\frac{n}{2})}$ are given by $f^*(v_{(k-1)m+i}) \equiv 4m(2k-1)+4i+2,\ 1 \le k \le \frac{n}{2},\ 1 \le i \le m$, the labels of the vertices of the outer circle $C_m^{(n)}$ are given by $f^*(v_{(n-1)m+i}) \equiv 2i+2,\ 1 \le i \le m$ and the labels of the vertices of the circles $C_m^{(\frac{n}{2}+1)}, \ldots, C_m^{(n-1)}$ are given by $f^*(v_{(k-1)m+i}) \equiv 8m(k-\frac{n}{2})+4i+2,\ 1 \le i \le m,\ \frac{n}{2}+1 \le k \le n-1$.

The center vertex v_0 is labeled as $f^*(v_0) \equiv 4mk$, and if $k=n$, we have $f^*(v_0) \equiv 0$.

Case (2): $m \equiv (8k-3)\text{mod}4n,\ 1 \le k \le \frac{n}{2}$.

In this case the vertex $v_{km(\frac{m+1}{2})}$ in the circle $C_m^{(k)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{km-(\frac{m+1}{2})}v_{km-(\frac{m-1}{2})}$ and $v_{km-(\frac{m-1}{2})}v_{km-(\frac{m-3}{2})}$. That is $f(v_{km-(\frac{m+1}{2})}v_{km-(\frac{m-1}{2})}) = 2mn+m(2k-1)+1$ and $f(v_{km-(\frac{m-1}{2})}v_{km-(\frac{m-3}{2})}) = 2mn+m(2k-1)-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{km-(\frac{m+1}{2})}) \equiv 2m(4k-1)+2,\ f^*(v_{km-(\frac{m-1}{2})}) \equiv 2m(4k-1)+4$ and $f^*(v_{km-(\frac{m-3}{2})}) \equiv 2m(4k-1)+6$. The center vertex v_0 is labeled as $f^*(v_0) = 2m(4k-1)$. The rest vertices are labeled as in case (1).

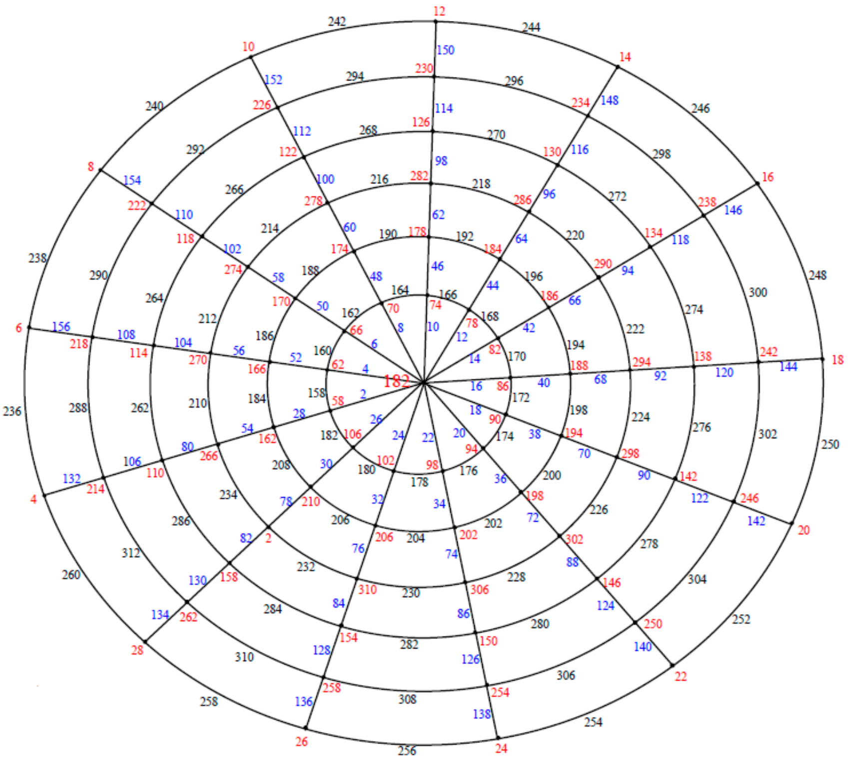
Case (3): $m \equiv (8k+1)\text{mod}4n,\ 1 \le k \le \frac{n}{2}-1$.

In this case the vertex $v_{m(\frac{n}{2}+k)\,(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{n}{2}+k)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{n}{2}+k)-(\frac{m+1}{2})}v_{m(\frac{n}{2}+k)\,(\frac{m-1}{2})}$ and $v_{m(\frac{n}{2}+k)-(\frac{m-1}{2})}v_{m(\frac{n}{2}+k)-(\frac{m-3}{2})}$. That is $f(v_{m(\frac{n}{2}+k)-(\frac{m+1}{2})}v_{m(\frac{n}{2}+k)-(\frac{m-1}{2})}) = 3mn+m(2k-1)+1$ and $f(v_{m(\frac{n}{2}+k)-(\frac{m-1}{2})}v_{m(\frac{n}{2}+k)-(\frac{m-3}{2})}) = 3mn+m(2k+1)-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{n}{2}+k)-(\frac{m+1}{2})}) \equiv 2m(4k+1)+2,\ f^*(v_{m(\frac{n}{2}+k)-(\frac{m-1}{2})}) \equiv 2m(4k+1)+4$, and $f^*(v_{m(\frac{n}{2}+k)-(\frac{m-3}{2})}) \equiv 2m(4k+1)+6$ and in this case the center vertex v_0 is labeled as $f^*(v_0) = 2m(4k+1)$. The rest vertices are labeled as in case (1).

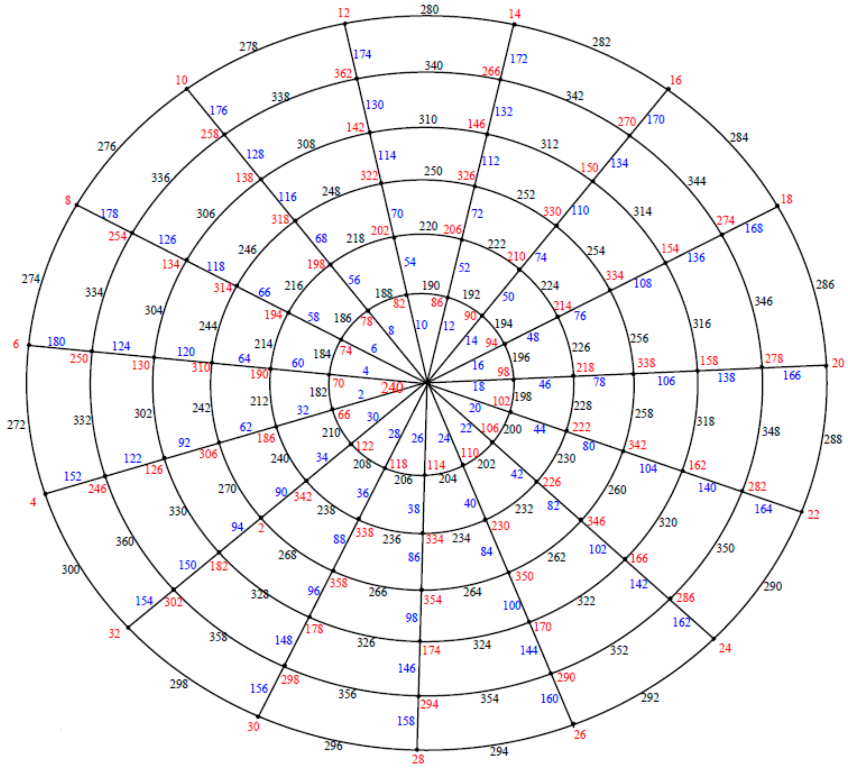
Case (4): $m \equiv 1\text{mod}4n$

In this case the vertex v_{mn-1} in the outer circle $C_m^{(n)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{mn-2}v_{mn-1}$ and $v_{mn}v_{m(n-1)+1}$. That is $f(v_{mn-2}v_{mn-1}) = m(3n+2)$ and $f(v_{mn}v_{m(n-1)+1}) = m(3n+2)-4$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{mn-2}) \equiv 2m+2,\ f^*(v_{mn-1}) \equiv 2m+4,\ f^*(v_{mn}) \equiv 2m-2$ and $f^*(v_{m(n-1)+1}) \equiv 0$, the center vertex v_0 is labeled as $f^*(v_0) \equiv 2m$. The rest vertices are labeled as in case (1).

Illustration. *The edge even graceful labeling of the polar grid graphs $P_{13,6}$, $P_{15,6}$, $P_{17,6}$ and $P_{25,6}$ respectively are shown in Figure 6.*

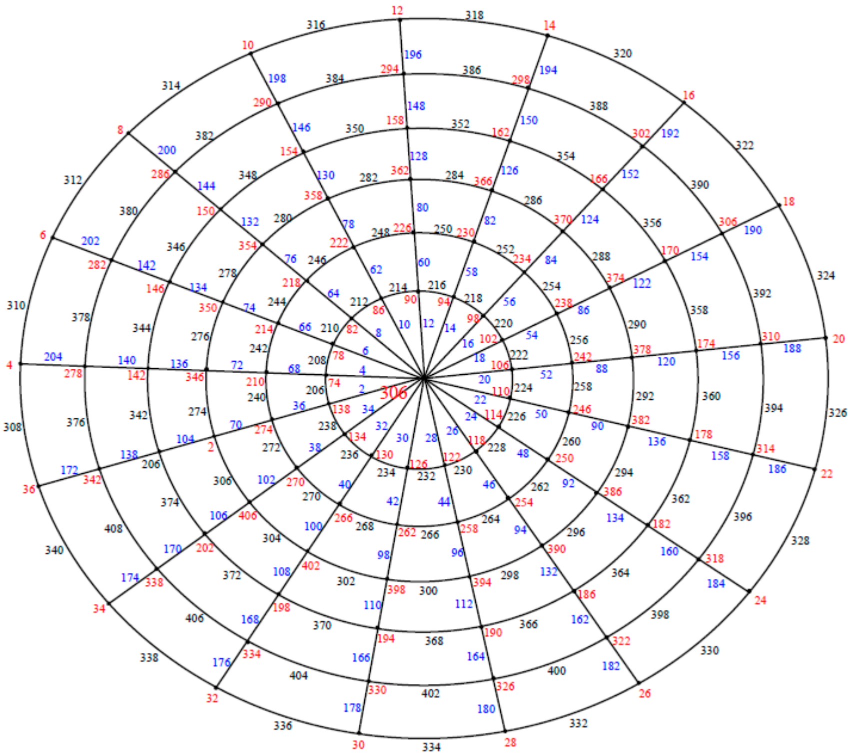


(**a**) $P_{13,6}$

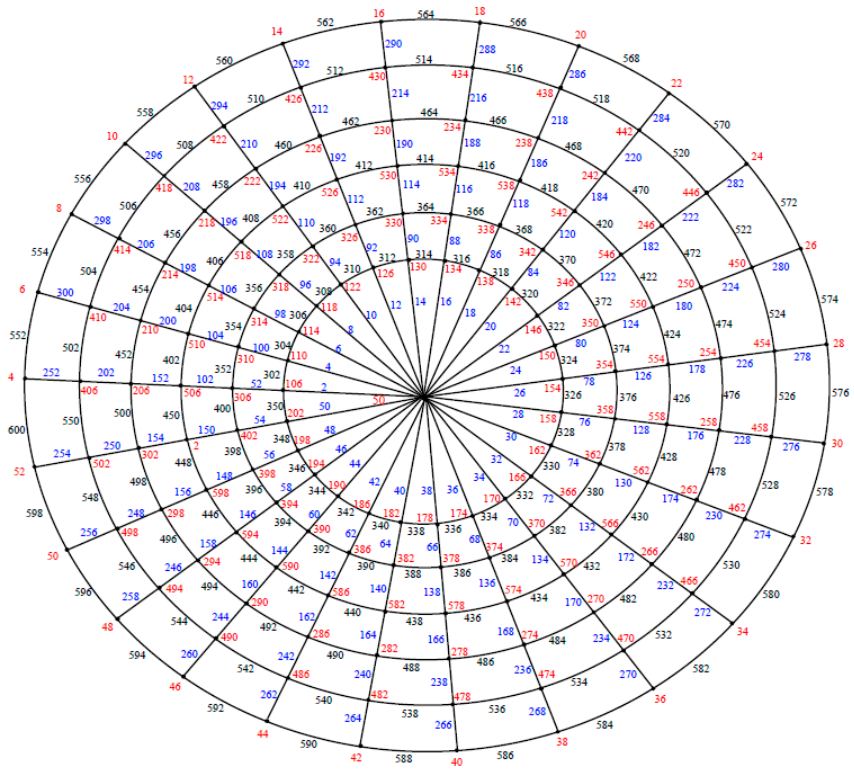


(**b**) $P_{15,6}$

Figure 6. *Cont.*



(c) $P_{17,6}$



(d) $P_{25,6}$

Figure 6. The edge even graceful labeling of the polar grid graphs $P_{13,6}$, $P_{15,6}$, $P_{17,6}$ and $P_{25,6}$.

Theorem 3. *If m is an even positive integer greater than or equal* 4 *and n is an odd positive integer greater than or equal* 3*. Then the polar grid graph $P_{m,n}$ is an edge even graceful graph.*

Proof. Let the polar grid graph $P_{m,n}$ be labeled as in Figure 7. Let $f : E(G) \rightarrow \{2, 4, \ldots, 2q\}$. □

First we label the edges of the circles $C_m^{(k)}$, $1 \leq k \leq n$ begin with the edges of the inner most circle $C_m^{(1)}$ to edges of the outer circle $C_m^{(n)}$ as follows:

$$f(v_{m(k-1)+i}v_{m(k-1)+i+1}) = 2m(k-1) + 2i,\ f(v_{km}v_{(k-1)m+1}) = 2km,\ 1 \leq i \leq m-1,\ 1 \leq k \leq n.$$

Second we label the edges of paths $P_{n+1}^{(k)}$, $1 \leq k \leq m$ begin with the edges of the path $P_{n+1}^{(1)}$ as follows: Move anticlockwise to label the edges v_0v_1, v_0v_m, v_0v_{m-1}, $\ldots$, v_0v_3, v_0v_2 by $2mn+2$, $2mn+4, 2mn+6$, $\ldots$, $2m(n+1)-2$, $2m(n+1)$, then move clockwise to label the edges v_1v_{m+1}, v_2v_{m+2}, $v_3v_{m+3}, \ldots$, $v_{m-1}v_{2m-1}$, v_mv_{2m} by $2m(n+1)+2, 2m(n+1)+4, 2m(n+1)+6, \ldots$, $2m(n+2)-2, 2m(n+2)$, then move anticlockwise to label the edges $v_{m+1}v_{2m+1}$, $v_{2m}v_{3m}$, $v_{2m-1}v_{3m-1}$, $\ldots$, $v_{m+3}v_{2m+3}$, $v_{m+2}v_{2m+2}$ by $2m(n+2)+2$, $2m(n+2)+4, 2m(n+2)+6, \ldots$, $2m(n+3)-2$, $2m(n+3)$ and so on. Finally move anticlockwise to label the edges $v_{m(n-2)+1}v_{m(n-1)+1}$, $v_{m(n-1)+1}v_{mn}$, $v_{m(n-1)-1}v_{mn-1}, \ldots$, $v_{m(n-2)+3}v_{m(m-1)+3}$, $v_{m(n-2)+2}v_{m(m-1)+2}$ by $2m(2n-1)+2, 2m(2n-1)+4, 2m(2n-1)+6, \ldots, 4mn-2, 4mn$.

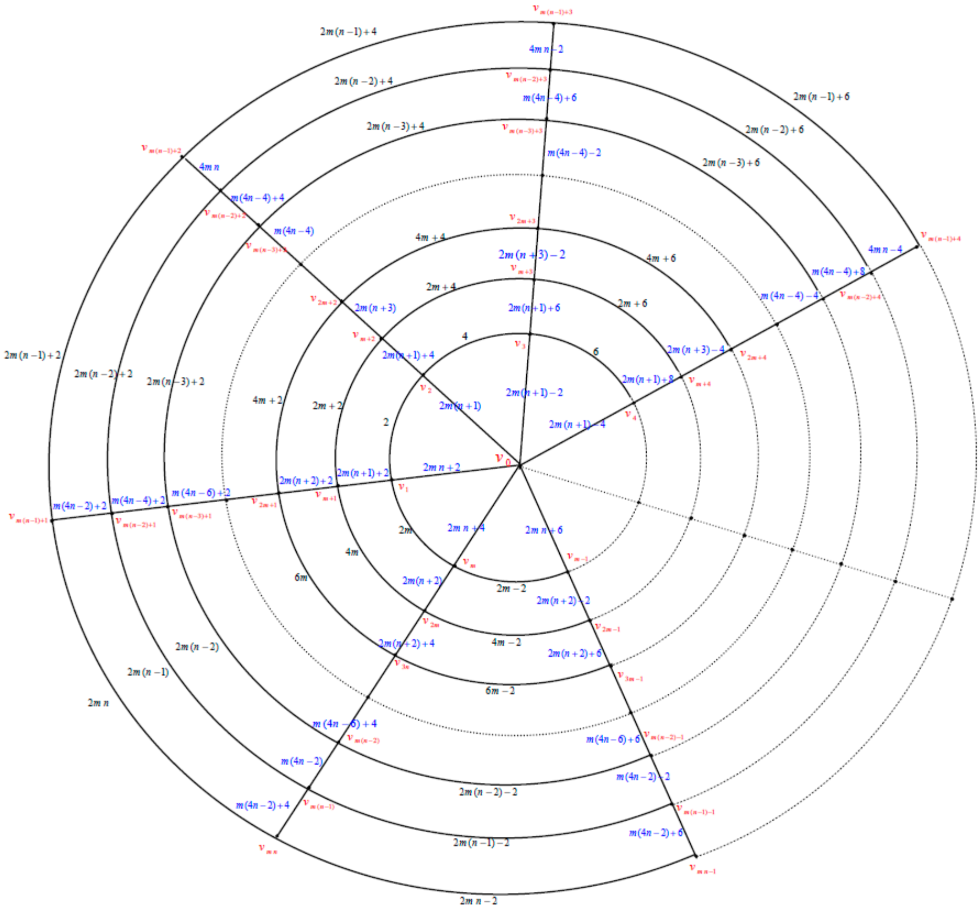


Figure 7. Labeling of the polar grid graph $P_{m,n}$ when n is odd and $n \geq 3$.

The corresponding labels of vertices mod$4mn$ are assigned as follows: There are four cases

Case (1) $m \equiv 4k \bmod 4n$, $1 \leq k \leq n-2$; $m \equiv (4n-2) \bmod 4n$ and $m \equiv 0 \bmod 4n$

$f^*(v_{(k-1)m+i}) \equiv 4m(2k-1) + 4i + 2$, $1 \leq i \leq m$, $1 \leq k \leq n-1$. That is the labels of the vertices in the most inner circle $C_m^{(1)}$ are assigned by $f^*(v_i) \equiv 4m + 4i + 2$, $1 \leq i \leq m$, the labels of the vertices in the circle $C_m^{(2)}$ are assigned by $f^*(v_{m+i}) \equiv 12m + 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n-1}{2})}$ are assigned by $f^*(\frac{v_{m(n-3)}}{2} + i) \equiv 4mn - 8m + 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n+1}{2})}$ are assigned by $f^*(\frac{v_{m(n-1)}}{2} + i) \equiv 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n+3}{2})}$ are assigned by $f^*(\frac{v_{m(n+1)}}{2}+i) \equiv 8m+4i+2, \ldots$, the labels of the vertices in the circle $C_m^{(n-1)}$ are assigned by $f^*(v_{m(n-2)+i}) \equiv 4mn-12m+4i+2$, $1 \le i \le m$ and the labels of the vertices of the outer circle $C_m^{(n)}$ are assigned by $f^*(v_{m(n-1)+i}) \equiv 4mn-4m+2i+2$, $1 \le i \le m$. The labels of the center vertex v_0 is assigned by $f^*(v_0) \equiv m^2(2n+1)$ when $m \equiv 4k \bmod 4n$, we have $f^*(v_0) \equiv m(4k+1)$, when $m \equiv (4n-2) \bmod 4n$, $f^*(v_0) \equiv 4mn-m$ and when $m \equiv 0 \bmod 4n$, $f^*(v_0) \equiv m$.

Case (2) $m \equiv (8k-2) \bmod 4n$, $1 \le k \le \frac{n-1}{2}$.

In this case the vertex $v_{km-(\frac{m+2}{4})}$ in the circle $C_m^{(k)}$ will repeat with the center vertex v_0. To avoid this problem we replace the label of two edges $v_{km-(\frac{m+2}{4})}v_{km-(\frac{m-2}{4})}$ and $v_{km-(\frac{m-2}{4})}v_{km-(\frac{m-6}{4})}$. That is $f(v_{km-(\frac{m+2}{4})}v_{km-(\frac{m-2}{4})}) = m(2k-1)+\frac{m+2}{2}$ and $f(v_{km-(\frac{m-2}{4})}v_{km-(\frac{m-6}{2})}) = m(2k-1)+\frac{m-2}{2}$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{km-(\frac{m+2}{4})}) \equiv m(8k-1)+2$, $f^*(v_{km-(\frac{m-2}{4})}) \equiv m(8k-1)+4$ and $f^*(v_{km-(\frac{m-6}{2})}) \equiv m(8k-1)+6$. In this case the center vertex v_0 is labeled as $f^*(v_0) \equiv m(2mn+m+1) \equiv m(8k-1)$. The rest vertices are labeled as in case (1).

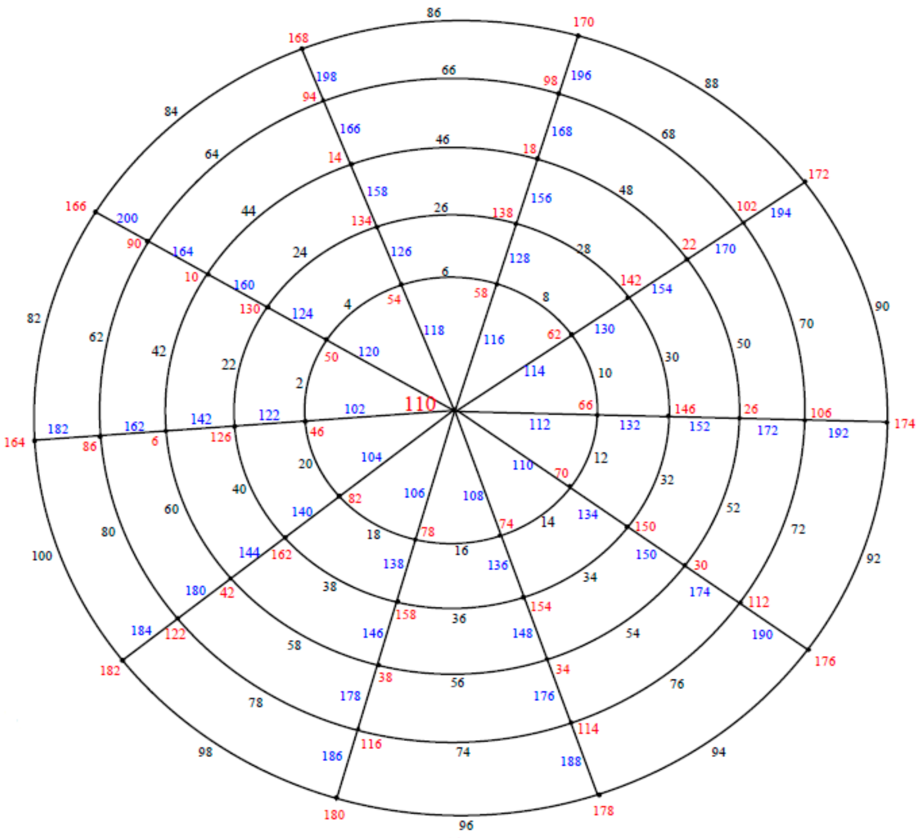
Case (3) $m \equiv (8k-6) \bmod 4n$, $1 \le k \le \frac{n-1}{2}$ and $m \neq 2$. In this case the vertex $v_{m(\frac{n+2k-1}{2})-(\frac{m+2}{4})}$ in the circle $C_m^{(\frac{n+2k-1}{2})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{n+2k-1}{2})-(\frac{m+2}{4})}v_{m(\frac{n+2k-1}{2})-(\frac{m-2}{4})}$ and $v_{m(\frac{n+2k-1}{2})-(\frac{m-2}{4})}v_{m(\frac{n+2k-1}{2})-(\frac{m-6}{4})}$. That is $f(v_{m(\frac{n+2k-1}{2})-(\frac{m+2}{4})}v_{m(\frac{n+2k-1}{2})-(\frac{m-2}{4})}) = mn+2m(k-1)+\frac{m+2}{2}$ and $f(v_{m(\frac{n+2k-1}{2})-(\frac{m-2}{4})}v_{m(\frac{n+2k-1}{2})-(\frac{m-6}{4})}) = mn+2m(k-1)+\frac{m-2}{2}$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{n+2k-1}{2})-(\frac{m+2}{4})}) \equiv m(8k-5)+2$, $f^*(v_{m(\frac{n+2k-1}{2})-(\frac{m-2}{4})}) \equiv m(8k-5)+4$ and $f^*(v_{m(\frac{n+2k-1}{2})-(\frac{m-6}{4})}) \equiv m(8k-5)+6$ and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv m(8k-5)$. The rest vertices are labeled as in case (1).

Case (4) $m \equiv (4n-4) \bmod 4n$. In this case the vertex $v_{mn-(\frac{m+2}{2})}$ in the outer circle $C_m^{(n)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{mn-(\frac{m+6}{4})}v_{mn-(\frac{m+2}{4})}$ and $v_{mn}v_{m(n-1)+1}$. That is $f(v_{mn-(\frac{m+6}{4})}v_{mn-(\frac{m+2}{4})}) = 2mn$ and $f(v_{mn}v_{m(n-1)+1}) = m(2n-1)-4$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{mn-(\frac{m+6}{4})}) \equiv 4mn-2m+2$, $f^*(v_{mn-(\frac{m+2}{4})}) \equiv 4mn-2m+8$, $f^*(v_{mn}) \equiv 4mn-3m-2$ and $f^*(v_{m(n-1)+1}) \equiv 4mn-5m$ and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv m(4n-3)$. The rest vertices are labeled as in case (1).

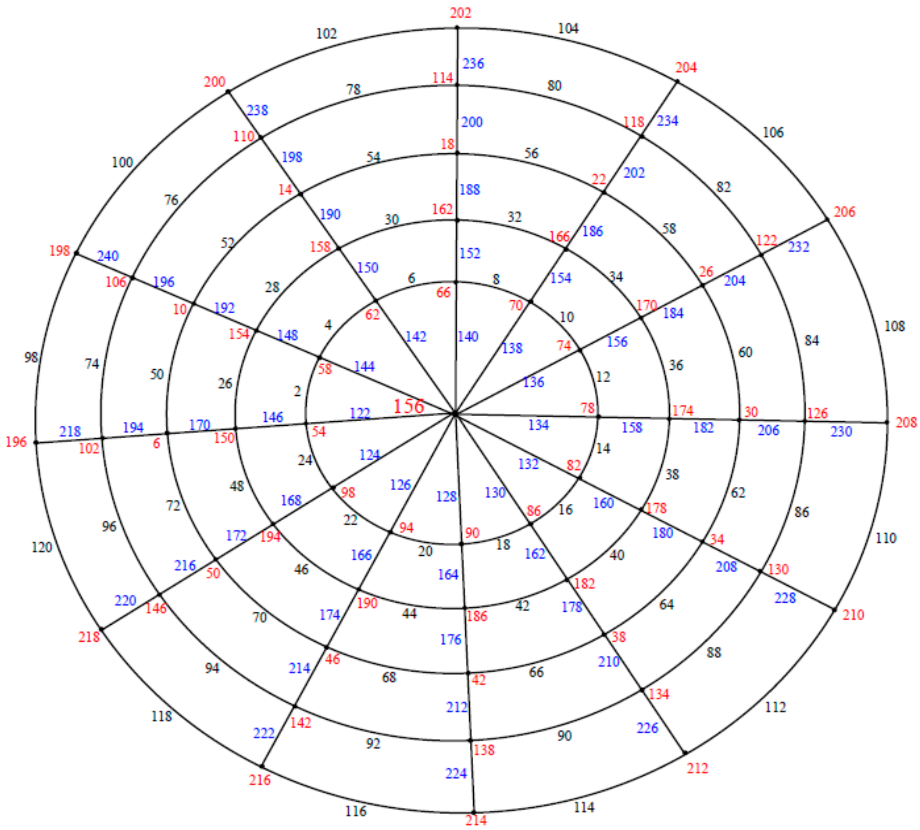
Illustration. *The edge even graceful labeling of the polar grid graphs $P_{10,5}$, $P_{12,5}$, $P_{14,5}$, $P_{16,5}$, $P_{18,5}$ and $P_{20,5}$ respectively are shown in Figure 8.*

Remark 2. *In case $m = 2$, n is odd, $n \ge 3$.*

Let the label of edges of the polar grid graph $P_{2,n}$ be as in Figure 9. Thus we have the labels of the corresponding vertices as follows: $f^*(v_1) \equiv 12$; $f^*(v_i) \equiv 16i-2$, $2 \le i \le \frac{n-1}{2}$; $f^*(v_{\frac{n+2i-1}{2}}) \equiv 16i-10, 1 \le i \le \frac{n-1}{2}$; $f^*(v_n) \equiv 8n-2$; $f^*(v'_i) \equiv 16i+2$, $1 \le i \le \frac{n-1}{2}$; $f^*(v'_{\frac{n+1}{2}}) \equiv 2$; $f^*(v'_{\frac{n+2i-1}{2}}) \equiv 16i-6$, $2 \le i \le \frac{n-1}{2}$; $f^*(v'_n) \equiv 0$ and $f^*(v_0) \equiv 4$.

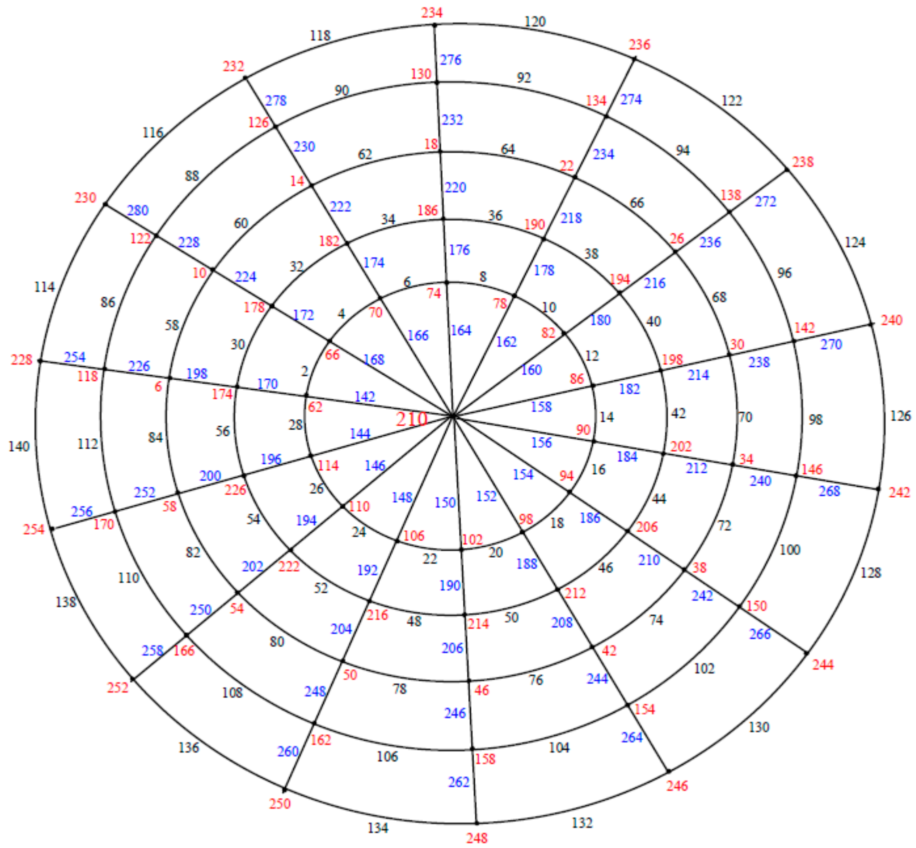


(**a**) $P_{10,5}$

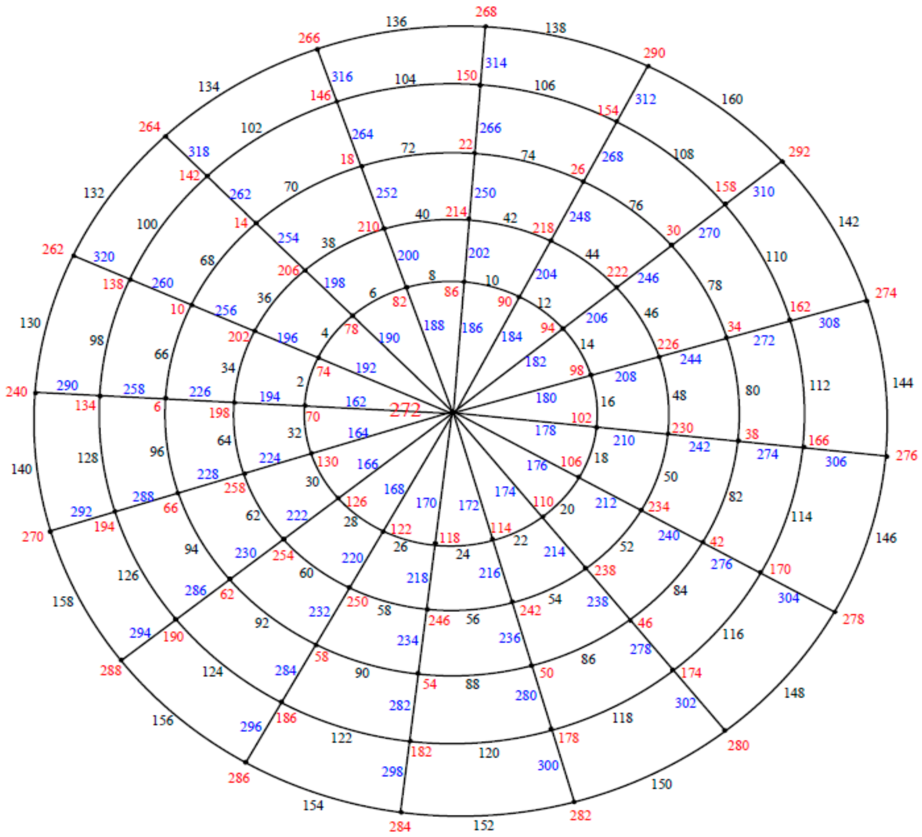


(**b**) $P_{12,5}$

Figure 8. *Cont.*

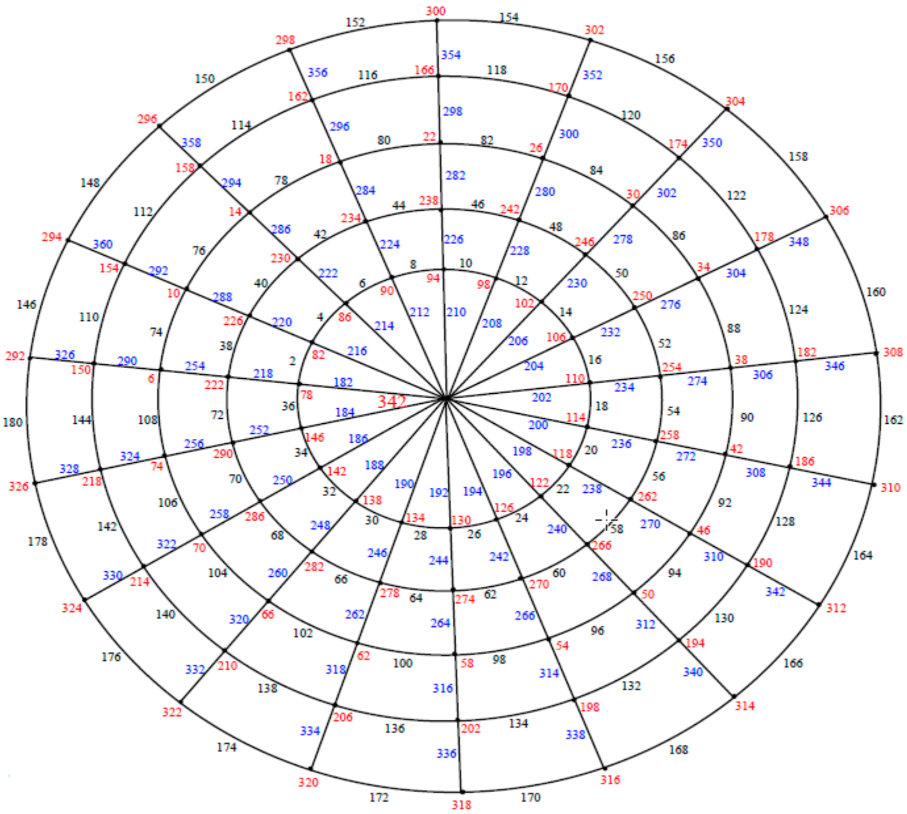


(**c**) $P_{14,5}$

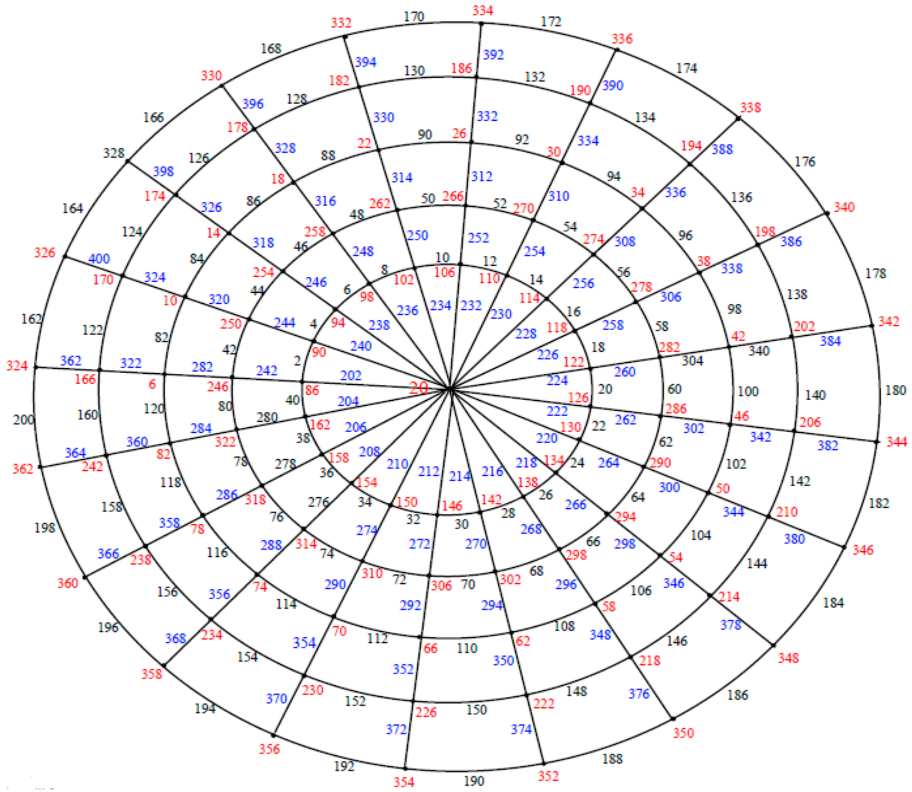


(**d**) $P_{16,5}$

Figure 8. *Cont.*



(e) $P_{18,5}$



(f) $P_{20,5}$

Figure 8. The edge even graceful labeling of the polar grid graphs $P_{10,5}$, $P_{12,5}$, $P_{14,5}$, $P_{16,5}$, $P_{18,5}$ and $P_{20,5}$.

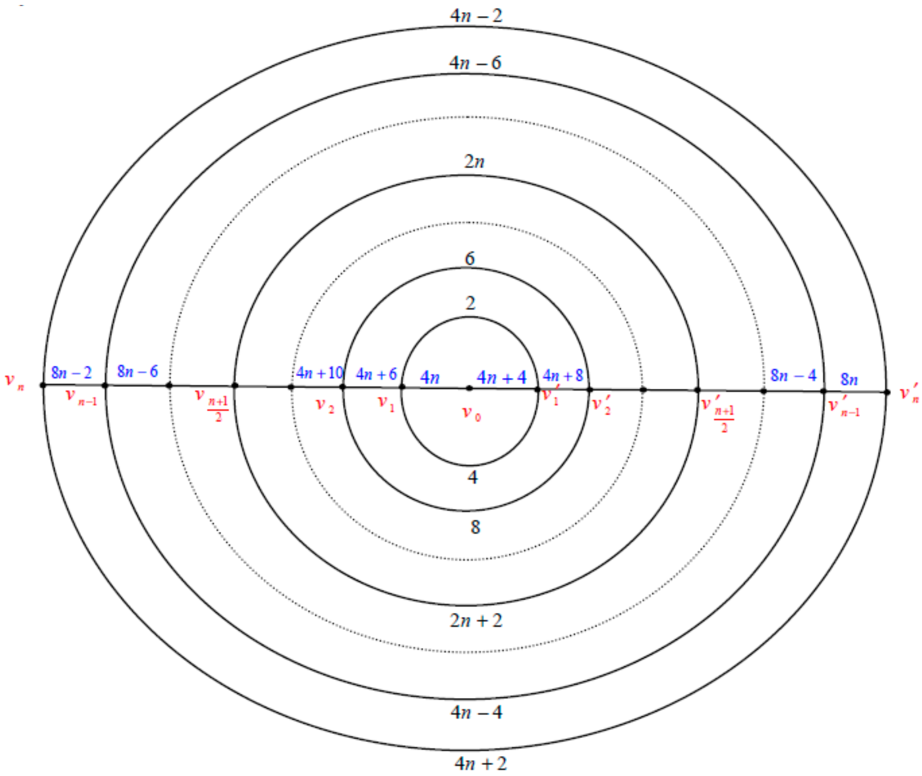


Figure 9. The labeling of the polar grid graph $P_{2,n}$, $n \geq 3$.

Illustration. *The edge even graceful labeling of the polar grid graphs $P_{2,5}$ is shown in Figure 10.*

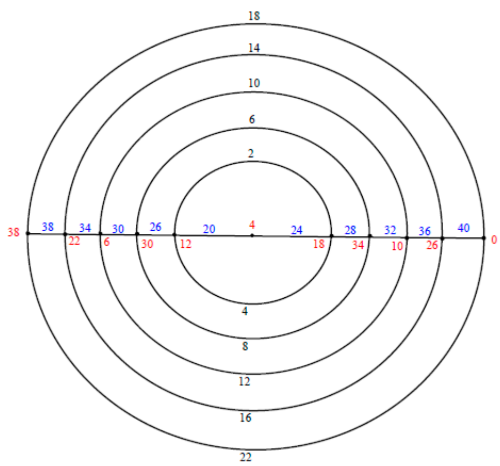


Figure 10. The labeling of the polar grid graph $P_{2,5}$.

Theorem 4. *If m and n are odd positive integers greater than 1. Then the polar grid graph $P_{m,n}$ is an edge even graceful graph.*

Proof. Let the polar grid graph $P_{m,n}$ be labeled as in Figure 7. Let $f : E(G) \to \{2, 4, \ldots, 2q\}$. □

The corresponding labels of vertices mod$4mn$ are assigned as follows: There are two cases:

Case (1) $n \equiv 1\,\mathrm{mod}4$, this case contains five subcases as follows:

SubCase (i) $m \equiv (4k-3)\mathrm{mod}\,4n$, $2 \leq k \leq n$

$f^*(v_{(k-1)m+i}) \equiv 4m(2k-1) + 4i + 2, 1 \leq k \leq n-1, 1 \leq i \leq m$. That is the labels of vertices of the most inner circle $C_m^{(1)}$ are assigned by $f^*(v_i) = 4m + 4i + 2$, the labels of vertices of the circle $C_m^{(2)}$ are assigned by $f^*(v_{m+i}) \equiv 12m+4i+2$, the labels of the vertices of the circle $C_m^{(\frac{n-1}{2})}$ are assigned by $f^*(v_{\frac{m(n-3)}{2}+i}) \equiv 4mn-8m+4i+2$, the labels of the vertices of the circle $C_m^{(\frac{n+1}{2})}$ are assigned by $f^*(v_{\frac{m(n-1)}{2}+i}) \equiv 4i+2$, the labels of the vertices of the circle $C_m^{(\frac{n+3}{2})}$ are assigned by $f^*(v_{\frac{m(n+1)}{2}+i}) \equiv 8m+4i+2, \ldots$, the labels of the vertices of the circle $C_m^{(n-1)}$ are assigned by $f^*(v_{m(n-2)+i}) \equiv 4mn-12m+4i+2$, $1 \le i \le m$ and the labels of the vertices of the outer circle $C_m^{(n)}$ are assigned by $f^*(v_{m(n-1)+i}) \equiv 4mn-12m+2i+2$, $1 \le i \le m$. The label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 2mn+2m(2k-1)$, when $k=\frac{n+1}{2}$, we have $f^*(v_0) \equiv 0$.

SubCase (ii) $m \equiv (8k-5) \bmod 4n$, $1 \le k \le \frac{n+1}{2}$, $m \ne 3$. In this subcase the vertex $v_{m(\frac{n+4k-1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{n+4k-1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{n+4k-1}{4})-(\frac{m+1}{2})}v_{m(\frac{n+4k-1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{n+4k-1}{4})-(\frac{m-1}{2})}v_{m(\frac{n+4k-1}{4})-(\frac{m-3}{2})}$. That is $f(v_{m(\frac{n+4k-1}{4})-(\frac{m+1}{2})}v_{m(\frac{n+4k-1}{4})-(\frac{m-1}{2})}) = m[2k+\frac{n-3}{2}]+1$ and $f(v_{m(\frac{n+4k-1}{4})-(\frac{m-1}{2})}v_{m(\frac{n+4k-1}{4})-(\frac{m-3}{2})}) = m[2k+\frac{n-3}{2}]-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{n+4k-1}{4})-(\frac{m+1}{2})}) \equiv 2mn+4m(2k-1)+2$, $f^*(v_{m(\frac{n+4k-1}{4})-(\frac{m-1}{2})}) \equiv 2mn+4m(2k-1)+4$, $f^*(v_{m(\frac{n+4k-1}{4})-(\frac{m-3}{2})}) \equiv 2mn+4m(2k-1)+6$, and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv 2mn+4m(2k-1)$. The rest vertices will be labeled as in subCase (i).

Remark 3. *When $n \equiv 1$ mod4 and $m=3$, in this case the vertex $v_{3(\frac{n-1}{4})+1}$ in the circle $C_3^{(\frac{n-1}{4}+1)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{3(\frac{n-1}{4})+2}v_{3(\frac{n-1}{4})+3}$ and $v_{3(\frac{n-1}{4})+1}v_{3(\frac{n-1}{4})+3}$. That is $f(v_{3(\frac{n-1}{4})+2}v_{3(\frac{n-1}{4})+3}) = 3(\frac{n-1}{2})+6$ and $f(v_{3(\frac{n-1}{4})+1}v_{3(\frac{n-1}{4})+3}) = 3(\frac{n-1}{2})+4$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{3(\frac{n-1}{4})+1}) \equiv 6n+10$, $f^*(v_{3(\frac{n-1}{4})+2}) \equiv 6n+18$, $f^*(v_{3(\frac{n-1}{4})+3}) \equiv 6n+16$ and the center vertex v_0 is labeld as $f^*(v_0) \equiv 6n+12$.*

SubCase (iii) $m \equiv (8k-1) \bmod 4n$, $1 \le k \le \frac{n-5}{4}$. In this subcase the vertex $v_{m(\frac{3n+4k+1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{3n+4k-1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{3n+4k+1}{4})-(\frac{m+1}{2})}v_{m(\frac{3n+4k+1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{3n+4k+1}{4})-(\frac{m-1}{2})}v_{m(\frac{3n+4k+1}{4})-(\frac{m-3}{2})}$. That is $f(v_{m(\frac{3n+4k+1}{4})-(\frac{m+1}{2})}v_{m(\frac{3n+4k+1}{4})-(\frac{m-1}{2})}) = 2mn+m[2k-\frac{n+1}{2}]+1$ and $f(v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})} v_{m(\frac{3n+4k-1}{4})-(\frac{m-3}{2})}) = 2mn+m[2k-\frac{n+1}{2}]-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{3n+4k+1}{4})-(\frac{m+1}{2})}) \equiv 2mn+8km+2$, $f^*(v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}) \equiv 2mn+8km+4$, $f^*(v_{m(\frac{3n+4k+1}{4})-(\frac{m-3}{2})}) \equiv 2mn+8km+6$, and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv 2mn+8km$. The rest vertices will be labeled as in subCase (i).

SubCase (iv) $m \equiv (8k-1) \bmod 4n$, $\frac{n+3}{4} \le k \le \frac{n-1}{2}$. In this case the vertex $v_{m(\frac{4k-n+1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{4n-k+1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{4k-n+1}{4})-(\frac{m+1}{2})}v_{m(\frac{4k-n+1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{4k-n+1}{4})-(\frac{m-1}{2})}v_{m(\frac{4k-n+1}{4})-(\frac{m-3}{2})}$. That is $f(v_{m(\frac{4k-n+1}{4})-(\frac{m+1}{2})}v_{m(\frac{4k-n+1}{4})-(\frac{m-1}{2})}) = m[2k-\frac{n+1}{2}]+1$ and $f(v_{m(\frac{4k-n+1}{4})-(\frac{m-1}{2})}v_{m(\frac{4k-n+1}{4})-(\frac{m-3}{2})}) = m[2k-\frac{n+1}{2}]-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{4k-n+1}{4})-(\frac{m+1}{2})}) \equiv 2mn+8km+2$, $f^*(v_{m(\frac{4k-n+1}{4})-(\frac{m-1}{2})}) \equiv 8km-2mn+4$, $f^*(v_{m(\frac{4k-n+1}{4})-(\frac{m-3}{2})}) \equiv 8km-2mn+6$, and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv 8km-2mn$. The rest vertices will be labeled as in subCase (i).

SubCase (v) $m \equiv (2n-3) \bmod 4n$. In this case the vertex v_{mn-1} in the outer circle $C_m^{(n)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{mn-2}v_{mn-1}$ and $v_{mn}v_{m(n-1)+1}$. That is $f(v_{mn-2}\,v_{mn-1}) = 2mn-4$, $f(v_{mn}v_{m(n-1)+1}) = 2mn$ and we obtain the labels of the corresponding vertices are as follows $f^*(v_{mn-2}) \equiv 4mn-2m+2$, $f^*(v_{mn-1}) \equiv 4mn-2m+4$,

$f^*(v_{mn}) \equiv 4mn - 2m - 2$ and $f^*(v_{m(n-1)+1}) \equiv 4mn - 4m$, and in this case the center vertex v_0 is labeled as $f^*(v_0) \equiv 4mn - 2m$. The rest vertices will be labeled as in subCase (i).

Illustration. *The edge odd graceful labeling of the polar grid graphs $P_{3,5}, P_{13,5}, P_{11,5}, P_{7,9}, P_{15,5}$ and $P_{7,5}$ respectively are shown in Figure 11.*

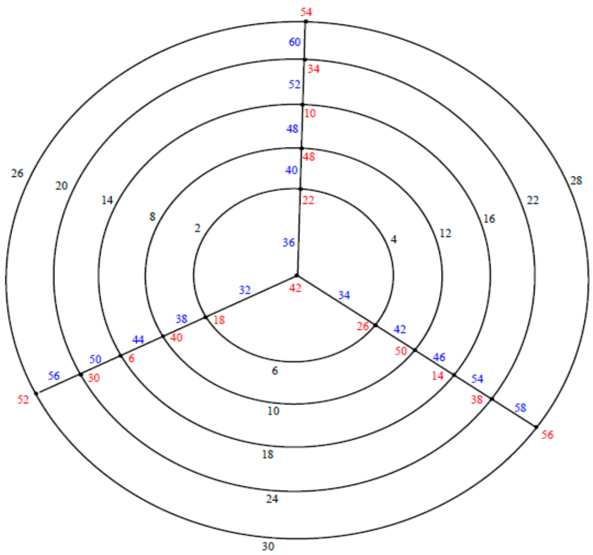
Case (2) $n \equiv 3 \bmod 4$. This case contains also five subcases as follows:

SubCase (i) $m \equiv (4k-3) \bmod 4n$, $2 \le k \le n$

$f^*(v_{(k-1)m+i}) \equiv 4m(2k-1) + 4i + 2, 1 \le k \le n-1, 1 \le i \le m$. That is the labels of vertices of the most inner circle $C_m^{(1)}$ are assigned by $f^*(v_i) \equiv 4m + 4i + 2$, the label of vertices of the circle $C_m^{(2)}$ are assigned by $f^*(v_{m+i}) \equiv 12m + 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n-1}{2})}$ are assigned by $f^*(v_{\frac{m(n-1)}{2}+i}) \equiv 4mn - 8m + 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n+1}{2})}$ are assigned by $f^*(v_{\frac{m(n-3)}{2}+i}) \equiv 4i + 2$, the labels of vertices of the circle $C_m^{(\frac{n+3}{2})}$ are assigned by $f^*(v_{\frac{m(n+1)}{2}+i}) \equiv 8m + 4i + 2, \ldots$, the labels of vertices of the circle $C_m^{(n-1)}$ are assigned by $f^*(v_{m(n-2)+i}) \equiv 4mn - 12m + 4i + 2$, $1 \le i \le m$ and the labels of the vertices of the outer circle $C_m^{(n)}$ are assigned by $f^*(v_{m(n-1)+i}) \equiv 4mn - 12m + 2i + 2$, $1 \le i \le m$. The label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 2mn + 2m(2k-1)$, when $k = \frac{n+1}{2}$, we have $f^*(v_0) \equiv 0$.

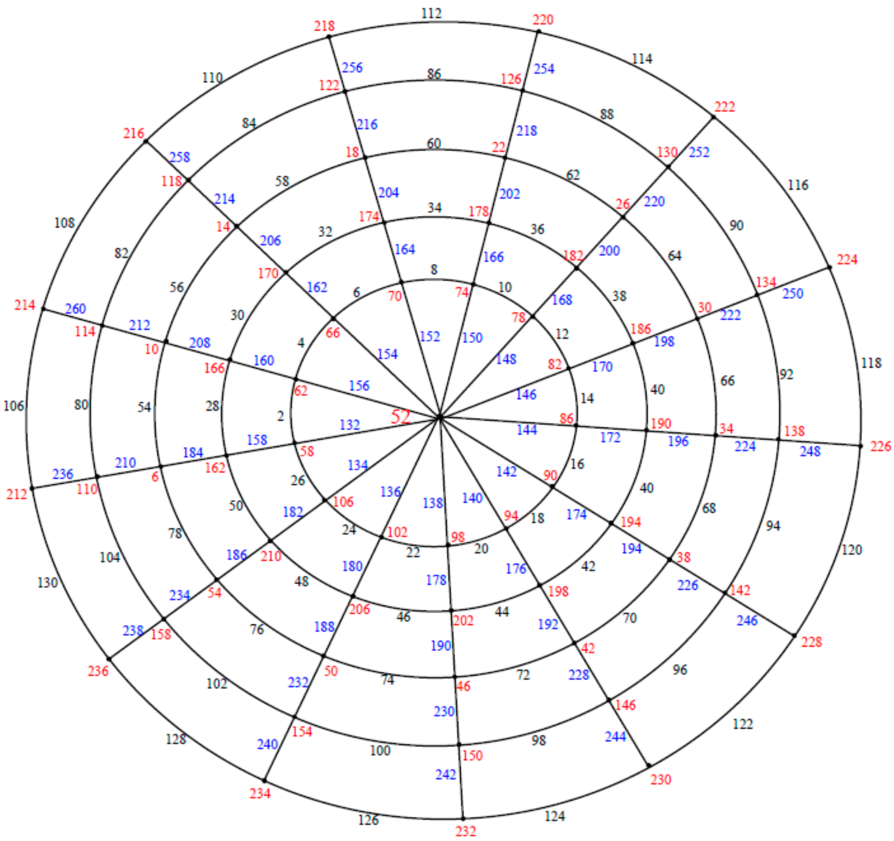
SubCase (ii) $m \equiv (8k-5) \bmod 4n$, $1 \le k \le \frac{n-3}{4}$, $m \ne 3$

In this subcase the vertex $v_{m(\frac{3n+4k-1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{3n+4k-1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{3n+4k-1}{4})-(\frac{m+1}{2})}v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}v_{m(\frac{3n+4k-1}{4})-(\frac{m-3}{2})}$. That is $f(v_{m(\frac{3n+4k-1}{4})-(\frac{m+1}{2})}v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}) = 2mn + m[2k - \frac{n+3}{2}] + 1$ and $f(v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}v_{m(\frac{3n+4k-1}{4})-(\frac{m-3}{2})}) = 2mn + m[2k - \frac{n+3}{2}] - 1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{3n+4k-1}{4})-(\frac{m+1}{2})}) \equiv 2mn + 4m(2k-1) + 2$, $f^*(v_{m(\frac{3n+4k-1}{4})-(\frac{m-1}{2})}) \equiv 2mn + 4m(2k-1) + 4$, $f^*(v_{m(\frac{3n+4k-1}{4})-(\frac{m-3}{2})}) \equiv 2mn + 4m(2k-1) + 6$, and the label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 2mn + 4m(2k-1)$. That rest vertices will be labeled as in subcase (i).

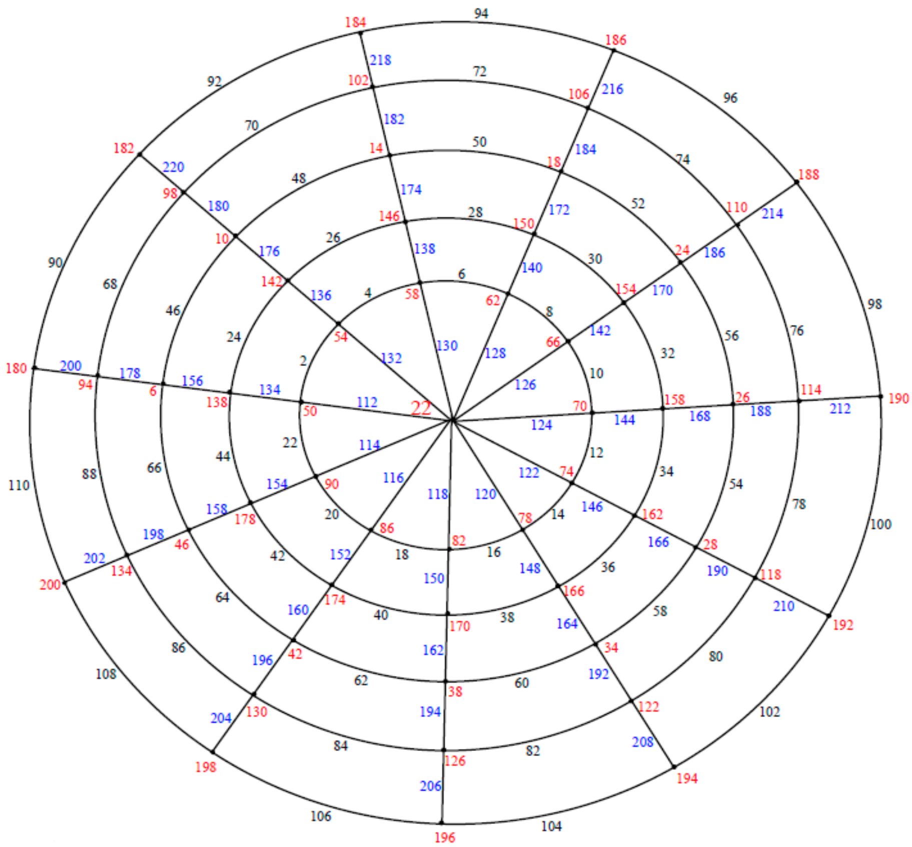


(a) $P_{3,5}$

Figure 11. *Cont.*

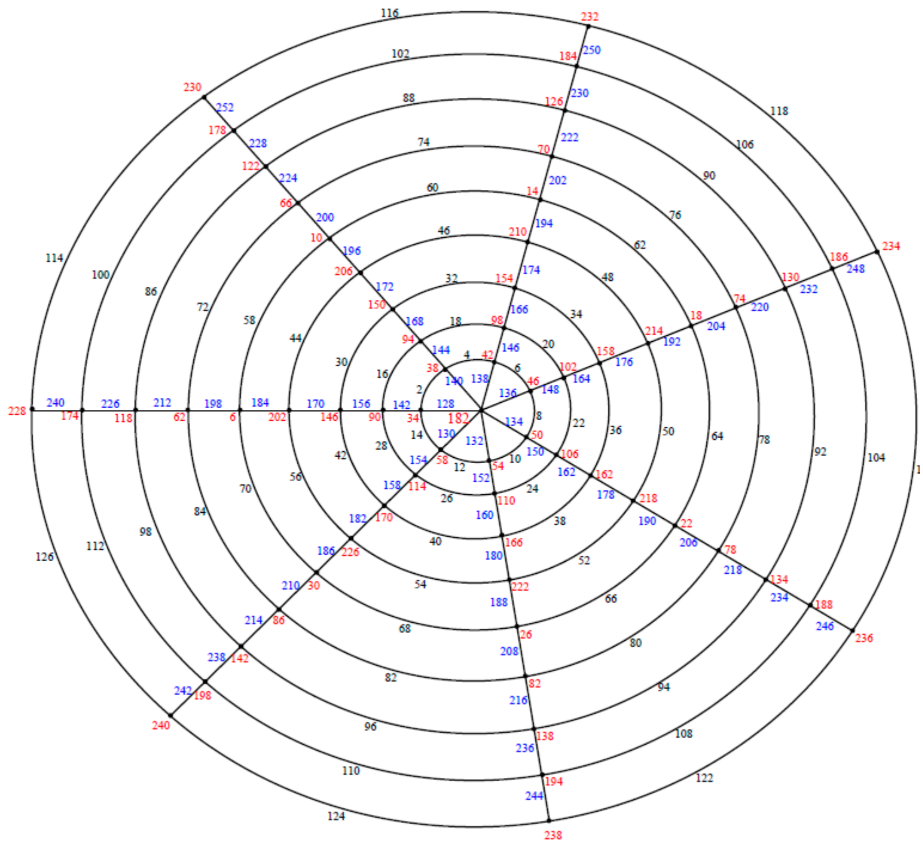


(**b**) $P_{13,5}$

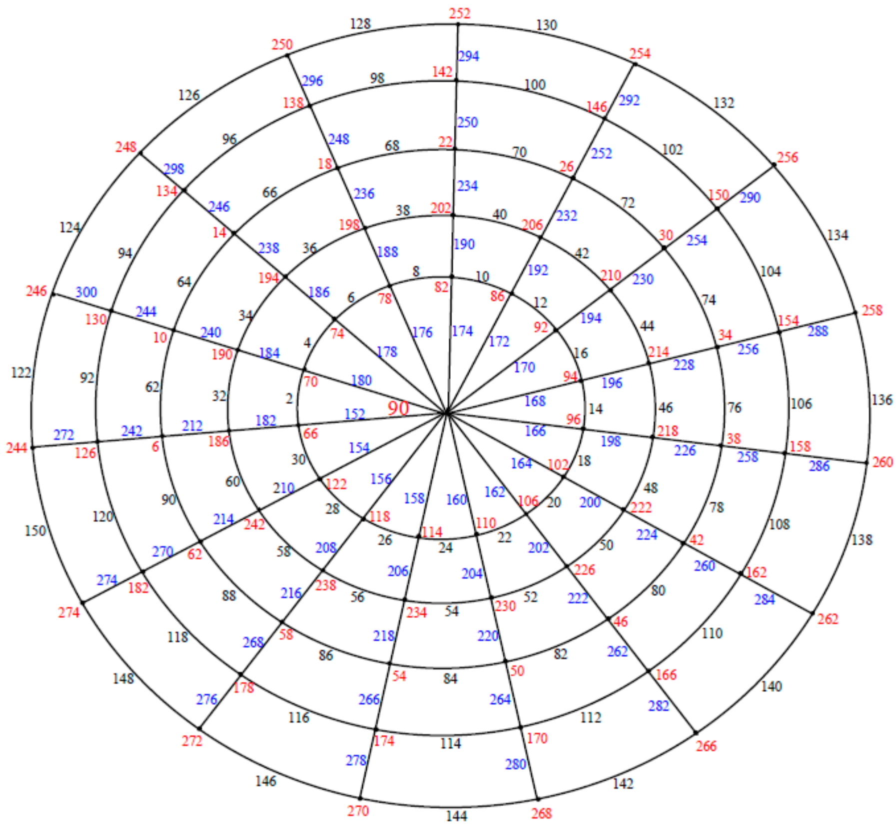


(**c**) $P_{11,5}$

Figure 11. *Cont.*

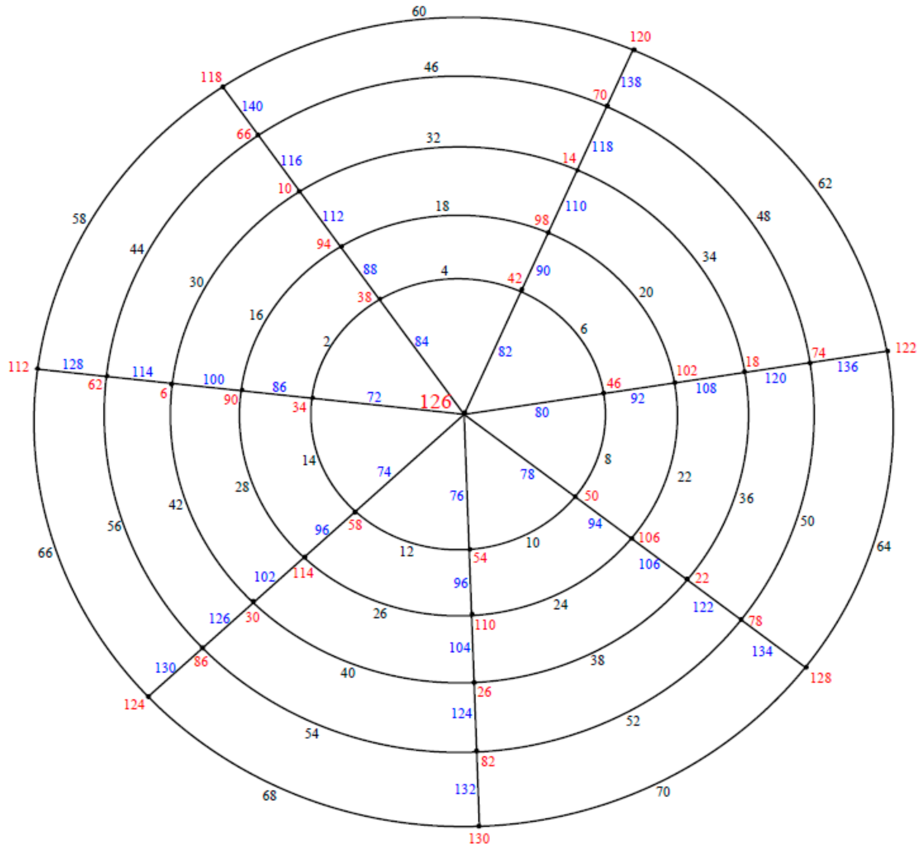


(**d**) $P_{7,9}$



(**e**) $P_{15,5}$

Figure 11. *Cont.*



(f) $P_{7,5}$

Figure 11. The polar grid graphs $P_{3,5}$, $P_{13,5}$, $P_{11,5}$, $P_{7,9}$, $P_{15,5}$ and $P_{7,5}$.

Remark 4. *When $n \equiv 3 \bmod 4$ and $m = 3$, we have the vertex $v_{3(\frac{3n-1}{4})+1}$ in the circle $C_m^{(\frac{3n-1}{4}+1)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{3(\frac{3n-1}{4})+2}v_{3(\frac{3n-1}{4})+3}$ and $v_{3(\frac{3n-1}{4})+1}v_{3(\frac{3n-1}{4})+3}$. That is $f(v_{3(\frac{3n-1}{4})+2}v_{3(\frac{3n-1}{4})+3}) = 3(\frac{3n-1}{2}) + 6$ and $f(v_{3(\frac{3n-1}{4})+1}v_{3(\frac{3n-1}{4})+3}) = 3(\frac{3n-1}{2}) + 4$ and we obtain the labes of the corresponding vertices mod$4mn$ are as follows: $f^*(v_{3(\frac{3n-1}{4})+1}) \equiv 6n+10$, $f^*(v_{3(\frac{3n-1}{4})+2}) \equiv 6n+18$, $f^*(v_{3(\frac{3n-1}{4})+3}) \equiv 6n+20$ and the label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 6n+12$.*

Note that $P_{3,3}$ is an edge even graceful grapg but not follow this rule. See Figure 12.

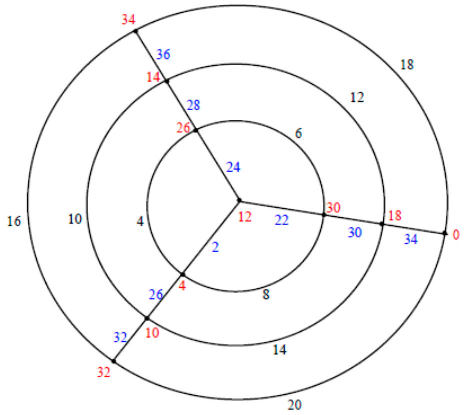


Figure 12. The polar grid graphs $P_{3,3}$.

SubCase (iii) $m \equiv (8k-1)\mathrm{mod}4n$, $\frac{n+5}{4} \leq k \leq \frac{n+1}{2}$,

In this subcase the vertex $v_{m(\frac{4k-n-1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{4k-n-1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{4k-n-1}{4})-(\frac{m+1}{2})}v_{m(\frac{4k-n-1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{4k-n-1}{4})-(\frac{m-1}{2})}v_{m(\frac{4k-n-1}{4})-(\frac{m-3}{2})}$ That is $f(v_{m(\frac{4k-n-1}{4})-(\frac{m+1}{2})}v_{m(\frac{4k-n-1}{4})-(\frac{m-1}{2})}) = (2k-1)m+1$ and $f(v_{m(\frac{4k-n-1}{4})-(\frac{m-1}{2})}v_{m(\frac{4k-n-1}{4})-(\frac{m-3}{2})}) = (2k-1)m-1$ and we obtain the labels of the corresponding vertices as follows: $f^*(v_{m(\frac{4k-n-1}{4})-(\frac{m+1}{2})}) \equiv 2m(4k-1)+2$, $f^*(v_{m(\frac{4k-n-1}{4})-(\frac{m-1}{2})}) \equiv 2m(4k-1)+4$ and $f^*(v_{m(\frac{4k-n-1}{4})-(\frac{m-3}{2})}) \equiv 2m(4k-1)+6$. The label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 2m(4k-1)$. The rest vertices will be labeled as in subCase (i).

SubCase (iv) $m \equiv (8k-1)\mathrm{mod}4n$, $1 \leq k \leq \frac{n-1}{2}$

In this subcase the vertex $v_{m(\frac{n+4k+1}{4})-(\frac{m+1}{2})}$ in the circle $C_m^{(\frac{n+4k+1}{4})}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{m(\frac{n+4k+1}{4})-(\frac{m+1}{2})}v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}$ and $v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}v_{m(\frac{n+4k+1}{4})-(\frac{m-3}{2})}$ That is $f(v_{m(\frac{n+4k+1}{4})-(\frac{m+1}{2})}v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}) = (2k+\frac{n-1}{2})m+1$ and $f(v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}v_{m(\frac{n+4k+1}{4})-(\frac{m-3}{2})}) = (2k+\frac{n-1}{2})m-1$ and we obtain the labels of the corresponding vertices as follows $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m+1}{2})}) = 2mn+8km+2$, $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}) = 2mn+8km+4$, $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m-3}{2})}) = 2mn+8km+6$ and the label of the center vertex v_0 is labeled as $f^*(v_0) \equiv 2mn+8km$. The rest vertices will be labeled as in subCase (i).

Remark 5. *If* $k = \frac{n+1}{4}$ *we have* $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m+1}{2})}) = 8km-2mn+2$, $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m-1}{2})}) = 8km-2mn+4$, $f^*(v_{m(\frac{n+4k+1}{4})-(\frac{m-3}{2})}) = 8km-2mn+6$ *and the center vertex* v_0 *is labeled as* $f^*(v_0) \equiv 8km-2mn$.

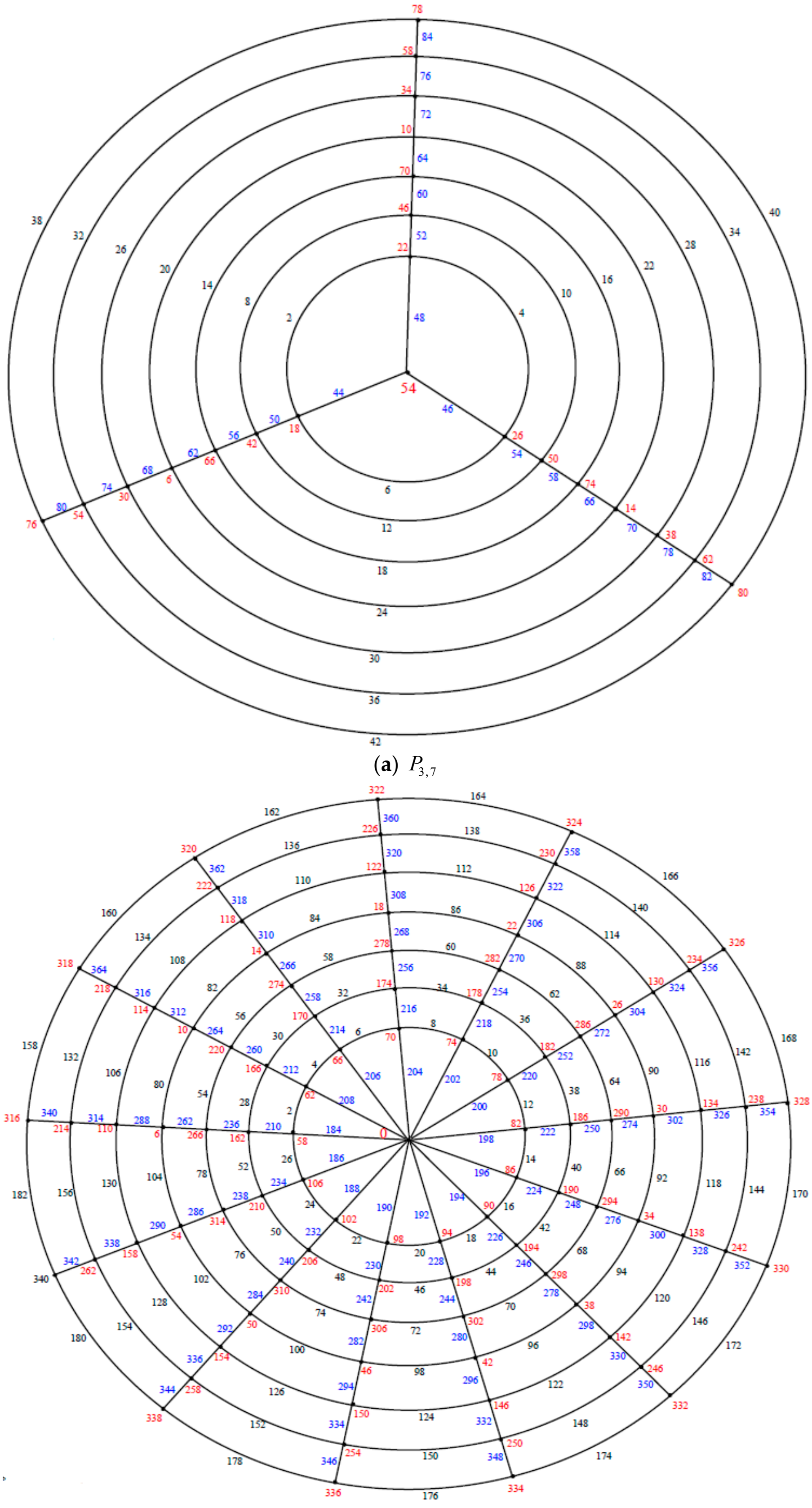
SubCase (v) $m \equiv (2n-3)\mathrm{mod}4n$

In this subcase the vertex v_{mn-1} in the outer circle $C_m^{(n)}$ will repeat with the center vertex v_0. To avoid this problem we replace the labels of the two edges $v_{mn-2}v_{mn-1}$ and $v_{mn}v_{m(n-1)+1}$. That is $f(v_{mn-2}v_{mn-1}) = 2mn-4$, $f(v_{mn}v_{m(n-1)+1}) = 2mn$ and we obtain the labes of the corresponding vertices as follows:

$f^*(v_{mn-2}) = 4mn-2m+2$, $f^*(v_{mn-1}) = 4mn-2m+4$, $f^*(v_{mn}) = 4mn-2m-2$ and $f^*(v_{m(n-1)+1}) = 4m(n-1)$ and the label of the center vertex v_0 is assigned by $f^*(v_0) \equiv 2m(2n-1)$.

The rest vertices will be labeled as in subCase (i).

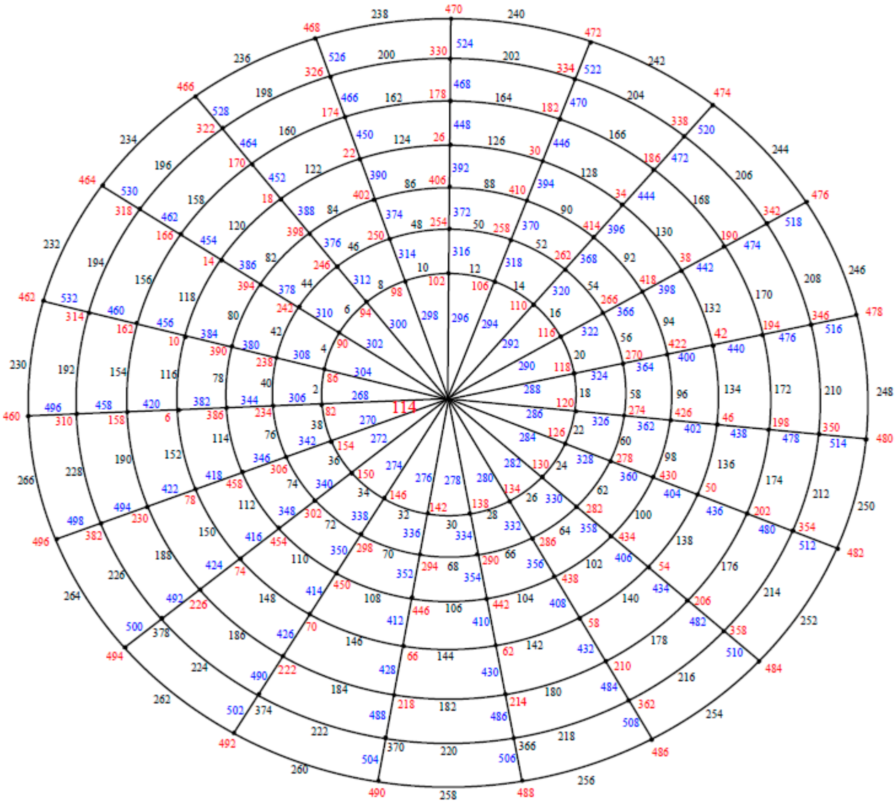
Illustration. *The edge odd graceful labeling of the polar grid graphs* $P_{3,7}$, $P_{13,7}$, $P_{19,7}$, $P_{11,7}$ *and* $P_{15,7}$ *respectively are shown in Figure 13.*



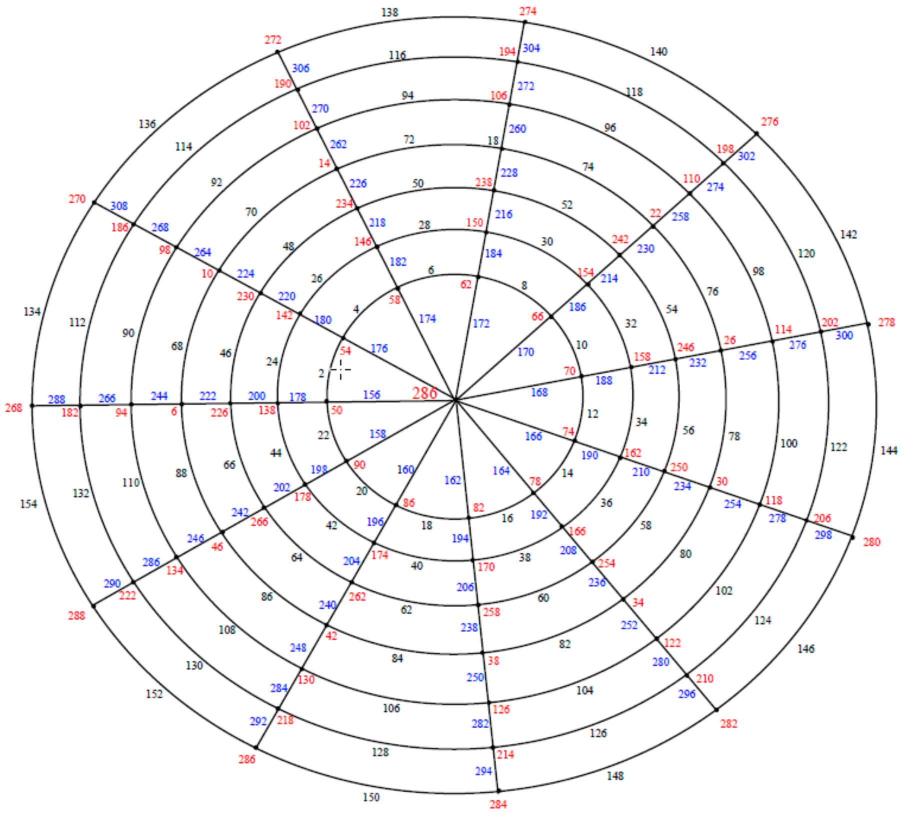
(**a**) $P_{3,7}$

(**b**) $P_{13,7}$

Figure 13. *Cont*.

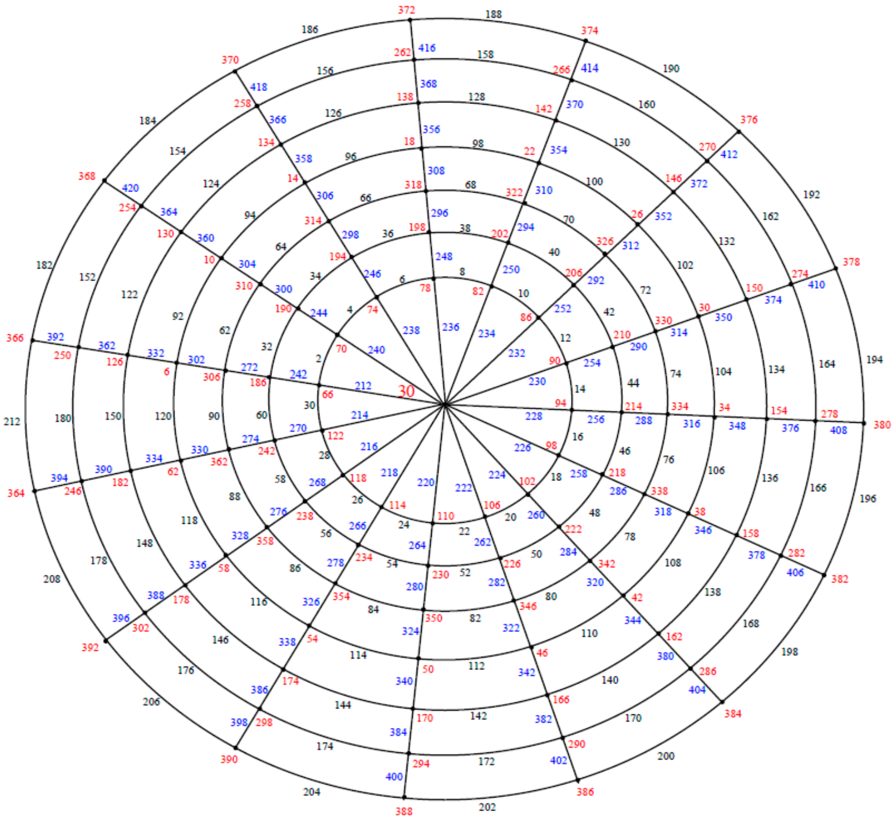


(**c**) $P_{19,7}$



(**d**) $P_{11,7}$

Figure 13. *Cont*.



(e) $P_{15,7}$

Figure 13. The polar grid graphs $P_{3,7}$, $P_{13,7}$, $P_{19,7}$, $P_{11,7}$ $P_{15,7}$.

3. Conclusions

This paper gives some basic knowledge about the application of Graph labeling and Graph Theory in real life which is the one branch of mathematics. It is designed for the researcher who research in graph labeling and graph Theory. In this paper, we give necessary and sufficient conditions for a polar grid graph to admit edge even labeling. In future work we will study the necessary and sufficient conditions for the cylinder $P_m \times C_n$, torus $C_m \times C_n$ and rectangular $P_m \times P_n$ grid graphs to be edge even graceful.

Funding: This work was supported by the deanship of Scientific Research, Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. Acharya, B.D.; Arumugam, S.; Rosa, A. *Labeling of Discrete Structures and Applications*; Narosa Publishing House: New Delhi, India, 2008; pp. 1–14.
2. Bloom, G.S. Numbered Undirected Graphs and Their Uses, a Survey of a Unifying Scientific and Engineering Concept and Its Use in Developing a Theory of Non-Redundant Homometric Sets Relating to Some Ambiguities in X-ray Diffraction Analysis. Ph.D. Thesis, University Southern California, Los Angeles, CA, USA, 1975.
3. Bloom, G.S.; Golomb, S.W. Numbered complete graphs, unusual rulers, and assorted applications. In *Theory and Applications of Graphs*; Lecture Notes in Math; Springer: New York, NY, USA, 1978; Volume 642, pp. 53–65.
4. Bloom, G.S.; Golomb, S.W. Applications of numbered undirected graphs. *Proc. IEEE* **1977**, *65*, 562–570. [CrossRef]

5. Bloom, G.S.; Hsu, D.F. On graceful digraphs and a problem in network addressing. *Congr. Numer.* **1982**, *35*, 91–103.
6. Graham, R.L.; Pollak, H.O. On the addressing problem for loop switching. *Bell Syst. Tech. J.* **1971**, *50*, 2495–2519. [CrossRef]
7. Sutton, M. Summable Graphs Labellings and their Applications. Ph.D. Thesis, Department Computer Science, The University of Newcastle, Callaghan, Australia, 2001.
8. Shang, Y. More on the normalized Laplacian Estrada index. *Appl. Anal. Discret. Math.* **2014**, *8*, 346–357. [CrossRef]
9. Shang, Y. Geometric Assortative Growth Model for Small-World Networks. *Sci. World J.* **2014**, *2014*, 759391. [CrossRef] [PubMed]
10. Gross, J.; Yellen, J. *Graph Theory and Its Applications*; CRC Press: London, UK, 1999.
11. Rosa, A. *Certain Valuations of the Vertices of a Graph*; Theory of Graphs (Internat. Symp, Rome, July 1966); Gordan and Breach: New York, NY, USA; Paris, France, 1967; pp. 349–355.
12. Golomb, S.W. How to Number a Graph. In *Graph Theory and Computing*; Read, R.C., Ed.; Academic Press: NewYork, NY, USA, 1972; pp. 23–37.
13. Gnanajothi, R.B. Topics in Graph Theory. Ph.D. Thesis, Madurai Kamaraj University, Tamil Nadu, India, 1991.
14. Seoud, M.A.; Abdel-Aal, M.E. On odd graceful graphs. *Ars Comb.* **2013**, *108*, 161–185.
15. Gao, Z. Odd graceful labelings of some union graphs. *J. Nat. Sci. Heilongjiang Univ.* **2007**, *24*, 35–39.
16. Lo, S.P. On edge-graceful1abelings of graphs. *Congr. Numerantium* **1985**, *50*, 231–241.
17. Kuan, Q.; Lee, S.; Mitchem, J.; Wang, A. On Edge-Graceful Unicyclic Graphs. *Congr. Numerantium* **1988**, *61*, 65–74.
18. Lee, L.; Lee, S.; Murty, G. On Edge-Graceful Labelings of Complete Graphs: Solutions of Lo's Conjecture. *Congr. Numerantium* **1988**, *62*, 225–233.
19. Solairaju, A.; Chithra, K. Edge-Odd Graceful Graphs. *Electron. Notes Discret. Math.* **2009**, *33*, 15–20. [CrossRef]
20. Daoud, S.N. Edge odd graceful labeling of some path and cycle related graphs. *AKCE Int. J. Graphs Comb.* **2017**, *14*, 178–203. [CrossRef]
21. Elsonbaty, A.; Daoud, S.N. Edge even graceful labeling of some path and cycle related graphs. *Ars Comb.* **2017**, *130*, 79–96.
22. Gallian, J.A. A Dynamic Survey of Graph Labeling. *Electron. J. Comb.* **2017**, *22*, #DS6.

© 2019 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).